THE ZELEVINSKY CLASSIFICATION OF UNRAMIFIED REPRESENTATIONS OF THE METAPLECTIC GROUP

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Abstract. In this paper a Zelevinsky type classification of genuine unramified irreducible representations of the metaplectic group over a p-adic field with $p \neq 2$ is obtained. The classification consists of three steps. Firstly, it is proved that every genuine irreducible unramified representation is a fully parabolically induced representation from unramified characters of general linear groups and a genuine irreducible negative unramified representation of a smaller metaplectic group. Genuine irreducible negative unramified representations are described in terms of parabolic induction from unramified characters of general linear groups and a genuine irreducible strongly negative unramified representation of a smaller metaplectic group. Finally, genuine irreducible strongly negative unramified representations are classified in terms of Jordan blocks. The main technical tool is the theory of Jacquet modules.

Introduction

In this paper we study the representation theory of the metaplectic group over a p-adic field with $p \neq 2$, that is, the unique non-trivial two-fold central extension of the p-adic symplectic group. Its importance among covering groups comes from the fact that it appears, together with a classical group, as a member of a dual pair in the theory of theta correspondence and the Weil representation, thus having applications in number theory.

On the other hand, the structure of this metaplectic group is almost like the structure of split classical groups [12], so that, in principle, the methods coming from representation theory of classical groups can be adjusted, with more or less difficulty, to the metaplectic group. In our case, the main technical tool is the theory of Jacquet modules, which we adjust for the application to the metaplectic group from the paper of Muić [18] classifying the unramified irreducible representations of split classical groups.

The reader should be aware that not all methods available for the study of representations of p-adic classical groups are extended to the metaplectic group. For example, the theory of R groups [10] is still not available. The basic properties of the Aubert–Schneider–Stuhler involution [1], [2], [20] are, at the time of writing this paper, under consideration by Dubravka Ban and Chris Jantzen, and it seems they hold in the setting of the metaplectic group. It is very likely that some of our proofs could be considerably simplified once these techniques are proved for the metaplectic group. However, for instance, the standard module conjecture [17], [6] fails for the metaplectic group, because the even Weil representation is generic and the associated standard module is reducible.

Hence, although we expect that most of the methods will eventually become available, we carefully restrict our tools in this paper to those techniques that are applicable for all l-groups countable at infinity (cf. [3], [4], [25]), such as Jacquet modules and induced representations, and the Langlands quotient theorem [14]. Classical groups, but also the metaplectic group, are examples of such groups.

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The goal of the paper is to provide the Zelevinsky type classification of (isomorphism classes) of unramified irreducible representations of the p-adic metaplectic group with $p \neq 2$. This type of classification for split classical groups over a p-adic field is obtained by Muić in [18]. The unramified representations are important in number theory, in particular the theory of automorphic forms, because the local component of an automorphic representation is unramified at all but finitely many places of the number field. For many number theoretic applications it is sufficient to work only with such "unramified places". Hence, having a good classification of unramified representations for the metaplectic group is of considerable interest, not only for the representation theory itself, but also in view of number theoretic applications.

The Zelevinsky classification consists of three steps. The first step says that every unramified representation is a fully parabolically induced representation from unramified characters of general linear groups and a negative unramified representation of a smaller metaplectic group. Then, negative unramified representations are described in terms of parabolic induction from unramified characters of general linear groups and a strongly negative unramified representation of a smaller metaplectic group. Finally, strongly negative unramified representations are classified in terms of Jordan blocks. For definitions of negative and strongly negative representations see the body of the paper.

The structure of the paper, after preliminary Sect. 1 on the metaplectic group and Sect. 2 on its representations, follows basically the three steps of the classification. The first two steps are made in Sect. 3, although the strong form of the first step cannot be proved until the final Sect. 5, as it requires the classification of negative and strongly negative representations. The third step is contained in Sect. 4. In Appendix A, to avoid interrupting the flow of arguments in the proof of classification, we provide a quite long proof of a certain technical lemma regarding reducibility of certain degenerate principal series representation.

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1. Metaplectic group

1.1. **Two-sheeted central extension.** Let F be a p-adic field of residual characteristic $p \neq 2$ with the ring of integers $\mathcal{O}$, containing q elements in its residue field. We denote by $|\,|_F$ the normalized absolute value on F. For an integer $n \geq 1$, let $Sp(n, F)$ be the group of F-points of the F-split symplectic group of F-rank n defined over F. When necessary, we always use the same matrix realization of $Sp(n, F)$ as in [13]. We fix, once for all, a maximal compact subgroup $Sp(n, \mathcal{O})$ of $Sp(n, F)$.

Let $\widetilde{Sp(n, F)}$ be the metaplectic group, that is, the unique non-trivial twofold central extension of $Sp(n, F)$. It fits into an exact sequence

$$1 \longrightarrow \mu_2 \overset{i}{\hookrightarrow} \widetilde{Sp(n, F)} \overset{p}{\twoheadrightarrow} Sp(n, F) \longrightarrow 1,$$

where $\mu_2 = \{\pm 1\}$ is the multiplicative group. As a set $\widetilde{Sp(n, F)} = Sp(n, F) \times \mu_2$, and the maps i and p are the obvious inclusion and projection. The multiplication is defined by

$$[h_1, \epsilon_1][h_2, \epsilon_2] = [h_1 h_2, \epsilon_1 \epsilon_2 \ c_{Rao}(h_1, h_2)], \quad h_i \in Sp(n, F),\ \epsilon_i \in \mu_2,\ i \in \{1, 2\},$$

where c_{Rao} is the Rao cocyle described in [13], [19].

For an integer $n \geq 1$, let $GL(n, F)$ be the general linear group of $n \times n$ regular matrices over F. We write $\nu = |\det|_F$. Consider the two-fold central extension $\widetilde{GL(n, F)}$ of $GL(n, F)$, given as the preimage, with respect to p, of the embedding of $GL(n, F)$ into $Sp(n, F)$ as the stabilizer of a maximal polarization of the underlying symplectic space. The multiplication is given by

$$[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 g_2, \epsilon_1 \epsilon_2 \ (\det g_1, \det g_2)_F], \quad g_i \in GL(n, F),\ \epsilon_i \in \mu_2,\ i \in \{1,\ 2\},$$

where $(\, , \,)_F$ is the quadratic Hilbert symbol of F [24].

By convention, for $n=0$ all the covering groups are considered to be μ_2, and all classical groups to be the trivial group.

1.2. **Parabolic subgroups.** We fix the Borel subgroup of $Sp(n,F)$ as in [13, Chap. III]. Then, as in *loc. cit.*, the standard parabolic subgroups of $Sp(n,F)$ are parameterized by ordered partitions of n into $s=(n_1,\dots,n_k;n_0)$, where $n_i \geq 1, \ i=1,\dots,k$, and $n_0 \geq 0$ are integers. We write P_s for the parabolic subgroup parameterized by s. In the case of $s=(-;n)$, i.e. $k=0$, we have $P_s = Sp(n,F)$. For $k=1$ we obtain maximal parabolic subgroups. Partition $s=(n;0)$, with $n_0=0$, gives the so-called Siegel parabolic subgroup, which we sometimes denote $P_S = P_{(n;0)}$. There is a Levi decomposition $P_s = M_s N_s$, where M_s is the Levi factor and N_s the unipotent radical.

Let $\widetilde{P_s}$ and $\widetilde{M_s}$ be the preimages of P_s and M_s in $\widetilde{Sp(n,F)}$ with respect to the projection p, and $N_s' = N_s \times \{1\}$. Then $\widetilde{P_s}$ are the standard parabolic subgroups of $\widetilde{Sp(n,F)}$, and there is a Levi decomposition $\widetilde{P_s} = \widetilde{M_s} N_s'$. For the Levi factor $\widetilde{M_s}$, according to [12], [11, p. 4], there is an epimorphism ϕ with finite kernel

$$\widetilde{GL(n_1,F)} \times \cdots \times \widetilde{GL(n_k,F)} \times \widetilde{Sp(n_0,F)} \xrightarrow{\phi} \widetilde{M_s}.$$

Similarly, fixing the Borel subgroup in $GL(n,F)$, the standard parabolic subgroups P_s of $GL(n,F)$ are parameterized by ordered partitions $s=(n_1,\dots,n_k)$ of n into positive integers. Let $P_s = M_s N_s$ be the Levi decomposition. Then the standard parabolic subgroups $\widetilde{P_s}$ of $\widetilde{GL(n,F)}$ are preimages of P_s with respect to p. We have the Levi decomposition $\widetilde{P_s} = \widetilde{M_s} \ltimes N_s'$, where $\widetilde{M_s}$ is the preimage of M_s and $N_s' = N_s \times \{1\}$. There is, again, an epimorphism with finite kernel

$$\widetilde{GL(n_1,F)} \times \cdots \times \widetilde{GL(n_k,F)} \xrightarrow{\phi} \widetilde{M_s} \leq \widetilde{GL(n,F)}.$$

1.3. **Splitting of the cover.** Recall that $\widetilde{Sp(n,F)}$ splits uniquely over $Sp(n,\mathcal{O})$ (cf. [16, Sect. 2.II.10]). We write $h \mapsto [h, i_n(h)]$, $h \in Sp(n,\mathcal{O})$, for the splitting. By [21, Lemma 2.1], the map i_n is trivial on $P_S \cap Sp(n,\mathcal{O})$. Directly, or embedding $GL(n,\mathcal{O})$ into $Sp(n,\mathcal{O})$, where $GL(n,\mathcal{O})$ is a fixed maximal compact subgroup of $GL(n,F)$, we obtain the splitting $g \mapsto [g,1]$, $g \in GL(n,\mathcal{O})$ in $\widetilde{GL(n,\mathcal{O})}$. Thus, the splitting from $GL(n,\mathcal{O})$ and from $Sp(n,\mathcal{O})$ restricted to $GL(n,\mathcal{O})$ match.

Lemma 1.1. *Let G be either $GL(n,F)$ or $Sp(n,F)$, and K the fixed maximal compact subgroup of G. Let $\overline{K}$ be the image of the splitting of K. In the notation as above, we have*

(1) $\widetilde{G} = \widetilde{P_s}\overline{K}$ *(Iwasawa decomposition)*
(2) $\widetilde{P_s} \cap \overline{K} = (\widetilde{M_s} \cap \overline{K})(N_s' \cap \overline{K})$
(3) *if* $G = GL(n,F)$ *then,* $\overline{GL(n_1,\mathcal{O})} \times \cdots \times \overline{GL(n_k,\mathcal{O})} \overset{\phi}{\cong} \widetilde{M_s} \cap \overline{K}$
(4) *if* $G = Sp(n,F)$ *then,* $\overline{GL(n_1,\mathcal{O})} \times \cdots \times \overline{GL(n_k,\mathcal{O})} \times \overline{Sp(n_0,\mathcal{O})} \overset{\phi}{\cong} \widetilde{M_s} \cap \overline{K}$.

Proof. Claim (1) directly follows from the Iwasawa decomposition $G = P_s K$. Claim (2) follows from the analogous decomposition for G and the fact that the splitting is trivial over N_s. Claim (3) is a direct computation using the formula for ϕ from [12] and the fact that the Hilbert symbol is trivial on units when the residual characteristic $p \neq 2$.

For claim (4), we need to check that the image of ϕ restricted to the group on the left-hand side really does lie in $\overline{K}$. It is enough to check this on some set of generators. Using the formula for ϕ from [12], we get

$$\begin{aligned} \phi([g_1,1],\dots,[g_k,1],[I_{n_0},1]) &= [(g_1,\dots,g_k,I_{n_0}),1], \quad g_i \in GL(n_i,\mathcal{O}), \ i=1,\dots,k, \\ \phi([I_{n_1},1],\dots,[I_{n_k},1],[h,i_{n_0}(h)]) &= [(I_m,h),i_{n_0}(h)], \qquad h \in Sp(n,\mathcal{O}), \end{aligned}$$

where I_j is the $j \times j$ identity matrix, and $m = n - n_0$. The image in the first formula is in $\overline{K}$, as we observed already that i_n is trivial on $P_S \cap K$. To show that the image in the second formula is in $\overline{K}$, it is sufficient to prove that $i_{n_0}(h) = i_n(I_m, h)$, that is, the splitting on $Sp(n_0, \mathcal{O})$ should coincide with the splitting on $Sp(n_0, \mathcal{O})$ coming from the embedding in $Sp(n, \mathcal{O})$ as a factor in a Levi subgroup. However, the splitting is unique, so the claim follows. □

2. Preliminaries from representation theory

2.1. Parabolic induction and Jacquet modules. In this entire section, G is either $Sp(n, F)$ or $GL(n, F)$, and $\widetilde{G}$ its two-fold cover defined in Sect. 1. In both cases we use the notation $P_s = M_s N_s$ for the parabolic subgroup of G attached to partition s. Since $\widetilde{G}$ is an l-group, we have the usual notions of smooth and admissible representations [3]. Representations that do not act trivially by μ_2 are called genuine, and only such are considered. With our choice of the Borel group, functors of the normalised parabolic induction and Jacquet module

$$\mathrm{Ind}_{\widetilde{M_s}}^{\widetilde{G}} : \mathrm{Alg}\, \widetilde{M_s} \to \mathrm{Alg}\, \widetilde{G},$$
$$r_s = \mathrm{Jacq}_s = \mathrm{Jacq}_{\widetilde{M_s}}^{\widetilde{G}} : \mathrm{Alg}\, \widetilde{G} \to \mathrm{Alg}\, \widetilde{M_s},$$

are defined as in [12], [11], where Alg stands for the category of smooth representations.

For σ in $\mathrm{Alg}\, \widetilde{G}$ and ρ in $\mathrm{Alg}\, \widetilde{M_s}$, we have the Frobenius reciprocity

$$\mathrm{Hom}_{\widetilde{G}}\left(\sigma, \mathrm{Ind}_{\widetilde{M_s}}^{\widetilde{G}}(\rho)\right) \cong \mathrm{Hom}_{\widetilde{M_s}}\left(\mathrm{Jacq}_{\widetilde{M_s}}^{\widetilde{G}}(\sigma), \rho\right). \tag{2.1}$$

Moreover, as remarked in [16, p. 59], all results of [4, Sect. 2] remain valid for the metaplectic group. Recall that σ is a cuspidal representation of G or $\widetilde{G}$ if the Jacquet module of σ is trivial with respect to any proper parabolic subgroup. Every irreducible representation can be embedded into a representation parabolically induced from a cuspidal one.

For the group $G = GL(n, F)$ the theory of genuine representations of $\widetilde{G}$ can be completely determined from the representation theory of G, as explained in [11, Sect. 4.1]. There is a bijection between smooth representations of finite length of G and $\widetilde{G}$, preserving irreducibility and commuting with the parabolic induction and Jacquet module. This bijection is given by twisting by a fixed genuine character χ_ψ of $\widetilde{G}$. Note that χ_ψ is not unique, and we now make our choice for it. Fix a non-trivial additive character ψ of F of even conductor. As in [12, p. 231], we define

$$\chi_\psi([g, \epsilon]) = \epsilon \gamma\left(\psi_{\frac{1}{2}}\right) \gamma\left(\psi_{\frac{\det g}{2}}\right)^{-1}, \quad g \in GL(n, F),\ \epsilon \in \mu_2,$$

where $\psi_a(x) = \psi(ax)$, and $\gamma(\eta)$ is the Weil index.

Now let $G = Sp(n, F)$. For $s = (n_1, \ldots, n_k; n_0)$ and ρ an irreducible genuine representation of $\widetilde{M_s}$, pulling back ρ with respect to ϕ enables us to think of it as $\chi_\psi \pi_1 \otimes \cdots \otimes \chi_\psi \pi_k \otimes \tau$ where π_i are irreducible representations of $GL(n_i, F)$, $i = 1, \ldots, k$, and τ a genuine irreducible representation of $\widetilde{Sp(n_0, F)}$. We use the notation, as for classical groups,

$$\mathrm{Ind}_{\widetilde{M_s}}^{\widetilde{G}}(\rho) = \mathrm{Ind}_{\widetilde{M_s}}^{\widetilde{G}}(\chi_\psi \pi_1 \otimes \cdots \otimes \chi_\psi \pi_k \otimes \tau) = \chi_\psi \pi_1 \times \cdots \times \chi_\psi \pi_k \rtimes \tau.$$

For $G = GL(n, F)$ we have the same argument and use the same notation, just without τ.

Calculation of the Weyl group action [11, Sect. 3] shows that composition series for $G = GL(n, F)$ remains the same after permuting π_i's. For $G = Sp(n, F)$, the composition series also remains the same after taking contragredients of $\chi_\psi \pi_i$. Note that $\chi_\psi^4 = 1$, so $(\chi_\psi \pi_i)^{\sim} \cong \chi_\psi\left(\chi_\psi^2 \widetilde{\pi}_i\right)$, where $\chi_\psi^2(g, \epsilon) = (\det g, -1)_F$, $g \in GL(n, F)$, $\epsilon \in \mu_2$.

By convention, the genuine irreducible representation of $\widetilde{Sp(0, F)} = \mu_2$ is denoted by ω_0, while the genuine irreducible representation of $\widetilde{GL(0, F)}$ is written as $\chi_\psi \mathbf{1}$, where $\mathbf{1}$ denotes the irreducible representation of the trivial group.

2.2. **Zelevinsky segment representation.** The following lemma summarizes some results of [25] transferred to $\widetilde{GL(n,F)}$ by twisting by χ_ψ. Its purpose is also to fix the notation.

Lemma 2.1. *Let χ be a character of $F^\times$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta+1 \in \mathbb{Z}_{>0}$. Recall that $\nu = |\det|_F$. Representation $\chi_\psi\chi\nu^{-\beta}\times\cdots\times\chi_\psi\chi\nu^{\alpha}$ has a unique irreducible subrepresentation $\zeta(-\beta,\alpha,\chi_\psi\chi)$ and a unique irreducible quotient $\delta(-\beta,\alpha,\chi_\psi\chi)$, so that*

$$\chi_\psi(\chi\circ\det)\nu^{\frac{-\beta+\alpha}{2}} \cong \zeta(-\beta,\alpha,\chi_\psi\chi) \hookrightarrow \chi_\psi\nu^{-\beta}\chi\times\cdots\times\chi_\psi\nu^{\alpha}\chi \twoheadrightarrow \delta(-\beta,\alpha,\chi_\psi\chi).$$

The representation $\zeta(-\beta,\alpha,\chi_\psi\chi)$ may be characterized as the unique subquotient such that $r_{(1,\ldots,1)}(\zeta(-\beta,\alpha,\chi_\psi\chi)) = \chi_\psi\nu^{-\beta}\chi\otimes\cdots\otimes\chi_\psi\nu^{\alpha}\chi$, while $\delta(-\beta,\alpha,\chi_\psi\chi)$ as the unique subquotient such that $r_{(1,\ldots,1)}(\delta(-\beta,\alpha,\chi_\psi\chi)) = \chi_\psi\nu^{\alpha}\chi\otimes\cdots\otimes\chi_\psi\nu^{-\beta}\chi$.

For the contragredient, we have $\zeta(-\beta,\alpha,\chi_\psi\chi)\tilde{} \cong \chi_\psi^2\zeta(-\alpha,\beta,\chi_\psi\chi^{-1})$ and $\delta(-\beta,\alpha,\chi_\psi\chi)\tilde{} \cong \chi_\psi^2\delta(-\alpha,\beta,\chi_\psi\chi^{-1})$.

Representation $\zeta(-\beta_1,\alpha_1,\chi_\psi\chi_1)\times\zeta(-\beta_2,\alpha_2,\chi_\psi\chi_2)$ reduces if and only if $\chi_1=\chi_2$, $\alpha_1-\alpha_2\in\mathbb{Z}$ and $-\beta_1\le-\beta_2-1\le\alpha_1<\alpha_2$ or $-\beta_2\le-\beta_1-1\le\alpha_2<\alpha$. In case of reducibility, induced representation has two non-isomorphic irreducible subquotients and for $\alpha_1<\alpha_2$ we have:

$$\zeta(-\beta_1,\alpha_2,\chi_\psi\chi)\times\zeta(-\beta_2,\alpha_1,\chi_\psi\chi) \hookrightarrow \zeta(-\beta_1,\alpha_1,\chi_\psi\chi_1)\times\zeta(-\beta_2,\alpha_2,\chi_\psi\chi_2)$$
$$\zeta(-\beta_2,\alpha_2,\chi_\psi\chi_2)\times\zeta(-\beta_1,\alpha_1,\chi_\psi\chi_2) \twoheadrightarrow \zeta(-\beta_1,\alpha_2,\chi_\psi\chi)\times\zeta(-\beta_2,\alpha_1,\chi_\psi\chi)$$

In case of irreducibility $\zeta(-\beta_1,\alpha_1,\chi_\psi\chi_1)\times\zeta(-\beta_2,\alpha_2,\chi_\psi\chi_2)\cong\zeta(-\beta_2,\alpha_2,\chi_\psi\chi_2)\times\zeta(-\beta_1,\alpha_1,\chi_\psi\chi_1)$. Interchanging and reversing the arrows, the above statements are valid for $\delta(-\beta_1,\alpha_1,\chi_\psi\chi_1)\times\delta(-\beta_2,\alpha_2,\chi_\psi\chi_2)$.

Representation $\zeta(-\beta,\alpha,\chi_\psi\chi)$ is called the Zelevinsky segment representation. It is convenient to agree that if $\alpha+\beta+1\notin\mathbb{Z}_{>0}$, then $\zeta(-\beta,\alpha,\chi_\psi\chi)$ and $\delta(-\beta,\alpha,\chi_\psi\chi)$ means $\chi_\psi\chi\mathbf{1}$. We write $\chi 1_n$ for the character $\chi\circ\det$ of $GL(n,F)$. We see that $\chi_\psi\chi 1_n\cong\zeta(-(n-1)/2,(n-1)/2,\chi_\psi\chi)$. Also, if $\chi=\nu^{e(\chi)}\chi^u$, where χ^u is unitary and $e(\chi)$ is a real number, then we have a uniform notation

$$\chi_\psi\chi 1_n\cong\zeta(-(n-1)/2+e(\chi),(n-1)/2+e(\chi),\chi_\psi\chi^u).$$

2.3. **Groups $\mathbf{R}^{gen}$ and $\mathbf{R}_1^{gen}$.** Let $n\ge 0$ be an integer and $R_{gen}(\widetilde{GL(n,F)})$ the Grothendieck group of the category of smooth genuine representations of $\widetilde{GL(n,F)}$ of a finite length. It is a free Abelian group with a basis of classes of irreducible smooth representations. Partial order $\le$ is defined as $\pi_1\le\pi_2$ if $\pi_2-\pi_1$ is a $\mathbb{Z}_{\ge0}$ linear combination of elements of the given basis. Let $R^{gen}=\bigoplus_{n\ge0}R_{gen}(\widetilde{GL(n,F)})$. We use s.s. to denote semisimplification. We have a map $m^*: R^{gen}\to R^{gen}\otimes R^{gen}$,

$$m^*(\pi)=\sum_{k=0}^{n}\text{s.s.}\left(\text{Jacq}_{(k,n-k)}(\pi)\right),\ \pi\in R^{gen},$$

where $\text{Jacq}_{(0,n)}(\pi)=\chi_\psi\mathbf{1}\otimes\pi$ and $\text{Jacq}_{(n,0)}(\pi)=\pi\otimes\chi_\psi\mathbf{1}$. We rewrite, for the case of $\widetilde{GL(n,F)}$, the results of Propositions 3.4 and 1.7 of [25], and Proposition 9.5 of [25], which are originally stated for $GL(n,F)$, as follows:

$$m^*(\zeta(-\beta,\alpha,\chi_\psi\chi))=\sum_{i=0}^{\alpha+\beta+1}\zeta(-\beta,-\beta-1+i,\chi_\psi\chi)\otimes\zeta(-\beta+i,\alpha,\chi_\psi\chi), \tag{2.2}$$

$$m^*(\delta(-\beta,\alpha,\chi_\psi\chi))=\sum_{i=0}^{\alpha+\beta+1}\delta(\alpha-i+1,\alpha,\chi_\psi\chi)\otimes\delta(-\beta,\alpha-i,\chi_\psi\chi), \tag{2.3}$$

$$m^*(\pi_1\times\pi_2)=(m\otimes m)\circ(id\otimes\kappa\otimes id)\circ(m^*(\pi_1)\otimes m^*(\pi_2)), \tag{2.4}$$

where $\kappa(x\otimes y)=y\otimes x$, $m(x\otimes y)=\text{s.s.}(x\times y)=\text{s.s.}(y\times x)$ and id is the identity. Similarly,

$$R_1^{gen}=\bigoplus_{n\geq 0} R_{gen}(\widetilde{Sp(n,F)}),$$

and we have a map $\mu^*: R_1^{gen}\to R^{gen}\otimes R_1^{gen}$,

$$\mu^*(\sigma)=\sum_{k=0}^{n}\text{s.s.}\left(\text{Jacq}_{(k,n-k)}(\sigma)\right), \sigma\in R_1^{gen},$$

where $\text{Jacq}_{(0;n)}(\sigma)=\chi_\psi\mathbf{1}\otimes\sigma,\ \sigma\in R_1^{gen}$. Using Proposition 4.5 of [11] and (2.2) and (2.3), we obtain

$$\begin{aligned}\mu^*(\zeta(-\beta,\alpha,\chi_\psi\chi)\rtimes\sigma)=\sum_{\zeta\otimes\sigma'\leq\mu^*(\sigma)}\sum_{i=0}^{\alpha+\beta+1}\sum_{j=0}^{i}&\zeta(-\alpha,\beta-i,\chi_\psi\chi^{-1})\times\zeta(-\beta,-\beta-1+j,\chi_\psi\chi)\times\zeta\\ &\otimes\zeta(-\beta+j,-\beta-1+i,\chi_\psi\chi)\rtimes\sigma'.\end{aligned}\tag{2.5}$$

$$\begin{aligned}\mu^*(\delta(-\beta,\alpha,\chi_\psi\chi)\rtimes\sigma)=\sum_{\zeta\otimes\sigma'\leq\mu^*(\sigma)}\sum_{i=0}^{\alpha+\beta+1}\sum_{j=0}^{i}&\delta(-\alpha+i,\beta,\chi_\psi\chi^{-1})\times\delta(\alpha-j+1,\alpha,\chi_\psi\chi)\times\zeta\\ &\otimes\delta(\alpha-i+1,\alpha-j,\chi_\psi\chi)\rtimes\sigma'.\end{aligned}\tag{2.6}$$

Note that because of the same composition series, we have in R_1^{gen}

$$\zeta(-\beta,\alpha,\chi_\psi\chi)\rtimes\sigma=\zeta(-\alpha,\beta,\chi_\psi\chi^{-1})\rtimes\sigma.\tag{2.7}$$

$$\delta(-\beta,\alpha,\chi_\psi\chi)\rtimes\sigma=\delta(-\alpha,\beta,\chi_\psi\chi^{-1})\rtimes\sigma.\tag{2.8}$$

3. Unramified representations

3.1. **Unramified representations of $\widetilde{GL(n,F)}$.** Representation of $\widetilde{GL(n,F)}$ is unramified if there exists a nontrivial vector fixed by $\overline{GL(n,\mathcal{O})}$. For the character ψ fixed in Sect. 2.1, twisting by χ_ψ provides a full correspondence with the theory of unramified representations of $GL(n,F)$. This is because χ_ψ is unramified, i.e., trivial on $\overline{GL(n,O)}$, for our choice of ψ, by [21, Lemma 3.4]. Hence, the following result of [25] and [5] is valid for the covering group.

Theorem 3.1. *Let the notation be as above.*

(1) *The induced representation $\zeta(-\beta_1,\alpha_1,\chi_\psi\chi_1)\times\cdots\times\zeta(-\beta_k,\alpha_k,\chi_\psi\chi_k)$ reduces if and only if $\zeta(-\beta_i,\alpha_i,\chi_\psi\chi_i)\times\zeta(-\beta_j,\alpha_j,\chi_\psi\chi_j)$ reduces for some i,j.*

(2) *Let $\chi_1,\ldots,\chi_k$ be a sequence of unramified characters of $F^\times$. Representation $\chi_\psi\chi_1\times\cdots\times\chi_\psi\chi_k$ has a unique unramified irreducible subquotient.*

(3) *Let π be a genuine unramified irreducible representation of $\widetilde{GL(n,F)}$. Then there exists a sequence of Zelevinsky segment representations, unique up to permutation, such that:*

$$\pi\cong\zeta(-\beta_1,\alpha_1,\chi_\psi\chi_1^u)\times\cdots\times\zeta(-\beta_k,\alpha_k,\chi_\psi\chi_k^u),$$

where $\chi_1^u,\ldots,\chi_k^u$ are unitary characters of $F^\times$.

3.2. **Unramified representations of $\widetilde{Sp(n,F)}$.** Representation of $\widetilde{Sp(n,F)}$ is unramified if there exists a nontrivial vector fixed by $\overline{Sp(n,\mathcal{O})}$. From [8, Sect. 2.6] (see also [9]), [12] and [11], using the uniqueness of the cuspidal support, we have the following theorem.

Theorem 3.2. *Let the notation be as above.*

(1) *Let $\chi_1,\dots,\chi_n$ be unramified characters of $F^\times$. Induced representation $\chi_\psi\chi_1\times\cdots\times\chi_\psi\chi_n\rtimes\omega_0$ contains a unique unramified irreducible subquotient, denoted by $\sigma_{(\chi_\psi\chi_1,\dots,\chi_\psi\chi_n)}$.*
(2) *Let $\chi_1,\dots,\chi_n$ and $\chi_1',\dots,\chi_n'$ be unramified characters of $F^\times$. Representations $\sigma_{(\chi_\psi\chi_1,\dots,\chi_\psi\chi_n)}$ and $\sigma_{(\chi_\psi\chi_1',\dots,\chi_\psi\chi_n')}$ are isomorphic if and only if there exists a permutation h of $\{1,\dots,n\}$ and a sequence $(\epsilon_1,\dots,\epsilon_n)\in\{\pm1\}^n$ such that $\chi_i'=\chi_{h(i)}^{\epsilon_i}$ $i=1,\dots,n$.*
(3) *Let σ be a genuine irreducible representation of $\widetilde{Sp(n,F)}$. Then there exist unramified characters $\chi_1,\dots,\chi_n$ of $F^\times$ such that $\sigma\cong\sigma_{(\chi_\psi\chi_1,\dots,\chi_\psi\chi_n)}$.*

A representation is said to be spherical with respect to some compact subgroup, if there exists nontrivial vector fixed by that subgroup. The following lemma shows that parabolic induction preserves unramified representations.

Lemma 3.3. *Let G be either $GL(n,F)$ or $Sp(n,F)$, and K its fixed maximal compact subgroup.*

(1) *Let σ be a smooth $\widetilde{M_s}\cap\overline{K}$-spherical representation of $\widetilde{M_s}$. Then $\mathrm{Ind}_{\widetilde{M_s}}^{\widetilde{G}}(\sigma)$ is $\overline{K}$-spherical.*
(2) *Let σ be a smooth representation of finite length of $\widetilde{M_s}$ such that $\mathrm{Ind}_{\widetilde{M_s}}^{\widetilde{G}}(\sigma)$ contains a $\overline{K}$-spherical subquotient. Then σ is $\widetilde{M_s}\cap\overline{K}$-spherical.*
(3) *Let $\pi_1,\dots,\pi_k$ be smooth genuine representations of finite length of $\widetilde{GL(n_i,F)}$, $i=1,\dots,k$, and ρ a smooth genuine representation of finite length of $\widetilde{Sp(n_0,F)}$. Then $\pi_1\times\cdots\times\pi_k$ (resp., $\pi_1\times\cdots\times\pi_k\rtimes\rho$) is unramified if and only if π_i's (resp., π_i's and ρ) are unramified.*

Proof. For claim (1) define a function on $\widetilde{G}$ by $f(\widetilde{m}n'\overline{k})=\delta_{P_s}(m)^{\frac12}\sigma(\widetilde{m})v$, where $\widetilde{m}=[m,\epsilon]\in\widetilde{M_s}$, $n'\in N_s'$, $\overline{k}\in\overline{K}$, δ_{P_s} the modular character, and $v\neq0$ an $\widetilde{M_s}\cap\overline{K}$-fixed vector of σ. It is easy to check that f is well-defined nontrivial $\overline{K}$-fixed vector of $\mathrm{Ind}_{\widetilde{M_s}}^{\widetilde{G}}(\sigma)$.

Taking $\overline{K}$-invariants is an exact functor, so the full induced representation $\mathrm{Ind}_{\widetilde{M_s}}^{\widetilde{G}}(\sigma)$ is $\overline{K}$-spherical. Choosing any nonzero $\overline{K}$-invariant function f in that induced representation, followed by direct calculation, and using the definition of induced representations shows that $f(1)$ is a nontrivial $\widetilde{M_s}\cap\overline{K}$-invariant vector for σ. Thus, (2) holds.

Claim (3) follows from (1) and (2) and Lemma 1.1. □

We end this subsection with two lemmas that are repeatedly used in the paper.

Lemma 3.4. *Let σ be a genuine irreducible unramified representation of $\widetilde{Sp(n,F)}$, σ' a genuine representation of finite length of $\widetilde{Sp(n',F)}$ with the unique unramified subquotient σ_0', and π_i, $i=1,\dots,l$, genuine representations of finite length of $\widetilde{GL(n_i,F)}$, such that the induced representation $\pi_1\times\cdots\times\pi_l$ has a unique unramified subquotient π and $\sigma\hookrightarrow\pi_1\times\cdots\times\pi_l\rtimes\sigma'$. Then*

$$\sigma\hookrightarrow\pi_1\times\cdots\times\pi_l\rtimes\sigma_0',\qquad\sigma\hookrightarrow\pi\rtimes\sigma'\qquad\textit{and}\qquad\sigma\hookrightarrow\pi\rtimes\sigma_0'.$$

Proof. Let (τ_i), $i=1,\dots,k$, be a composition series of the $\pi_1\times\cdots\times\pi_l$ and $\sigma\hookrightarrow\tau_{i_0+1}\rtimes\sigma'$ the first possible embedding. Then $\sigma\hookrightarrow(\tau_{i_0+1}\rtimes\sigma')/(\tau_{i_0}\rtimes\sigma')\cong(\tau_{i_0+1}/\tau_{i_0})\rtimes\sigma'\cong\pi\rtimes\sigma'$, where the last isomorphism follows from Lemma 3.3. Other claims can be proved in the same way. □

Lemma 3.5. *Let χ be a character of $F^\times$, $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \in \mathbb{Z}_{\geq 0}$, and σ a genuine unramified representation of $\widetilde{Sp(n,F)}$ such that $\mu^*(\sigma) \geq \zeta(-\beta, \alpha, \chi_\psi \chi) \otimes \sigma''$, where σ'' is an irreducible genuine representation of the metaplectic group of appropriate size. Then there exists a unique genuine irreducible unramified representation σ' of the metaplectic group such that $\mu^*(\sigma) \geq \zeta(-\beta, \alpha, \chi_\psi \chi) \otimes \sigma'$. Moreover, σ' has the same cuspidal support as σ'' and $\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'$.*

Proof. By Theorem 3.2, cuspidal support of σ consists of characters. Unless $\alpha + \beta + 1 = n$, which is a trivial case, there are characters $\chi_1, \ldots, \chi_{n-\alpha-\beta-1}$ of $F^\times$ such that $\sigma'' \hookrightarrow \chi_\psi \chi_1 \times \cdots \times \chi_\psi \chi_{n-\alpha-\beta-1} \rtimes \omega_0$. By Frobenius reciprocity and transitivity of Jacquet module, we have

$$\mathrm{Jacq}_{(1,\ldots,1;0)}(\sigma) \geq \chi_\psi \chi \nu^{-\beta} \otimes \cdots \otimes \chi_\psi \chi \nu^{\alpha} \otimes \chi_\psi \chi_1 \otimes \cdots \otimes \chi_\psi \chi_{n-\alpha-\beta-1} \otimes \omega_0.$$

By [4, Thm. 2.4], a cuspidal subquotient of an admissible representation is a quotient. Thus,

$$\mathrm{Jacq}_{(1,\ldots,1;0)}(\sigma) \twoheadrightarrow \chi_\psi \chi \nu^{-\beta} \otimes \cdots \otimes \chi_\psi \chi \nu^{\alpha} \otimes \chi_\psi \chi_1 \otimes \cdots \otimes \chi_\psi \chi_{n-\alpha-\beta-1} \otimes \omega_0.$$

Frobenius reciprocity gives

$$\sigma \hookrightarrow \chi_\psi \chi \nu^{-\beta} \times \cdots \times \chi_\psi \chi \nu^{\alpha} \times \chi_\psi \chi_1 \times \cdots \times \chi_\psi \chi_{n-\alpha-\beta-1} \rtimes \omega_0.$$

Now, by Lemma 3.4, $\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'$, so that $\mu^*(\sigma) \geq \zeta(-\beta, \alpha, \chi_\psi \chi) \otimes \sigma'$. □

3.3. **Weak form of Zelevinsky classification.** Using [11], we obtain a weak form of Zelevinsky classification for unramified representations. We first define negative and strongly negative genuine irreducible unramified representations of the metaplectic group. For a character χ of $F^\times$, let $e(\chi)$ be the real number such that $\chi = \nu^{e(\chi)} \chi^u$, where χ^u is a unitary character of $F^\times$. Let σ be a genuine irreducible unramified representation of $\widetilde{Sp(n,F)}$. We call σ negative if for every embedding of form $\sigma \hookrightarrow \chi_1 \chi_\psi \times \cdots \times \chi_n \chi_\psi \rtimes \omega_0$, where $\chi_1, \ldots, \chi_n$ are characters of $F^\times$, we have

$$\begin{aligned} e(\chi_1) &\leq 0 \\ e(\chi_1) + e(\chi_2) &\leq 0 \\ &\cdots \\ e(\chi_1) + \cdots + e(\chi_n) &\leq 0. \end{aligned}$$

If above inequalities are strict, σ is said to be strongly negative. We classify genuine irreducible unramified representations of the metaplectic group in terms of negative ones.

Theorem 3.6. *Let σ be a genuine irreducible unramified representation of $\widetilde{Sp(n,F)}$. Then, either σ is negative, or there exist $k \in \mathbb{Z}_{>0}$, $\alpha_i, \beta_i \in \mathbb{R}$ such that $\alpha_i - \beta_i$, $\alpha_i + \beta_i + 1 \in \mathbb{Z}_{>0}$, unitary unramified characters χ_i of $F^\times$, $i = 1, \ldots, k$, and a genuine unramified irreducible negative representation σ_{neg} of the metaplectic group such that*

1. *$\sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k) \rtimes \sigma_{neg}$ as unique irreducible subrepresentation, and*
2. *$\zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k)$ is irreducible.*

Data $\zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1), \ldots, \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k)$ are unique up to permutation, while σ_{neg} is unique up to isomorphism.

Proof. By Lemma 3.3, all representations that participate in Zelevinsky classification (cf. [11, Thms. 4.6 and 4.7]) are unramified, and thus, described in Theorems 3.1 and 3.2. Reducibility of the representation $\zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k)$ is not possible, because by Theorem 3.1 and Lemmas 3.4 and 2.1, Zelevinsky data would change. □

3.4. **Negative representations.** Having proved the weak form of Zelevinsky classification, we are ready to describe negative representations in terms of strongly negative ones.

Theorem 3.7. *Let σ be a genuine irreducible unramified negative representation of $\widetilde{Sp(n,F)}$. Then, either σ is strongly negative, or there exist $k \in \mathbb{Z}_{>0}$, unramified unitary characters $\chi_1, \dots, \chi_k$ of $F^\times$, $\beta_i \in \mathbb{R}$ such that $2\beta_i + 1 \in \mathbb{Z}_{>0}$, $i = 1, \dots, k$, and a genuine irreducible unramified strongly negative representation σ_{sn} of the metaplectic group such that*

$$\sigma \hookrightarrow \zeta(-\beta_1, \beta_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \beta_k, \chi_\psi\chi_n) \rtimes \sigma_{sn}.$$

Data $\zeta(-\beta_1, \beta_1, \chi_\psi\chi_1), \dots, \zeta(-\beta_k, \beta_k, \chi_\psi\chi_k)$ are unique up to permutation and replacing χ_i with χ_i^{-1}, while σ_{sn} is unique up to isomorphism.

Proof. Unless σ is strongly negative, there exist unramified characters $\chi_1, \dots, \chi_n$ of $F^\times$ and an integer $1 \le t \le n$ such that $\mathrm{Jacq}_{(1,\dots,1;0)}(\sigma) \ge \chi_\psi\chi_1 \otimes \cdots \otimes \chi_\psi\chi_n \otimes \omega_0$ and $e(\chi_1) + \cdots + e(\chi_t) = 0$. Since a cuspidal subquotient of an admissible representation is a quotient, using Frobenius reciprocity we have $\sigma \hookrightarrow \chi_\psi\chi_1 \times \cdots \times \chi_\psi\chi_n \rtimes \omega_0$. Let σ_0 be an irreducible unramified subquotient of $\chi_\psi\chi_{t+1} \times \cdots \times \chi_\psi\chi_n \rtimes \omega_0$ if $n \ge 2$, or else ω_0. By lemma 3.4, $\sigma \hookrightarrow \chi_\psi\chi_1 \times \cdots \times \chi_\psi\chi_t \rtimes \sigma_0$. Thus σ_0 must be negative. Classifying unramified irreducible subquotient of $\chi_\psi\chi_1 \times \cdots \times \chi_\psi\chi_t$ by Theorem 3.1 and using Lemma 3.4, we obtain:

$$\sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1^u) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k^u) \rtimes \sigma_0.$$

Since $\zeta(-\beta_i, \alpha_i, \chi_\psi\chi_i^u) \times \zeta(-\beta_j, \alpha_j, \chi_\psi\chi_j^u)$ commute for $i, j = 1, \dots, k$, σ is negative and $e(\chi_1) + \cdots + e(\chi_t) = 0$, we must have $\alpha_i = \beta_i$, $i = 1, \dots, k$. Thus, using Lemma 3.4 the proof is obtained by induction, with uniqueness of the classifying data proven in the next lemma. □

Lemma 3.8. (1) *Let $l \in \frac{1}{2}\mathbb{Z}_{\ge 0}$, χ a unitary unramified character of $F^\times$ and σ a genuine irreducible unramified negative representation of $\widetilde{Sp(n,F)}$. Then, the irreducible unramified subquotient of $\zeta(-l, l, \chi_\psi\chi) \rtimes \sigma$ is negative.*

(2) *Let $l_1, \dots, l_k \in \frac{1}{2}\mathbb{Z}_{\ge 0}$, $\chi_1, \dots, \chi_k$ unitary unramified characters of $F^\times$ and σ a genuine irreducible unramified strongly negative representation of $\widetilde{Sp(n,F)}$. Then, the irreducible unramified subquotient τ of $\zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_k, l_k, \chi_\psi\chi_k) \rtimes \sigma$ is a subrepresentation and negative. Given τ, representations $\zeta(-l_1, l_1, \chi_\psi\chi_1), \dots, \zeta(-l_k, l_k, \chi_\psi\chi_k)$ are determined up to permutation and replacing χ_i with χ_i^{-1}, while σ is determined up to isomorphism.*

Proof. (1) Let τ be the unramified irreducible subquotient of $\zeta(-l, l, \chi_\psi\chi) \rtimes \sigma$. Unless τ is negative, by Theorem 3.6 and Lemma 3.4, there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \in \mathbb{Z}_{\ge 0}$, $-\beta + \alpha > 0$, there exists a unitary unramified character χ_1 of $F^\times$, and a genuine irreducible unramified representation σ' of the metaplectic group, such that $\tau \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi_1) \rtimes \sigma'$. Thus, $\zeta(-\beta, \alpha, \chi_\psi\chi_1) \otimes \sigma' \le \mu^*(\tau) \le \mu^*(\zeta(-l, l, \chi_\psi\chi) \rtimes \sigma)$. By formula (2.5) there exist $0 \le j \le i \le 2l + 1$ and an irreducible representation $\zeta_1 \otimes \sigma_1 \le \mu^*(\sigma)$ such that

$$\zeta(-\beta, \alpha, \chi_\psi\chi_1) \otimes \sigma' \le \zeta(-l, l - i, \chi_\psi\chi^{-1}) \times \zeta(-l, j - l - 1, \chi_\psi\chi) \times \zeta_1 \otimes \zeta(j - l, i - l - 1, \chi_\psi\chi) \rtimes \sigma_1.$$

We compare the cuspidal support to the left of $\otimes$. Since a cuspidal subquotient of a Jacquet module is a quotient, as used in Theorem 3.7, and σ is negative, sum of exponents of ν contained in ζ_1 cannot be positive, and the same for $\zeta(-l, l - i, \chi_\psi\chi^{-1})$ and $\zeta(-l, j - l - 1, \chi_\psi\chi)$. The sum of exponents of ν in $\zeta(-\beta, \alpha, \chi_\psi\chi_1)$ is positive. We have a contradiction.

(2) The first claim shows that τ is negative, so by Theorem 3.7, there exist $t_1, \dots, t_r \in \mathbb{Z}_{\ge 0}$, unitary unramified characters $\chi'_1, \dots, \chi'_r$ of $F^\times$ and σ_s a genuine, irreducible, unramified and strongly negative representation, such that $\tau \hookrightarrow \zeta(-t_1, t_1, \chi_\psi\chi'_1) \times \cdots \times \zeta(-t_r, t_r, \chi_\psi\chi'_r) \rtimes \sigma_s$. Thus

$$\zeta(-t_1, t_1, \chi_\psi\chi'_1) \times \cdots \times \zeta(-t_r, t_r, \chi_\psi\chi'_r) \otimes \sigma_s \le \mu^*(\zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_k, l_k, \chi_\psi\chi_k) \rtimes \sigma).$$

By formula (2.5), there exist $0 \leq j_m \leq i_m \leq 2l_m+1,\ m=1,\dots,k$, and an irreducible representation $\zeta_1 \otimes \sigma_1 \leq \mu^*(\sigma)$ such that

$$\begin{aligned}\zeta(-t_1,t_1,\chi_\psi\chi_1') \times \cdots \times \zeta(-t_r,t_r,\chi_\psi\chi_r') \otimes \sigma_s \leq \\ \zeta(-l_1,l_1-i_1,\chi_\psi\chi_1^{-1}) \times \zeta(-l_1,j_1-l_1-1,\chi_\psi\chi_1) \times \cdots \times \zeta(-l_k,l_k-i_k,\chi_\psi\chi_k^{-1}) \times \\ \zeta(-l_k,j_k-l_k-1,\chi_\psi\chi_k) \times \zeta_1 \\ \bigotimes \zeta(j_1-l_1,i_1-l_1-1,\chi_\psi\chi_1) \times \cdots \times \zeta(j_k-l_k,i_k-l_k-1,\chi_\psi\chi_k) \rtimes \sigma_1.\end{aligned}$$

We compare the cuspidal support to the left of $\otimes$. Because σ is negative and the sum of exponents of ν on the left hand side is 0, we must have $\zeta_1 = \chi_\psi \mathbf{1}$, $i_m = j_m = 0$ or $2l_m+1$, $m=1,\dots,k$. Thus $\zeta(-t_1,t_1,\chi_\psi\chi_1') \times \cdots \times \zeta(-t_r,t_r,\chi_\psi\chi_r') = \zeta(-l_1,l_1,\chi_\psi\chi_1^{\pm 1}) \times \cdots \times \zeta(-l_k,l_k,\chi_\psi\chi_k^{\pm 1})$ and $\sigma_s \cong \sigma$. Comparing the largest segments yields the claim.

□

4. Unramified strongly negative representations

In this section we classify genuine irreducible unramified strongly negative representations of the metaplectic group in terms of Jordan blocks.

4.1. **Jordan blocks.** Let $\chi_0 = \nu^{\pi\sqrt{-1}/\ln q}$ be the unique unramified character of order two, and 1 the trivial character of $F^\times$. Jordan block is a pair $(m, \chi_\psi\chi)$, where m is a positive integer and $\chi \in \{1, \chi_0\}$. Jord is a set built of Jordan blocks. Given $\chi \in \{1,\chi_0\}$ we denote $\mathrm{Jord}(\chi_\psi\chi) = \{m \mid (m,\chi_\psi\chi) \in \mathrm{Jord}\}$. Let $k,l \in \mathbb{Z}_{\geq 0}$ and

$$\mathrm{Jord}(\chi_\psi) = \{2m_1+1 < 2m_2+1 < \cdots < 2m_l+1\},\ m_i \in \frac{1}{2}+\mathbb{Z}_{\geq 0},\ i=1,\dots,l,$$
$$\mathrm{Jord}(\chi_\psi\chi_0) = \{2n_1+1 < 2n_2+1 < \cdots < 2n_k+1\},\ n_j \in \frac{1}{2}+\mathbb{Z}_{\geq 0},\ j=1,\dots,k.$$

We denote by $\sigma(\mathrm{Jord})$ the unique unramified irreducible subquotient (cf. Theorem 3.2 and [8]) of the induced representation

$$\begin{aligned}\zeta(-m_{l-1},m_l,\chi_\psi) \times \zeta(-m_{l-3},m_{l-2},\chi_\psi) \times \cdots \times \zeta(-n_{k-1},n_k,\chi_\psi\chi_0) \\ \times \zeta(-n_{k-3},n_{k-2},\chi_\psi\chi_0) \times \cdots \rtimes \sigma_0(\mathrm{Jord}),\end{aligned}$$

where $\sigma_0(\mathrm{Jord})$ is the unique unramified irreducible subquotient of

$$\begin{array}{rl} \zeta(\frac{1}{2},m_1,\chi_\psi) \times \zeta(\frac{1}{2},n_1,\chi_\psi\chi_0) \rtimes \omega_0 & \text{if } k,l \in 2\mathbb{Z}+1, \\ \zeta(\frac{1}{2},m_1,\chi_\psi) \rtimes \omega_0 & \text{if } k \in 2\mathbb{Z},\ l \in 2\mathbb{Z}+1, \\ \zeta(\frac{1}{2},n_1,\chi_\psi\chi_0) \rtimes \omega_0 & \text{if } k \in 2\mathbb{Z}+1,\ l \in 2\mathbb{Z}, \\ \omega_0 & \text{if } k,l \in 2\mathbb{Z}. \end{array}$$

When $k=l=0$, we have $\mathrm{Jord}=\emptyset$, and $\sigma(\mathrm{Jord}) = \omega_0$, which is by definition strongly negative.

For $n \in \mathbb{Z}_{\geq 0}$, we denote by $\mathrm{Jord}(n)$ set of all Jord, such that

$$\sum_{(m,\chi_\psi\chi)\in \mathrm{Jord}} m = 2n.$$

So, given $\mathrm{Jord} \in \mathrm{Jord}(n)$, $\sigma(\mathrm{Jord})$ is a representation of $\widetilde{Sp(n,F)}$

4.2. **Construction of unramified strongly negative representations.** We begin with two simple cases of unramified strongly negative representations in the following two lemmas.

Lemma 4.1. *Let $\chi \in \{1, \chi_0\}$, $\alpha \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$. Let σ_α be the unique irreducible unramified subquotient of $\zeta(\frac{1}{2}, \alpha, \chi_\psi \chi) \rtimes \omega_0$, and put $\sigma_{-\frac{1}{2}} = \omega_0$. The representation σ_α is strongly negative, and*

$$\mu^*(\sigma_\alpha) = \sum_{i=0}^{\alpha+\frac{1}{2}} \zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi \chi) \otimes \sigma_{i-\frac{1}{2}}, \tag{4.1}$$

$$\sigma_\alpha \hookrightarrow \zeta(-\alpha, -\frac{1}{2}, \chi_\psi \chi) \rtimes \omega_0, \tag{4.2}$$

$$r_{(1,\dots,1;0)}(\sigma_\alpha) = \chi_\psi \nu^{-\alpha} \chi \otimes \chi_\psi \nu^{-\alpha+1} \otimes \cdots \otimes \chi_\psi \nu^{-\frac{1}{2}} \otimes \omega_0 \tag{4.3}$$

Proof. We use induction on α. Case $\alpha = \frac{1}{2}$ follows from (2.5) and Theorem A.1. As induction hypothesis, assume that claims are valid for $\alpha' \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ such that $\frac{1}{2} \leq \alpha' < \alpha$. We must prove that they are valid for α. Compare $\chi_\psi \nu^{-\alpha} \chi \rtimes \sigma_{\alpha-1}$ and $\zeta(\frac{1}{2}, \alpha, \chi_\psi \chi) \rtimes \omega_0$. As both are subquotients of $\chi_\psi \nu^{\frac{1}{2}} \chi \times \cdots \times \chi_\psi \nu^{\alpha-1} \chi \times \chi_\psi \nu^{\alpha} \chi \rtimes \omega_0$, the $\overline{K}$-fixed subquotient σ_α appears in both. In the minimal Jacquet module, using the induction hypothesis and (2.5), one has

$$\begin{aligned} s.s.r_{(1,\dots,1;0)}(\chi_\psi \nu^{-\alpha} \chi \rtimes \sigma_{\alpha-1}) =& \chi_\psi \nu^{-\alpha} \chi \otimes \chi_\psi \nu^{-\alpha+1} \chi \otimes \cdots \otimes \chi_\psi \nu^{-\frac{1}{2}} \chi \otimes \omega_0 \\ &+ \text{additional terms, all having } \chi_\psi \nu^{-\frac{1}{2}} \chi \end{aligned}$$

and

$$\begin{aligned} s.s.r_{(1,\dots,1;0)}(\zeta(\frac{1}{2}, \alpha, \chi_\psi \chi) \rtimes \omega_0) =& \chi_\psi \nu^{-\alpha} \chi \otimes \chi_\psi \nu^{-\alpha+1} \chi \otimes \cdots \otimes \chi_\psi \nu^{-\frac{1}{2}} \chi \otimes \omega_0 \\ &+ \text{additional terms, all having } \chi_\psi \nu^{\frac{1}{2}} \chi \end{aligned}$$

Thus $r_{(1,\dots,1;0)}(\sigma_\alpha) = \chi_\psi \nu^{-\alpha} \chi \otimes \chi_\psi \nu^{-\alpha+1} \chi \otimes \cdots \otimes \chi_\psi \nu^{-\frac{1}{2}} \chi \otimes \omega_0$, that is, we proved (4.3). Using Frobenius reciprocity, we have $\sigma_\alpha \hookrightarrow \chi_\psi \nu^{-\alpha} \chi \times \chi_\psi \nu^{-\alpha+1} \chi \times \cdots \times \chi_\psi \nu^{-\frac{1}{2}} \chi \rtimes \omega_0$. Now (4.2) is a consequence of Theorem 3.1 and Lemma 3.4. To prove (4.1), we use the transitivity of the Jacquet module, (4.3) and Lemma 2.1. □

Lemma 4.2. *Let $\alpha, \alpha' \in -\frac{1}{2} + \mathbb{Z}_{\geq 0}$, let $\sigma_{\alpha,\alpha'}$ be the unramified irreducible subquotient of $\zeta(\frac{1}{2}, \alpha, \chi_\psi) \times \zeta(\frac{1}{2}, \alpha', \chi_\psi \chi_0) \rtimes \omega_0$, let σ_α be the unramified irreducible subquotient of $\zeta(\frac{1}{2}, \alpha, \chi_\psi) \rtimes \omega_0$, and let $\sigma'_{\alpha'}$ be the unramified irreducible subquotient of $\zeta(\frac{1}{2}, \alpha', \chi_\psi \chi_0) \rtimes \omega_0$. Then $\sigma_{\alpha,\alpha'} \leq \zeta(\frac{1}{2}, \alpha, \chi_\psi) \rtimes \sigma'_{\alpha'}$, and $\sigma_{\alpha,\alpha'} \leq \zeta(\frac{1}{2}, \alpha', \chi_\psi \chi_0) \rtimes \sigma_\alpha$. The representation $\sigma_{\alpha,\alpha'}$ is strongly negative, and*

$$\mu^*(\sigma_{\alpha,\alpha'}) = \sum_{i=0}^{\alpha+\frac{1}{2}} \sum_{i'=0}^{\alpha'+\frac{1}{2}} \zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi \chi_0) \otimes \sigma_{i-\frac{1}{2}, i'-\frac{1}{2}}, \tag{4.4}$$

$$\sigma_{\alpha,\alpha'} \hookrightarrow \zeta(-\alpha, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi \chi_0) \rtimes \omega_0. \tag{4.5}$$

Proof. Lemma 3.3 implies $\sigma_{\alpha,\alpha'} \leq \zeta(\frac{1}{2}, \alpha, \chi_\psi) \rtimes \sigma'_{\alpha'}$ and $\sigma_{\alpha,\alpha'} \leq \zeta(\frac{1}{2}, \alpha', \chi_\psi \chi_0) \rtimes \sigma_\alpha$. We prove by induction on $\alpha + \alpha'$ that, for an irreducible representation σ, if $\sigma \leq \zeta(\frac{1}{2}, \alpha, \chi_\psi) \rtimes \sigma'_{\alpha'}$ and $\sigma \leq \zeta(\frac{1}{2}, \alpha', \chi_\psi \chi_0) \rtimes \sigma_\alpha$, then $\sigma \cong \sigma_{\alpha,\alpha'}$.

Claims are valid if $\alpha = -\frac{1}{2}$ or $\alpha' = -\frac{1}{2}$ by Lemma 4.1. Let $n \in \mathbb{Z}_{>0}$ and assume that the lemma holds for $\alpha + \alpha' < n$. Let $\alpha + \alpha' = n$. From Lemma 4.1 and (2.5) we have

$$\mu^*(\sigma) \leq \mu^*(\zeta(\frac{1}{2}, \alpha, \chi_\psi) \rtimes \sigma'_{\alpha'}) = \sum_{i'=0}^{\alpha'+\frac{1}{2}} \sum_{i=0}^{\alpha+\frac{1}{2}} \sum_{j=0}^{i}$$
$$\zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \times \zeta(\frac{1}{2}, j - \frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \otimes \zeta(j + \frac{1}{2}, i - \frac{1}{2}, \chi_\psi) \rtimes \sigma'_{i'-\frac{1}{2}}$$

$$\mu^*(\sigma) \leq \mu^*(\zeta(\frac{1}{2}, \alpha', \chi_\psi\chi_0) \rtimes \sigma_\alpha) = \sum_{i=0}^{\alpha+\frac{1}{2}} \sum_{i'=0}^{\alpha'+\frac{1}{2}} \sum_{j'=0}^{i'}$$
$$\zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \times \zeta(\frac{1}{2}, j' - \frac{1}{2}, \chi_\psi\chi_0) \times \zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \otimes \zeta(j' + \frac{1}{2}, i' - \frac{1}{2}, \chi_\psi\chi_0) \rtimes \sigma_{i-\frac{1}{2}}.$$

Since in the first formula positive powers of ν before $\otimes$ appear with $\chi_\psi \mathbf{1}$, and in the second with $\chi_\psi\chi_0$, we should keep only terms with $j = j' = 0$. Thus,

$$\mu^*(\sigma) \leq \sum_{i=0}^{\alpha+\frac{1}{2}} \sum_{i'=0}^{\alpha'+\frac{1}{2}} \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \times \zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \otimes \zeta(\frac{1}{2}, i - \frac{1}{2}, \chi_\psi) \rtimes \sigma'_{i'-\frac{1}{2}}.$$
$$\mu^*(\sigma) \leq \sum_{i=0}^{\alpha+\frac{1}{2}} \sum_{i'=0}^{\alpha'+\frac{1}{2}} \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \times \zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \otimes \zeta(\frac{1}{2}, i' - \frac{1}{2}, \chi_\psi\chi_0) \rtimes \sigma_{i-\frac{1}{2}}.$$

Induction hypothesis implies

$$\mu^*(\sigma) \leq \chi_\psi \mathbf{1} \otimes \sigma \;+ \sum_{0 \leq i+i' < \alpha+\alpha'+1} \zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \otimes \sigma_{i-\frac{1}{2}, i'-\frac{1}{2}}.$$

We see that $r_{(n;0)}(\sigma) \cong \zeta(-\alpha, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \otimes \omega_0$ and it is easy to prove that this appears with multiplicity one in $\mu^*(\zeta(-\alpha, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0)$. Hence, σ is uniquely determined, and thus $\sigma \cong \sigma_{\alpha,\alpha'}$. Looking at the cuspidal support of $r_{(n;0)}(\sigma)$, we see that $\sigma_{\alpha,\alpha'}$ is strongly negative. Let $i = 0, \ldots, \alpha + \frac{1}{2},\ i' = 0, \ldots, \alpha' + \frac{1}{2}$. Frobenius reciprocity and Lemma 3.4 imply

$$\sigma_{\alpha,\alpha'} \hookrightarrow \zeta(-\alpha, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0 \hookrightarrow$$
$$\zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \times \zeta(\frac{1}{2} - i, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \times \zeta(\frac{1}{2} - i', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0 \cong$$
$$\zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \times \zeta(\frac{1}{2} - i, -\frac{1}{2}, \chi_\psi) \times \zeta(\frac{1}{2} - i', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0,$$

and therefore

$$\sigma_{\alpha,\alpha'} \hookrightarrow \zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \rtimes \sigma_{i-\frac{1}{2}, i'-\frac{1}{2}}$$
$$\mu^*(\sigma_{\alpha,\alpha'}) \geq \zeta(-\alpha, -\frac{1}{2} - i, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2} - i', \chi_\psi\chi_0) \otimes \sigma_{i-\frac{1}{2}, i'-\frac{1}{2}},$$

proving (4.4) and (4.5). □

The following lemma is crucial in showing that $\sigma(\mathrm{Jord})$ are strongly negative and that they exhaust all such representations.

Lemma 4.3. *Let σ be a genuine irreducible unramified strongly negative representation of $\widetilde{Sp(n, F)}$.*

(1) *Then there exists an unramified unitary character* χ *of* $F^\times$*, there exist* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha + \beta \in \mathbb{Z}_{\geq 0}$*, and there exists an irreducible unramified representation* σ' *of the metaplectic group such that* $\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'$*. Also* $\alpha - \beta < 0$ *and* $\zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'$ *reduces. If* α *is the largest possible for such embedding, then* σ' *is strongly negative.*
(2) *Let* β *be the maximum of* $|e(\chi')|$ *over all* $\chi_\psi \chi'$ *in the cuspidal support of* σ*, achieved for* $\nu^{\pm\beta} \chi_\psi \chi$*, where* χ *is a unitary character of* $F^\times$*. Then there exist* $\alpha \in \mathbb{R}$ *such that* $\alpha + \beta \in \mathbb{Z}_{\geq 0}$*, and there exists an irreducible unramified representation* σ' *of the metaplectic group, such that*

$$\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'.$$

Proof. We first prove (1). Write the cuspidal support so that $\sigma \hookrightarrow \nu^{k_1} \chi_\psi \chi_1 \times \cdots \times \nu^{k_n} \chi_\psi \chi_n \rtimes \omega_0$. Note that $\zeta(k_1, k_1, \chi_\psi \chi_1) \cong \nu^{k_1} \chi_\psi \chi_1$. If $n > 1$, we use Lemma 3.4 to take the unramified irreducible subquotient σ' of $\nu^{k_2} \chi_\psi \chi_2 \times \cdots \times \nu^{k_n} \chi_\psi \chi_n \rtimes \omega_0$, and get an embedding of the required form.

Strong negativity of σ and Frobenius reciprocity imply $-\beta + \alpha < 0$ and reducibility of $\zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'$, since otherwise, by (2.7), $\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma' \cong \zeta(-\alpha, \beta, \chi_\psi \chi^{-1}) \rtimes \sigma'$, contradicting strong negativity of σ. Let α be the largest possible with such embedding. Assume that σ' is not strongly negative. By Theorems 3.6 and 3.7 and Lemma 3.4, there exists a unitary unramified character χ' of $F^\times$, there exist $\alpha', \beta' \in \mathbb{R}$ with $\alpha' + \beta' \in \mathbb{Z}_{\geq 0}$ and $\alpha' - \beta' \geq 0$, and there exists an irreducible representation σ'', such that $\sigma' \hookrightarrow \zeta(-\beta', \alpha', \chi_\psi \chi') \rtimes \sigma''$. There is a nontrivial intertwining

$$\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \times \zeta(-\beta', \alpha', \chi_\psi \chi') \rtimes \sigma'' \to \zeta(-\beta', \alpha', \chi_\psi \chi') \times \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma''.$$

Because $\alpha' - \beta' \geq 0$ and σ is strongly negative, σ must be in the kernel of the second map. Thus, $\zeta(-\beta, \alpha, \chi_\psi \chi) \times \zeta(-\beta', \alpha', \chi_\psi \chi')$ reduces, and by Lemma 2.1, $\chi = \chi'$, $\alpha - \alpha' \in \mathbb{Z}$. Lemma 3.4 implies $\sigma \hookrightarrow \zeta(-\beta, \alpha', \chi_\psi \chi) \times \zeta(-\beta', \alpha, \chi_\psi \chi) \rtimes \sigma''$. But the maximality of α and Lemma 3.4 imply $-\beta' + \alpha' < -\beta + \alpha < 0$, a contradiction. Thus σ' is strongly negative.

Now we prove (2). Write the cuspidal support of σ so that $\sigma \hookrightarrow \chi_\psi \chi'_1 \times \cdots \times \chi_\psi \chi'_n \rtimes \omega_0$. By Theorem 3.1, let $\zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k)$ be the irreducible unramified subquotient of $\chi_\psi \chi'_1 \times \cdots \times \chi_\psi \chi'_n$, where k is an integer, $\alpha_i, \beta_i \in \mathbb{R}$ with $\alpha_i + \beta_i \in \mathbb{Z}_{>0}$ and χ_i are unitary unramified characters of $F^\times$, $i = 1, \ldots, k$. By Lemma 3.4,

$$\sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi \chi_k) \rtimes \omega_0.$$

Because Zelevinsky segment representations commute and σ is strongly negative, Lemma 2.1 gives $-\beta \in \{-\beta_1, \ldots, -\beta_k\}$. Again, Lemma 3.4 finishes the proof. □

Now we are ready to prove that all $\sigma(\mathrm{Jord})$, defined in Sect. 4.1, are strongly negative.

Theorem 4.4. *The representation* $\sigma(\mathrm{Jord})$*, attached to a set* Jord *of Jordan blocks, is strongly negative, and we have*

$$\begin{aligned} \sigma(\mathrm{Jord}) \hookrightarrow & \zeta(-m_l, m_{l-1}, \chi_\psi) \times \zeta(-m_{l-2}, m_{l-3}, \chi_\psi) \times \\ & \cdots \times \zeta(-n_k, n_{k-1}, \chi_\psi \chi_0) \times \zeta(-n_{k-2}, n_{k-3}, \chi_\psi \chi_0) \times \cdots \rtimes \sigma_0(\mathrm{Jord}). \end{aligned} \tag{4.6}$$

If $\chi \in \{1, \chi_0\}$ *and* $\mathrm{card}(\mathrm{Jord}(\chi_\psi \chi)) \geq 2$*, let* $2\beta + 1 > 2\alpha + 1$ *be two largest elements in* $\mathrm{Jord}(\chi_\psi \chi)$*. Put* $\mathrm{Jord}' = \mathrm{Jord} \backslash \{(2\beta + 1, \chi_\psi \chi), (2\alpha + 1, \chi_\psi \chi)\}$ *and* $\sigma' = \sigma(\mathrm{Jord}'), \sigma = \sigma(\mathrm{Jord})$*. Then:*

$$\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'. \tag{4.7}$$

Proof. We use induction on card(Jord). If $(l, k) = (0, 0)$, then ω_0 is by definition strongly negative. Lemmas 4.1 and 4.2 prove cases $(0, 1)$, $(1, 0)$ and $(1, 1)$. Let $t \geq 2$ be an integer. Suppose that claims are valid for Jord with less than t elements. Take Jord such that $l + k = t$. Since $(l, k) = (1, 1)$ is settled, we may assume that there exists $\chi \in \{1, \chi_0\}$ such that $\mathrm{card}(\mathrm{Jord}(\chi_\psi \chi)) \geq 2$. Let σ, σ', α, β be as in the theorem. Comparing the cuspidal support, and using the uniqueness of irreducible unramified subquotient, we have

$$\sigma \leq \zeta(-\beta, \alpha, \chi_\psi \chi) \rtimes \sigma'.$$

By induction hypothesis σ' is strongly negative.

First, we shall prove that σ is strongly negative. Assume that σ is negative but not strongly negative. By Theorem 3.7 and Lemma 3.4 there exist $2m \in \mathbb{Z}_{\geq 0}$, an unramified unitary character χ' of $F^\times$, and an irreducible unramified negative representation σ'', such that $\sigma \hookrightarrow \zeta(-m, m, \chi_\psi\chi') \rtimes \sigma''$. Frobenius reciprocity implies $\zeta(-m, m, \chi_\psi\chi') \otimes \sigma'' \leq \mu^*(\sigma) \leq \mu^*(\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma')$. By (2.5), there exist $0 \leq j \leq i \leq \alpha + \beta + 1$ and an irreducible representation $\zeta \otimes \sigma'_1 \leq \mu^*(\sigma')$ such that

$$\zeta(-m, m, \chi_\psi\chi') \otimes \sigma'' \leq \zeta(-\alpha, \beta - i, \chi_\psi\chi) \times \zeta(-\beta, j - \beta - 1, \chi_\psi\chi) \times \zeta \otimes \zeta(j - \beta, i - \beta - 1, \chi_\psi\chi) \rtimes \sigma'_1$$

Cuspidal support of σ does not contain $\nu^{\pm\gamma}\chi_\psi\chi$ twice, for $\gamma \in \alpha + \mathbb{Z}_{>0}$. Hence $j = 0$. Because σ' is strongly negative, if $\zeta \neq \chi_\psi\mathbf{1}$, it has a negative sum of powers of ν. Thus we must have $-m = -\alpha, m = \beta - i, \zeta = \chi_\psi\mathbf{1}$ and $\chi = \chi'$. Now $\sigma'_1 = \sigma'$ and

$$\sigma'' \leq \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \rtimes \sigma'. \tag{4.8}$$

Let us show that the representation on the right hand side is irreducible. There exist an integer h, characters $\chi_1, \ldots, \chi_h \in \{1, \chi_0\}$ and $r_1, \ldots, r_h \in \mathbb{R}$ with $|r_s| < \alpha$ if $\chi_s = \chi$, $s = 1, \ldots, h$, such that

$$\sigma' \hookrightarrow \nu^{r_1}\chi_\psi\chi_1 \times \cdots \times \nu^{r_h}\chi_\psi\chi_h \rtimes \omega_0.$$

By Lemma 2.1, representations $\zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \times \nu^{r_s}\chi_\psi\chi_s$ and $\zeta(\alpha + 1, \beta, \chi_\psi\chi) \times \nu^{r_s}\chi_\psi\chi_s$, $s = 1, \ldots, h$ are irreducible. By Theorems A.1 and A.7, $\zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \rtimes \omega_0$ is also irreducible. Using Lemma 2.1 and (2.7), we have

$$\begin{aligned}
&\zeta(\alpha + 1, \beta, \chi_\psi\chi) \rtimes \sigma' \hookrightarrow \zeta(\alpha + 1, \beta, \chi_\psi\chi) \times \nu^{r_1}\chi_\psi\chi_1 \times \cdots \times \nu^{r_h}\chi_\psi\chi_h \rtimes \omega_0 \cong \\
&\nu^{r_1}\chi_\psi\chi_1 \times \cdots \times \nu^{r_h}\chi_\psi\chi_h \times \zeta(\alpha + 1, \beta, \chi_\psi\chi) \rtimes \omega_0 \cong \nu^{r_1}\chi_\psi\chi_1 \times \cdots \times \nu^{r_h}\chi_\psi\chi_h \times \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \rtimes \omega_0 \\
&\cong \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \times \nu^{r_1}\chi_\psi\chi_1 \times \cdots \times \nu^{r_h}\chi_\psi\chi_h \rtimes \omega_0.
\end{aligned}$$

Now,

$$\zeta(\alpha + 1, \beta, \chi_\psi\chi) \rtimes \sigma' \hookrightarrow \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \times \nu^{r_1}\chi_\psi\chi_1 \times \cdots \times \nu^{r_h}\chi_\psi\chi_h \rtimes \omega_0. \tag{4.9}$$

$$\zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \rtimes \sigma' \hookrightarrow \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \times \nu^{r_1}\chi_\psi\chi_1 \times \cdots \times \nu^{r_h}\chi_\psi\chi_h \rtimes \omega_0. \tag{4.10}$$

By Theorem 3.2, the representations in (4.9) and (4.10) have the same unramified irreducible subquotient, so images of embeddings have a nontrivial intersection. By [11, Thm. 4.6], $\zeta(\alpha + 1, \beta, \chi_\psi\chi) \rtimes \sigma'$ has a unique irreducible subrepresentation and it appears with multiplicity one, but at the same time, by [15, Lemma 3.1], it is a quotient in $\zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \rtimes \sigma'$. Thus $\zeta(\alpha + 1, \beta, \chi_\psi\chi) \rtimes \sigma'$ is irreducible and $\zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \rtimes \sigma' \cong \zeta(\alpha + 1, \beta, \chi_\psi\chi) \rtimes \sigma'$ contradicts negativity of σ''. Thus, we have shown that σ cannot be negative, but not strongly negative.

Now, assume that σ is not negative. By Theorem 3.6, there exist an irreducible negative representation σ_{neg}, an integer h and $\chi_i \in \{1, \chi_0\}$, for $i = 1, \ldots, h$, and there exist $\alpha_i, \beta_i \in \mathbb{R}$ such that $\alpha_i - \beta_i \in \mathbb{Z}_{>0}$, $\alpha_i + \beta_i \in \mathbb{Z}_{\geq 0}$, and $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_h, \alpha_h, \chi_\psi\chi_h)$ is irreducible and

$$\sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_h, \alpha_h, \chi_\psi\chi_h) \rtimes \sigma_{neg}. \tag{4.11}$$

Frobenius reciprocity gives $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_h, \alpha_h, \chi_\psi\chi_h) \otimes \sigma_{neg} \leq \mu^*(\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma')$. By (2.5), there exist $0 \leq j \leq i \leq \alpha + \beta + 1$ and an irreducible representation $\zeta \otimes \sigma'_1 \leq \mu^*(\sigma')$ for which

$$\begin{aligned}
&\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_h, \alpha_h, \chi_\psi\chi_h) \otimes \sigma_{neg} \leq \\
&\zeta(-\alpha, \beta - i, \chi_\psi\chi) \times \zeta(-\beta, j - \beta - 1, \chi_\psi\chi) \times \zeta \otimes \zeta(j - \beta, i - \beta - 1, \chi_\psi\chi) \rtimes \sigma'_1
\end{aligned}$$

Because of the cuspidal support of σ' and $\alpha_s - \beta_s > 0$, $s = 1, \ldots, h$, we have $j = 0$ and

$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_h, \alpha_h, \chi_\psi\chi_h) \otimes \sigma_{neg} \leq \zeta(-\alpha, \beta - i, \chi_\psi\chi) \times \zeta \otimes \zeta(-\beta, i - \beta - 1, \chi_\psi\chi) \rtimes \sigma'_1.$$

Since σ' is strongly negative, ζ cannot have a positive sum of powers of ν. Thus $\beta - i > \alpha$. But then $i-\beta-1 < -\alpha-1$ and if $i > 0$, exactly the same argument as after (4.8) proves that $\zeta(-\beta, i-\beta-1, \chi_\psi\chi) \rtimes \sigma_1'$ is irreducible, contradicting negativity of σ_{neg}. Thus, $i = 0$.

Suppose $\zeta \neq \chi_\psi \mathbf{1}$. Then ζ is unramified by Lemma 3.3, and by Theorem 3.1 there exists an integer h', and for $s = 1, \dots, h'$, there exist a unitary unramified character χ_s' of $F^\times$, and $\alpha_s', \beta_s' \in \mathbb{R}$ with $\alpha_s' + \beta_s' \geq 0$, such that $\zeta \cong \zeta(-\beta_1', \alpha_1', \chi_\psi\chi_1') \times \cdots \times \zeta(-\beta_{h'}', \alpha_{h'}', \chi_\psi\chi_{h'}')$. Comparing cuspidal supports, by Theorem 3.1, we see that $\zeta(-\alpha, \beta, \chi_\psi\chi) \times \zeta$ is irreducible. Since ζ cannot produce positive sum of powers of ν and $\alpha_s - \beta_s > 0,\ s = 1, \dots, h$, using the uniqueness of classification from Theorem 3.1, we have a contradiction.

Thus $i = j = h = 0,\ \zeta = \chi_\psi \mathbf{1},\ \sigma_{neg} \cong \sigma_1' \cong \sigma'$ and (4.11) becomes

$$\sigma \hookrightarrow \zeta(-\alpha, \beta, \chi_\psi\chi) \rtimes \sigma'. \tag{4.12}$$

The argument just after (4.8) again shows that $\zeta(-\beta, -\alpha-1, \chi_\psi\chi) \rtimes \sigma'$ is irreducible, so

$$\zeta(-\alpha, \beta, \chi_\psi\chi) \rtimes \sigma' \hookrightarrow \zeta(-\alpha, \alpha, \chi_\psi\chi) \times \zeta(\alpha+1, \beta, \chi_\psi\chi) \rtimes \sigma' \cong \zeta(-\alpha, \alpha, \chi_\psi\chi) \times \zeta(-\beta, -\alpha-1, \chi_\psi\chi) \rtimes \sigma',$$

which together with (4.12) and Lemma 3.4 implies

$$\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'. \tag{4.13}$$

By [11, Thm. 4.6] and (4.12), σ is the unique irreducible subrepresentation of $\zeta(-\alpha, \beta, \chi_\psi\chi) \rtimes \sigma'$ and appearing with multiplicity one, but at the same time it is a quotient, due to (4.13) and [15, Lemma 3.1]. Thus, if we prove that $\zeta(-\alpha, \beta, \chi_\psi\chi) \rtimes \sigma'$ reduces, it will give a contradiction and finish the proof that σ is strongly negative.

To show that $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$ reduces, we assume that $\chi = 1$ and $k-2, l \geq 3$ are odd. Otherwise, the proof goes in the same way, only the notation has to be changed.

It is easy to see that $\zeta(-\alpha, \alpha, \chi_\psi) \otimes \omega_0$ appears with multiplicity two in $\mu^*(\zeta(-\alpha, \alpha, \chi_\psi) \rtimes \omega_0)$, and since admissible unitarizable representations are completely reducible, we can write

$$\zeta(-\alpha, \alpha, \chi_\psi) \rtimes \omega_0 \cong \pi_1 \oplus \pi_2, \tag{4.14}$$

where π_1 and π_2 are irreducible and not isomorphic, and one of them must be unramified. Using induction hypothesis, Lemmas 4.2, 4.1 and 2.1, we have

$$\begin{aligned}
&\zeta(-\beta, \alpha, \chi_\psi) \rtimes \sigma' \hookrightarrow \zeta(-\beta, -\alpha-1, \chi_\psi) \times \zeta(-\alpha, \alpha, \chi_\psi) \times \sigma' \hookrightarrow \\
&\zeta(-\beta, -\alpha-1, \chi_\psi) \times \zeta(-m_{l-2}, m_{l-3}, \chi_\psi) \times \cdots \times \zeta(-m_1, -\frac{1}{2}, \chi_\psi) \times \\
&\zeta(-n_k, n_{k-1}, \chi_\psi\chi_0) \times \cdots \times \zeta(-n_1, -\frac{1}{2}, \chi_\psi\chi_0) \rtimes (\pi_1 \oplus \pi_2).
\end{aligned}$$

Let us denote

$$\begin{aligned}
\rho \cong &\zeta(-\beta, -\alpha-1, \chi_\psi) \times \zeta(-m_{l-2}, m_{l-3}, \chi_\psi) \times \cdots \times \zeta(-m_1, -\frac{1}{2}, \chi_\psi) \times \\
&\zeta(-n_k, n_{k-1}, \chi_\psi\chi_0) \times \cdots \times \zeta(-n_1, -\frac{1}{2}, \chi_\psi\chi_0).
\end{aligned}$$

It is irreducible, and we have

$$\zeta(-\beta, \alpha, \chi_\psi) \rtimes \sigma' \hookrightarrow \rho \rtimes \pi_1 \oplus \rho \rtimes \pi_2.$$

To prove that $\zeta(-\beta, \alpha, \chi_\psi) \rtimes \sigma'$ reduces, it is enough to see $\mu^*(\zeta(-\beta, \alpha, \chi_\psi) \rtimes \sigma') \geq \rho \otimes \pi_1 + \rho \otimes \pi_2$, $\mu^*(\rho \rtimes \pi_1) \ngeq \rho \otimes \pi_2$, and $\mu^*(\rho \rtimes \pi_2) \ngeq \rho \otimes \pi_1$. First, by (2.5), we have

$$\begin{aligned}
\mu^*(\zeta(-\beta, \alpha, \chi_\psi) \rtimes \sigma') = &\sum_{\zeta \otimes \sigma_1 \leq \mu^*(\sigma')} \sum_{i=0}^{\alpha+\beta+1} \sum_{j=0}^{i} \zeta(-\alpha, \beta - i, \chi_\psi) \\
&\times \zeta(-\beta, -\beta-1+j, \chi_\psi) \times \zeta \otimes \zeta(-\beta+j, -\beta-1+i, \chi_\psi) \rtimes \sigma_1
\end{aligned} \tag{4.15}$$

By induction hypothesis, Lemma 2.1 and Frobenius reciprocity, we have

$$\begin{aligned}\mu^*(\sigma') \geq & \zeta(-m_{l-2}, m_{l-3}, \chi_\psi) \times \cdots \times \zeta(-l_1, -\frac{1}{2}, \chi_\psi) \times \\ & \zeta(-n_k, n_{k-1}, \chi_\psi\chi_0) \times \zeta(-n_{k-2}, n_{k-3}, \chi_\psi\chi_0) \times \cdots \times \zeta(-k_1, -\frac{1}{2}, \chi_\psi\chi_0) \otimes \omega_0.\end{aligned} \tag{4.16}$$

Picking $i = \alpha + \beta + 1$ and $j = \beta - \alpha$ in (4.15), and using (4.14) and (4.16), we have

$$\mu^*(\zeta(-\beta, \alpha, \chi_\psi) \rtimes \sigma') \geq \rho \otimes \pi_1 + \rho \otimes \pi_2.$$

We now show that $\mu^*(\rho \rtimes \pi_1) \not\geq \rho \otimes \pi_2$. Suppose the contrary, and apply (2.5) to $\rho \rtimes \pi_1$. There exist an irreducible representation $\zeta \otimes \sigma_1 \leq \mu^*(\pi_1)$ and indices

$$\begin{aligned}0 \leq j \leq i \leq -\alpha + \beta, \ & 0 \leq j_{l-2} \leq i_{l-2} \leq m_{l-3} + m_{l-2} + 1, \ldots, 0 \leq j_1 \leq i_1 \leq m_1 + \frac{1}{2}, \\ & 0 \leq j'_k \leq i'_k \leq n_{k-1} + n_k + 1, \ldots, 0 \leq j'_1 \leq i'_1 \leq n_1 + \frac{1}{2},\end{aligned}$$

such that

$$\begin{aligned}
&\zeta(-\beta, -\alpha - 1, \chi_\psi) \times \zeta(-m_{l-2}, m_{l-3}, \chi_\psi) \times \cdots \times \zeta(-m_1, -\frac{1}{2}, \chi_\psi) \times \\
&\zeta(-n_k, n_{k-1}, \chi_\psi\chi_0) \times \zeta(-n_{k-2}, n_{k-3}, \chi_\psi\chi_0) \times \cdots \times \zeta(-n_1, -\frac{1}{2}, \chi_\psi\chi_0) \otimes \pi_2 \\
&\leq \\
&\zeta(\alpha + 1, \beta - i, \chi_\psi) \times \zeta(-\beta, -\beta - 1 + j, \chi_\psi) \times \\
&\zeta(-m_{l-3}, m_{l-2} - i_{l-2}, \chi_\psi) \times \zeta(-m_{l-2}, -m_{l-2} - 1 + j_{l-2}, \chi_\psi) \times \\
&\cdots \\
&\zeta(\frac{1}{2}, m_1 - i_1, \chi_\psi) \times \zeta(-m_1, -m_1 - 1 + j_1, \chi_\psi) \times \\
&\zeta(-n_{k-1}, n_k - i'_k, \chi_\psi\chi_0) \times \zeta(-n_k, -n_k - 1 + j'_k, \chi_\psi\chi_0) \times \\
&\cdots \\
&\zeta(\frac{1}{2}, n_1 - i'_1, \chi_\psi\chi_0) \times \zeta(-n_1, -n_1 - 1 + j'_1, \chi_\psi\chi_0) \times \zeta \\
&\otimes \\
&\zeta(-\beta + j, -\beta - 1 + i, \chi_\psi) \times \\
&\zeta(-m_{l-2} + j_{l-2}, -m_{l-2} - 1 + i_{l-2}, \chi_\psi) \times \cdots \times \zeta(-m_1 + j_1, -m_1 - 1 + i_1, \chi_\psi) \times \\
&\zeta(-n_k + j'_k, -n_k - 1 + i'_k, \chi_\psi\chi_0) \times \cdots \times \zeta(-n_1 + j'_1, -n_1 - 1 + i'_1, \chi_\psi\chi_0) \rtimes \sigma_1.
\end{aligned}$$

Comparing cuspidal supports, we see that the cuspidal support of ζ cannot contain $\nu^{-\alpha}\chi_\psi$, and

$$\begin{aligned}&\mu^*(\pi_1 + \pi_2) = \mu^*(\zeta(-\alpha, \alpha, \chi_\psi) \rtimes \omega_0) = \\ &\sum_{u=0}^{2\alpha+1} \sum_{v=0}^{u} \zeta(-\alpha, \alpha - u, \chi_\psi) \times \zeta(-\alpha, -\alpha - 1 + v, \chi_\psi) \otimes \zeta(-\alpha + v, -\alpha - 1 + u, \chi_\psi) \rtimes \omega_0\end{aligned}$$

implies $\zeta \cong \chi_\psi \mathbf{1}$, so $\sigma_1 \cong \pi_1$. As π_1 and π_2 have the same cuspidal support, we must have

$$j = i, \ j_{l-2} = i_{l-2}, , \ldots, j_1 = i_1, \ j'_k = i'_k, , \ldots, j'_1 = i'_1$$

and $\pi_1 \cong \pi_2$, a contradiction. Thus, we showed $\mu^*(\rho \rtimes \pi_1) \not\geq \rho \otimes \pi_2$. In the same way one gets $\mu^*(\rho \rtimes \pi_2) \not\geq \rho \otimes \pi_1$. Thus $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$ reduces. As we already explained this proves that σ is strongly negative.

Now, we prove formula (4.7), that is, $\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$, and (4.6) is a consequence, obtained using induction hypothesis. Since σ is strongly negative, by Lemma 4.3 (2), there exist a real number α', $\alpha' + \beta \in \mathbb{Z}_{\geq 0}$, and an irreducible unramified representation σ'', such that $\sigma \hookrightarrow \zeta(-\beta, \alpha', \chi_\psi\chi) \rtimes \sigma''$. Take α' the largest possible, so σ'' is strongly negative. The case $\alpha' > \alpha$ would imply that $\nu^{\pm\alpha'}\chi_\psi\chi$ appears two times in the cuspidal support of σ, which is not possible. Hence, $\alpha' \leq \alpha$, and let σ'_n be the unramified irreducible subquotient of $\zeta(-\alpha, \alpha, \chi_\psi\chi) \rtimes \sigma'$. Suppose $\alpha' < \alpha$. We have

$$\sigma'_n \leq \zeta(-\alpha, \alpha, \chi_\psi\chi) \rtimes \sigma' \leq \zeta(-\alpha, \alpha', \chi_\psi\chi) \times \zeta(\alpha' + 1, \alpha, \chi_\psi\chi) \rtimes \sigma'.$$

Cuspidal support implies $\sigma'' \leq \zeta(\alpha' + 1, \alpha, \chi_\psi\chi) \rtimes \sigma'$. By Lemma 3.3, $\sigma'_n \leq \zeta(-\alpha, \alpha', \chi_\psi\chi) \rtimes \sigma''$. Now

$$\sigma \hookrightarrow \zeta(-\beta, \alpha', \chi_\psi\chi) \rtimes \sigma'' \hookrightarrow \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \times \zeta(-\alpha, \alpha', \chi_\psi\chi) \rtimes \sigma''$$

and therefore

$$\sigma \hookrightarrow \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \rtimes \sigma'_n.$$

Lemma 3.8 implies $\sigma'_n \hookrightarrow \zeta(-\alpha, \alpha, \chi_\psi\chi) \rtimes \sigma'$. Thus

$$\sigma \hookrightarrow \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \times \zeta(-\alpha, \alpha, \chi_\psi\chi) \rtimes \sigma'.$$

By Lemma 3.4, $\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$. So $\alpha' = \alpha$, proving (4.7) and (4.6). $\square$

4.3. **Classification of strongly negative unramified representations.** We now prove that representations of Theorem 4.4 exhaust all genuine irreducible strongly negative unramified representations of the metaplectic group. We first have a proposition.

Proposition 4.5. *Let χ_1 be an unramified unitary character of $F^\times$ and $\alpha_1, \beta_1 \in \mathbb{R}$, such that $\alpha_1 + \beta_1 \in \mathbb{Z}_{\geq 0}$ and $\alpha_1 - \beta_1 > 0$. Then*

(1) *Representation $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \omega_0$ reduces if and only if $\nu^i\chi_1\chi_\psi \rtimes \omega_0$ reduces for some i with $-\beta_1 \leq i \leq \alpha_1$ and $\alpha_1 - i \in \mathbb{Z}$, i.e., $\chi_1 \in \{1, \chi_0\}$, $-\beta_1 \in \frac{1}{2} - \mathbb{Z}_{\geq 0}$. If it reduces, the unique subrepresentation is not unramified.*

(2) *Let $\sigma = \sigma_{\alpha,\alpha'}$ be as in Lemma 4.2. Then, representation $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ reduces if and only if one of the following five representations reduces*

$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\alpha, -\frac{1}{2}, \chi_\psi),\ \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \times \zeta(-\alpha, -\frac{1}{2}, \chi_\psi),$$
$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0),\ \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0),$$
$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \omega_0.$$

If $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ reduces, its unique subrepresentation is not unramified.

(3) *Let $\sigma(\mathrm{Jord})$ be as in Sect. 4.1. Then $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma(\mathrm{Jord})$ reduces if and only if one of the following representations reduces ($i = l, l-2, \ldots,\ j = k, k-2, \ldots$)*

$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-m_i, m_{i-1}, \chi_\psi),\ \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \times \zeta(-m_i, m_{i-1}, \chi_\psi),$$
$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-n_j, n_{j-1}, \chi_\psi\chi_0),\ \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \times \zeta(-n_j, n_{j-1}, \chi_\psi, \chi_0)$$
$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma_0(\mathrm{Jord}).$$

If $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma(\mathrm{Jord})$ reduces, its unique subrepresentation is not unramified.

Proof. (1) Theorems A.1 and A.7 solve reducibility. Theorem 4.4 shows that in case of reducibility, the irreducible unramified subquotient is strongly negative, so cannot be a subrepresentation.

(2) Let all representations from the list be irreducible, and $\zeta = \zeta(-\alpha, -\frac{1}{2}, \chi_\psi)$, $\zeta' = \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0)$, $\zeta_1 = \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1)$ and $\zeta_2 = \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1})$. By Lemma 4.2 and (2.7),

$$\zeta_1 \rtimes \sigma \hookrightarrow \zeta_1 \times \zeta \times \zeta' \rtimes \omega_0 \cong \zeta \times \zeta' \times \zeta_1 \rtimes \omega_0 \cong \zeta \times \zeta' \times \zeta_2 \rtimes \omega_0 \cong \zeta_2 \times \zeta \times \zeta' \rtimes \omega_0,\ \text{thus}$$

$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma \hookrightarrow \zeta_2 \times \zeta \times \zeta' \rtimes \omega_0,$$
$$\zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \rtimes \sigma \hookrightarrow \zeta_2 \times \zeta \times \zeta' \rtimes \omega_0.$$

Because of the uniqueness of the irreducible unramified subquotient, the intersection of images of the embeddings is nontrivial. By [11, Thm. 4.6], $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ has a unique irreducible subrepresentation, appearing with multiplicity one in its composition series, but it is also a quotient of $\zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \rtimes \sigma$ by [15, Lemma 3.1]. Thus, $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ must be irreducible.

Let us suppose that one of the representations from the list reduces, and denote by τ the unramified irreducible subquotient of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$. Whenever τ is negative, it cannot be the unique subrepresentation of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ and thus this representation must reduce, proving the claim. We use this argument repeatedly below.

(i) If $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\alpha, -\frac{1}{2}, \chi_\psi)$ reduces, then $\chi_1 = 1$ and its unramified irreducible subquotient is $\zeta(-\alpha, \alpha_1, \chi_\psi) \times \zeta(-\beta_1, -\frac{1}{2}, \chi_\psi)$. By Lemmas 2.1 and 3.3,

$$\tau \leq \zeta(-\alpha, \alpha_1, \chi_\psi) \times \zeta(-\beta_1, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0,$$
$$\tau \leq \zeta(-\alpha, \alpha_1, \chi_\psi) \rtimes \sigma_{\beta_1, \alpha'}.$$

By Lemma 4.2, $\sigma_{\beta_1,\alpha'}$ is negative. If $\alpha = \alpha_1$, Lemma 3.8 implies that τ is negative. Else, $\alpha \neq \alpha_1$ and because of the reducibility $-\beta_1 = \frac{1}{2}$ or $-\alpha < -\beta_1 \leq -\frac{1}{2}$. Together with $-\beta_1 + \alpha_1 > 0$, Lemmas 4.2 and 3.8, and Theorem 4.4 imply negativity of τ. Since τ is negative, it cannot be the unique subrepresentation of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$, and thus, this representation must reduce, proving the claim.

(ii) If $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0)$ reduces, the proof is the same as in (i).

(iii) If $\zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \times \zeta(-\alpha, -\frac{1}{2}, \chi_\psi)$ reduces, we may assume that $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\alpha, -\frac{1}{2}, \chi_\psi)$ is irreducible. Lemma 2.1 implies $\chi_1 = 1$, $\alpha + 1 \geq -\beta_1 > \frac{1}{2}$, $\alpha_1 > \alpha$. Because of (2.7) and Lemmas 2.1, 3.3 and 4.2, we have

$$\tau \leq \zeta(-\alpha_1, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha, \beta_1, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0,$$
$$\tau \leq \zeta(-\beta_1, \alpha, \chi_\psi) \rtimes \sigma_{\alpha_1, \alpha'}.$$

If $-\beta_1 = \alpha + 1$, Lemma 4.2 implies negativity of τ. If $-\beta_1 = \alpha$, Lemma 3.8 implies negativity of τ. In both cases, because τ is negative, it cannot be the unique subrepresentation of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$, and thus, this representation must reduce, proving the claim. Otherwise, $\alpha > -\beta_1$ and for representation $\zeta(-\beta_1, \alpha, \chi_\psi) \rtimes \sigma_{\alpha_1,\alpha'}$ we have irreducibility of all representations listed in (2) (because $\alpha_1 > \alpha > -\beta_1 > \frac{1}{2}$), and so it is irreducible, as we already proved at the beginning. Thus, $\tau \cong \zeta(-\beta_1, \alpha, \chi_\psi) \rtimes \sigma_{\alpha_1,\alpha'}$. But then, because of the uniqueness of the Zelevinsky classification (Theorem 3.6) and $\alpha \neq \alpha_1$, τ is not a subrepresentation of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$. Thus $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ reduces and its unique subrepresentation is not unramified.

(iv) If $\zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0)$ reduces, the proof is the same as in (iii).

(v) If $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \omega_0$ reduces, assume that all other representations from the list are irreducible. We have $\chi_1 \in \{1, \chi_0\}$, and take $\chi_1 = 1$, the other case being the same.
If $\zeta(-\alpha, -\frac{1}{2}, \chi_\psi) = \chi_\psi \mathbf{1}$, then $\tau \leq \zeta(-\alpha_1, \beta_1, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0$ and it is negative by Theorem 4.4 or Lemma 4.2. Else, $-\beta_1 \leq -\alpha$ so $\alpha_1 > \alpha$ and

$$\tau \leq \zeta(-\alpha_1, \beta_1, \chi_\psi) \times \zeta(-\alpha, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0,$$

If $-\beta_1 < -\alpha$, then τ is negative by Lemma 4.4. Otherwise, $-\beta_1 = -\alpha$ and

$$\tau \leq \zeta(-\alpha_1, -\frac{1}{2}, \chi_\psi) \times \zeta(\frac{1}{2}, \alpha, \chi_\psi) \times \zeta(-\alpha, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0,$$

$$\tau \leq \zeta(-\alpha, \alpha, \chi_\psi) \times \zeta(-\alpha_1, -\frac{1}{2}, \chi_\psi) \times \zeta(-\alpha', -\frac{1}{2}, \chi_\psi\chi_0) \rtimes \omega_0.$$

Now, Lemmas 3.8, 4.2 and 3.3 imply that τ is negative. In all cases, because τ is negative, it cannot be the unique subrepresentation of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ and thus this representation must reduce, proving the claim.

(3) We use induction on card(Jord). Case card(Jord) $= 0$ is trivial and card(Jord) $= 1$ is a consequence of (1). Because of (2), we may assume that there exist $\chi \in \{1, \chi_0\}$ such that $\mathrm{card}(\mathrm{Jord}(\chi_\psi\chi)) \geq 2$. Let $2\beta+1 > 2\alpha+1$ be two largest elements of $\mathrm{Jord}(\chi_\psi\chi)$. Put $\mathrm{Jord}' = \mathrm{Jord}\backslash\{(2\beta+1, \chi_\psi\chi), (2\alpha+1, \chi_\psi\chi)\}$, $\sigma = \sigma(\mathrm{Jord})$ and $\sigma' = \sigma(\mathrm{Jord}')$. Note that $\beta > \alpha > \frac{1}{2}$. Let τ be the unramified irreducible subquotient of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$. By Theorem 4.4, $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$.

(i) If $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\beta, \alpha, \chi_\psi\chi)$ reduces, then $\chi_1 = \chi$, $\beta > \beta_1$, $\alpha_1 > \alpha$. Suppose

$$\tau \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma.$$

Then, by Lemma 3.4 $\tau \hookrightarrow \zeta(-\beta, \alpha_1, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha, \chi_\psi\chi) \rtimes \sigma'$. If $|-\beta_1| \geq \alpha$, τ is negative by Theorem 4.4 and Lemma 3.8 (Lemma 3.8 applies if $\beta = \alpha_1$ or $\beta_1 = \alpha$). Thus, it cannot be the unique subrepresentation of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ and this representation must reduce, proving the claim. Else, $|-\beta_1| < \alpha$ and by Theorem 4.4 and Lemma 3.8 (Lemma 3.8 applies if $\beta = \alpha_1$) the irreducible unramified subquotient σ'' of $\zeta(-\beta, \alpha_1, \chi_\psi\chi) \rtimes \sigma'$ is negative. Now we have $\tau \hookrightarrow \zeta(-\beta_1, \alpha, \chi_\psi\chi) \rtimes \sigma''$, a contradiction with the Zelevinsky classification (Theorem 3.6 and $\alpha_1 \neq \alpha$). Thus, τ is not a subrepresentation and $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ reduces.

(ii) If $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma'$ reduces, by induction hypothesis $\chi_1 \in \{1, \chi_0\}$ and $\beta_1 \in \frac{1}{2} + \mathbb{Z}$. Having proved (i), we may also assume $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\beta, \alpha, \chi_\psi\chi)$ is irreducible. Thus

$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma'.$$

Let π be the unique irreducible subrepresentation of $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma'$. By induction hypothesis π is not unramified. If $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \otimes \sigma'$ appears with the same multiplicity in $\mu^*(\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma')$ and $\mu^*(\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \pi)$, then $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ reduces and τ is not a subrepresentation.

First, we calculate the multiplicity of $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \otimes \sigma'$ in $\mu^*(\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma')$. By (2.5), there exist $0 \leq j \leq i \leq \alpha + \beta + 1$, $0 \leq j_1 \leq i_1 \leq \alpha_1 + \beta_1 + 1$ and an irreducible representation $\zeta_1 \otimes \sigma_1 \leq \mu^*(\sigma')$ such that

$$\begin{aligned}
&\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \otimes \sigma' \leq \\
&\zeta(-\alpha, \beta - i, \chi_\psi\chi) \times \zeta(-\beta, j - \beta - 1, \chi_\psi\chi)\times \\
&\zeta(-\alpha_1, \beta_1 - i_1, \chi_\psi\chi_1) \times \zeta(-\beta_1, j_1 - \beta_1 - 1, \chi_\psi\chi_1) \times \zeta_1 \\
&\otimes \zeta(j - \beta, i - \beta - 1, \chi_\psi\chi) \times \zeta(j_1 - \beta_1, i_1 - \beta_1 - 1, \chi_\psi\chi_1) \rtimes \sigma_1.
\end{aligned}$$

Note that the cuspidal support of ζ_1 does not contain $\nu^k\chi_\psi\chi$ for $|k| \geq \alpha$.

If $\chi_1 = \chi$, we have:

- If $-\beta_1 \leq -\beta$, then $\alpha < \beta \leq \beta_1 < \alpha_1$, so $i_1 = j_1 = \alpha_1 + \beta_1 + 1$. Now $i = j = \alpha + \beta + 1$ or $i = j = \beta - \alpha$ and $\zeta_1 = \chi_\psi\mathbf{1}$ and $\sigma_1 \cong \sigma'$.
- If $\alpha_1 \leq \alpha$, then $|\beta_1| < \alpha_1 \leq \alpha < \beta$. Since $\zeta(-\alpha_1, \beta_1 - i_1, \chi_\psi\chi) \times \zeta(-\beta_1, j_1 - \beta_1 - 1, \chi_\psi\chi)$ cannot produce $\nu^\alpha\chi_\psi\chi$ two times if $\alpha_1 = \alpha$ or once if $\alpha_1 < \alpha$, we must have $i = j = \alpha + \beta + 1$ or $i = \beta - \alpha$, and since $\nu^{-\alpha-1}\chi_\psi\chi$ cannot be obtained by a choice of i_1 and j_1, we must have $j = \beta - \alpha$. In both cases, because of $-\beta_1 + \alpha_1 > 0$, $\zeta(-\alpha_1, \beta_1 - i_1, \chi_\psi\chi_1)$

should not appear, so $i_1 = \alpha_1 + \beta_1 + 1$. As σ' is strongly negative, ζ_1 cannot have in cuspidal support a positive sum of powers of ν. Thus $j_1 = \alpha_1 + \beta_1 + 1$, $\zeta_1 = \chi_\psi \mathbf{1}$, $\sigma_1 \cong \sigma'$.

- If $-\beta_1 > \alpha + 1$, we look how to get $\nu^\alpha \chi_\psi \chi$. One possibility is $j = \alpha + \beta + 1$. Then $i = j$, $i_1 = j_1 = \alpha_1 + \beta_1 + 1$, $\zeta_1 = \chi_\psi \mathbf{1}$ and $\sigma_1 \cong \sigma'$. Another possibility is $i = \beta - \alpha$. Since $\nu^{-\alpha-1}\chi_\psi\chi$ cannot be obtained by a choice of i_1 and j_1, we must have $j = \beta - \alpha$, $i_1 = j_1 = \alpha_1 + \beta_1 + 1$, $\zeta_1 = \chi_\psi \mathbf{1}$ and $\sigma_1 \cong \sigma'$.

If $\chi \neq \chi_1$, since ζ_1 does not contain $\nu^\alpha \chi_\psi \chi$, we must have $i = j = \alpha + \beta + 1$ or $i = j = \beta - \alpha$. Because $-\beta_1 + \alpha_1 > 0$, $\zeta(-\alpha_1, \beta_1 - i_1, \chi_\psi \chi_1)$ cannot appear, and $i_1 = \alpha_1 + \beta_1 + 1$. As σ' is strongly negative, ζ_1 cannot have in the cuspidal support a positive sum of powers of ν. Thus, $j_1 = \alpha_1 + \beta_1 + 1$, $\zeta_1 = \chi_\psi \mathbf{1}$ and $\sigma_1 \cong \sigma'$.
We proved that $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \otimes \sigma'$ appears in $\mu^*(\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma')$ two times.
Now we show that $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \otimes \sigma'$ appears in $\mu^*(\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \pi)$ at least two times. Take $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \otimes \sigma' \leq \mu^*(\pi)$. Now

$$\begin{aligned}\mu^*(\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \pi) \geq \sum_{i=0}^{\alpha+\beta+1} \sum_{j=0}^{i} & \zeta(-\alpha, \beta - i, \chi_\psi\chi) \times \zeta(-\beta, j - \beta - 1, \chi_\psi\chi) \times \\ & \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \otimes \zeta(j - \beta, i - \beta - 1, \chi_\psi\chi) \rtimes \sigma'.\end{aligned}$$

Choices of indices $i = j = \alpha + \beta + 1$ and $i = j = \beta - \alpha$ prove the claim.

(iii) If $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1})$ reduces, we may assume that $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\beta, \alpha, \chi_\psi\chi)$ and $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma'$ are irreducible. Now $\chi = \chi_1$, $-\beta_1 > \alpha + 1$, $\beta_1 \geq -\beta - 1$, $-\alpha_1 < -\beta$ and

$$\begin{aligned}\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma \hookrightarrow & \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma' \\ \cong & \zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma' \\ \cong & \zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1) \rtimes \sigma'.\end{aligned}$$

If $\tau \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$, Lemma 3.4 implies

$$\tau \hookrightarrow \zeta(-\alpha_1, \alpha, \chi_\psi\chi) \times \zeta(-\beta, \beta_1, \chi_\psi\chi) \rtimes \sigma'.$$

If $\zeta(-\beta, \beta_1, \chi_\psi\chi) = \chi_\psi \mathbf{1}$, then τ is negative by Theorem 4.4, a contradiction. Otherwise, by induction hypothesis, $\zeta(-\beta, \beta_1, \chi_\psi\chi) \rtimes \sigma'$ is irreducible, so

$$\tau \hookrightarrow \zeta(-\alpha_1, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \beta, \chi_\psi\chi) \rtimes \sigma'.$$

Because $-\beta_1 > \alpha + 1$, $\zeta(-\alpha_1, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \beta, \chi_\psi\chi)$ is irreducible, we have

$$\tau \hookrightarrow \zeta(-\beta_1, \beta, \chi_\psi\chi) \times \zeta(-\alpha_1, \alpha, \chi_\psi\chi) \rtimes \sigma'.$$

Let σ'' be the unramified irreducible subquotient of $\zeta(-\alpha_1, \alpha, \chi_\psi\chi) \rtimes \sigma'$. By Theorem 4.4, it is strongly negative. Lemma 3.4 gives $\tau \hookrightarrow \zeta(-\beta_1, \beta, \chi_\psi\chi) \rtimes \sigma''$, a contradiction with Theorem 3.6 (uniqueness of embedding).

(iv) If all representations from the list are irreducible, denote $\zeta = \zeta(-\beta, \alpha, \chi_\psi)$, $\zeta_1 = \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1)$ and $\zeta_2 = \zeta(-\alpha_1, \beta_1, \chi_\psi\chi^{-1})$. Now

$$\begin{aligned}\zeta_1 \rtimes \sigma &\hookrightarrow \zeta_1 \times \zeta \rtimes \sigma' \\ \zeta_2 \rtimes \sigma &\hookrightarrow \zeta_2 \times \zeta \rtimes \sigma' \cong \zeta \times \zeta_2 \rtimes \sigma' \cong \zeta \times \zeta_1 \rtimes \sigma' \cong \zeta_1 \times \zeta \rtimes \sigma'.\end{aligned}$$

By the uniqueness of the irreducible unramified subquotient, the intersection of the images of embeddings is nontrivial. By [11, Thm. 4.6], $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \rtimes \sigma$ has a unique irreducible subrepresentation appearing with multiplicity one in the composition series, but it is also a quotient of $\zeta(-\alpha_1, \beta_1, \chi_\psi \chi_1^{-1}) \rtimes \sigma$, by [15, Lemma 3.1]. Thus, $\zeta(-\beta_1, \alpha_1, \chi_\psi \chi_1) \rtimes \sigma$ must be irreducible.

$\square$

For $n \in \mathbb{Z}_{\geq 0}$, let $\mathrm{Irr}_{gen,unr,sn}(\widetilde{Sp(n,F)})$ (resp., $[\mathrm{Irr}_{gen,unr,sn}(\widetilde{Sp(n,F)})]$) be the set (resp., the set of isomorphism classes) of genuine irreducible strongly negative unramified representations of $\widetilde{Sp(n,F)}$. Now we finally prove their classification, up to isomorphism. Brackets [ ] are used to denote an isomorphism class.

Theorem 4.6. *Let $n \in \mathbb{Z}_{\geq 0}$. The map given by the assignment* $\mathrm{Jord} \mapsto [\sigma(\mathrm{Jord})]$ *is a bijection between* $\mathrm{Jord}(n)$ *and* $[Irr_{gen,unr,sn}(\widetilde{Sp(n,F)})]$.

Proof. Injectivity is obvious. We prove surjectivity by induction on n. Case $n = 0$ is trivial. Let $n \in \mathbb{Z}_{>0}$, and suppose that the claim is valid for all $0 \leq m < n$. Take $\sigma \in \mathrm{Irr}_{gen,unr,sn}(\widetilde{Sp(n,F)})$. Let β be the largest such that $\nu^{\pm\beta}\chi_\psi\chi$ is in cuspidal support of σ, where χ is a unitary character. As in Lemma 4.3, let α be the largest such that there is an embedding $\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$, where σ' is irreducible. Lemma 4.3 implies that $-\beta + \alpha < 0$, σ' is strongly negative and $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$ reduces. By induction hypothesis and Proposition 4.5, we have $\sigma' = \sigma(\mathrm{Jord}')$, $\chi \in \{1, \chi_0\}$ and $\beta \in \frac{1}{2} + \mathbb{Z}$. If $\mathrm{Jord}'(\chi_\psi\chi) = \emptyset$, then, by Theorem 4.4,

$$\sigma = \sigma(\mathrm{Jord}) \text{ for } \mathrm{Jord} = \mathrm{Jord}' \cup \{(2\beta + 1, \chi_\psi\chi), (2\alpha + 1, \chi_\psi\chi)\}.$$

Otherwise, we show that $\alpha > |i|$ for every $\nu^i\chi_\psi\chi$ appearing in the cuspidal support of σ'. Let $2\beta' + 1$ be the largest in $\mathrm{Jord}'(\chi_\psi\chi)$, and $2\alpha' + 1$ the second largest, if it exists, or else $2\alpha' + 1 = 0$. Let $\mathrm{Jord}'' = \mathrm{Jord}' \backslash \{(2\beta' + 1, \chi_\psi\chi), (2\alpha' + 1, \chi_\psi\chi)\}$. By Theorem 4.4

$$\sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta', \alpha', \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}''). \tag{4.17}$$

If $\beta' = \beta$, then $\sigma \hookrightarrow \zeta(-\beta', \alpha', \chi_\psi\chi) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}'')$. By Lemma 3.4 and the choice of α, we have $\alpha \geq \alpha'(\geq -\frac{1}{2})$. Also $\sigma \leq \zeta(-\beta', \beta, \chi_\psi\chi) \times \zeta(-\alpha, \alpha', \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}'')$. By Theorem 4.4 and Lemma 3.8, the irreducible unramified subquotient of $\zeta(-\alpha, \alpha', \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}'')$ is negative, so, by Lemma 3.8, σ is negative, but not strongly negative, a contradiction. Thus, $\beta' < \beta$.

If $\beta' > \alpha$, we have several cases:

- If $\alpha' > \alpha$, then $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-\beta', \alpha', \chi_\psi\chi)$ reduces. By Lemma 3.4, $\sigma \hookrightarrow \zeta(-\beta, \alpha', \chi_\psi\chi) \times \zeta(-\beta', \alpha, \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}'')$. Lemma 3.4 gives a contradiction with the choice of α.
- If $\alpha > \alpha'$, then $\sigma \leq \zeta(-\beta, \beta', \chi_\psi\chi) \times \zeta(-\alpha, \alpha', \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}'')$. By Theorem 4.4, σ embeds in this product, and Lemma 3.4 gives a contradiction with the choice of α.
- If $\alpha = \alpha'$, then $\sigma \leq \zeta(-\alpha, \alpha', \chi_\psi\chi) \times \zeta(-\beta, \beta', \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}'')$. By Theorem 4.4 and Lemma 3.8, σ is negative, but not strongly negative, a contradiction.

So $\alpha \geq \beta'$. If $\alpha = \beta'$, $\sigma \leq \zeta(-\beta, -\alpha - 1, \chi_\psi\chi) \times \zeta(-\alpha, \alpha, \chi_\psi\chi) \times \zeta(-\alpha, \alpha', \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}'')$, and

$$\sigma \leq \zeta(-\alpha, \alpha, \chi_\psi\chi) \times \zeta(-\beta, \alpha', \chi_\psi\chi) \rtimes \sigma(\mathrm{Jord}'').$$

By Lemma 3.8 and Theorem 4.4, σ is not strongly negative, a contradiction. Thus, $\beta > \alpha > \beta' > \alpha'$ and $\sigma \cong \sigma(\mathrm{Jord})$ for $\mathrm{Jord} = \mathrm{Jord}' \cup \{(2\beta + 1, \chi_\psi\chi), (2\alpha + 1, \chi_\psi\chi)\}$. $\square$

5. The Zelevinsky classification of unramified representations

We are finally ready to prove the strong form of the Zelevinsky classification of genuine unramified irreducible representations of $\widetilde{Sp(n,F)}$. But first we need a lemma.

Lemma 5.1. *Let $\chi, \chi_1, \ldots, \chi_t$ be unramified unitary characters of $F^\times$, $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta + 1 \in \mathbb{Z}_{>0}$ and $-\beta + \alpha > 0$, and $l_1, \ldots, l_t \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let σ' be a genuine irreducible unramified strongly negative representation of the metaplectic group and σ the irreducible unramified subquotient of $\zeta(-l_1, l_1, \chi_\psi\chi_1) \times$*

$\cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \sigma'$. *Then* $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma$ *reduces if and only if one of the following representations reduces* $(i = 1, \dots, t)$

$$\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-l_i, l_i, \chi_\psi\chi_i),\ \zeta(-\alpha, \beta, \chi_\psi\chi^{-1}) \times \zeta(-l_i, l_i, \chi_\psi\chi_i),\ \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'.$$

If $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma$ *reduces, its unique subrepresentation is not unramified.*

Proof. Lemma 3.8 implies $\sigma \hookrightarrow \zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \sigma'$ and the negativity of σ. If all representations from the list are irreducible, we have

$$\begin{aligned}\zeta(-\alpha, \beta, \chi_\psi\chi^{-1}) \rtimes \sigma \hookrightarrow & \zeta(-\alpha, \beta, \chi_\psi\chi^{-1}) \times \zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \sigma' \\ \cong & \zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \times \zeta(-\alpha, \beta, \chi_\psi\chi^{-1}) \rtimes \sigma' \\ \cong & \zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma' \\ \cong & \zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \sigma'.\end{aligned}$$

$$\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \sigma'$$

Because of the uniqueness of the irreducible unramified subquotient, the intersection of the images of embeddings is nontrivial. By [11, Thm. 4.6], $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma$ has a unique irreducible subrepresentation appearing with multiplicity one, which is also a quotient of $\zeta(-\alpha, \beta, \chi_\psi\chi^{-1}) \rtimes \sigma$, by [15, Lemma 3.1]. Thus $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma$ must be irreducible.

Now, we consider the cases when one of the representations from the list reduces. Let τ be the unramified irreducible subquotient of $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma$. Suppose that

(i) $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-l_i, l_i, \chi_\psi\chi_i)$ reduces for some $i = 1, \dots, t$. We may assume $i = 1$. Reducibility implies $\alpha > l_1 > \beta \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\chi = \chi_1$. Assume $\tau \hookrightarrow \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma$. Now

$$\tau \hookrightarrow \zeta(-l_1, \alpha, \chi_\psi\chi) \times \zeta(-\beta, l_1, \chi_\psi\chi) \times \zeta(-l_2, l_2, \chi_\psi\chi_2) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \sigma'$$

Let $\sigma_1 \leq \zeta(-\beta, l_1, \chi_\psi\chi) \times \zeta(-l_2, l_2, \chi_\psi\chi_2) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \sigma'$ be its irreducible unramified subquotient. By Lemma 3.4, $\tau \hookrightarrow \zeta(-l_1, \alpha, \chi_\psi\chi) \rtimes \sigma_1$. For σ_1, there exists an embedding $\sigma_1 \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}$, as in Theorem 3.6. Now

$$\tau \hookrightarrow \zeta(-l_1, \alpha, \chi_\psi\chi) \times \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}.$$

Since $\zeta(-\beta, \alpha, \chi_\psi\chi)$ does not contain $\nu^{-l_1}\chi_\psi\chi$ in the cuspidal support, we have a contradiction with the uniqueness of the embedding of τ (Theorem 3.6). Thus, τ is not a subrepresentation of $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma$, which must then reduce.

(ii) $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$ reduces. By Theorem 4.6 and Proposition 4.5, we have $\chi \in \{1, \chi_0\}$. Having proved (i), we may assume that $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-l_i, l_i, \chi_\psi\chi_i)$ is irreducible for every i. Now

$$\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma \hookrightarrow \zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'.$$

Let π be the unique irreducible subrepresentation of $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma'$. By Proposition 4.5, π is not unramified. If $\zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \otimes \sigma'$ is contained with the same multiplicity in $\mu^*(\zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma')$ and $\mu^*(\zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \pi)$, then τ cannot be a subrepresentation of $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \sigma$ and it must then reduce. Let us calculate these multiplicities.

First, by (2.5), there exist $0 \leq j \leq i \leq \alpha+\beta+1$, $0 \leq j_1 \leq i_1 \leq 2l_1+1, \dots, 0 \leq j_t \leq i_t \leq 2l_t+1$, and an irreducible representation $\zeta_1 \otimes \sigma_1 \leq \mu^*(\sigma')$, such that

$$\begin{aligned}
&\zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \otimes \sigma' \leq \\
&\zeta(-l_1, l_1-i_1, \chi_\psi\chi_1^{-1}) \times \zeta(-l_1, j_1-l_1-1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t-i_t, \chi_\psi\chi_t^{-1}) \times \\
&\zeta(-l_t, j_t-l_t-1, \chi_\psi\chi_t) \times \zeta(-\alpha, \beta-i, \chi_\psi\chi) \times \zeta(-\beta, j-\beta-1, \chi_\psi\chi) \times \zeta_1 \otimes \\
&\zeta(j_1-l_1, i_1-l_1-1, \chi_\psi\chi_1) \times \\
&\quad \cdots \times \zeta(j_t-l_t, i_t-l_t-1, \chi_\psi\chi_t) \times \zeta(j-\beta, i-\beta-1, \chi_\psi\chi) \rtimes \sigma_1.
\end{aligned}$$

The sum of exponents of ν in the cuspidal support of $\zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \times \zeta(-\beta, \alpha, \chi_\psi\chi)$ is $-\beta + \cdots + \alpha > 0$. On the right hand side only $\zeta(-\beta, j-\beta-1, \chi_\psi\chi)$ can have a positive sum, at most $-\beta + \cdots + \alpha$, achieved for $j = \alpha+\beta+1$, while ζ_1 has a negative sum if different from $\chi_\psi\mathbf{1}$. Thus $j = i = \alpha+\beta+1$, $\zeta_1 = \chi_\psi\mathbf{1}$, $\sigma_1 \cong \sigma'$ and if $\chi_s \neq \chi_s^{-1}$, $i_s = j_s = 2l_s+1$, while if $\chi_s = \chi_s^{-1}$, then $i_s = j_s = 2l_s+1$ or $i_s = j_s = 0$, for all $s = 1, \dots, t$.

For the second multiplicity, note that $\zeta(-\beta, \alpha, \chi_\psi\chi) \otimes \sigma' \leq \mu^*(\pi)$. Now

$$\begin{aligned}
&\mu^*(\zeta(-l_1, l_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-l_t, l_t, \chi_\psi\chi_t) \rtimes \pi) \geq \\
&\sum_{s=1}^{t} \sum_{i_s=0}^{2l_s+1} \sum_{j_s=0}^{i_s} \zeta(-l_1, l_1-i_1, \chi_\psi\chi_1^{-1}) \times \zeta(-l_1, j_1-l_1-1, \chi_\psi\chi_1) \times \dots \\
&\times \zeta(-l_t, l_t-i_t, \chi_\psi\chi_t^{-1}) \times \zeta(-l_t, j_t-l_t-1, \chi_\psi\chi_t) \times \zeta(-\beta, \alpha, \chi_\psi\chi) \\
&\otimes \zeta(j_1-l_1, i_1-l_1-1, \chi_\psi\chi_1) \times \cdots \times \zeta(j_t-l_t, i_t-l_t-1, \chi_\psi\chi_t) \rtimes \sigma'.
\end{aligned}$$

We can make choices for i_s and j_s, $s = 1, \dots, t$ as above, so multiplicities are equal.

(iii) $\zeta(-\alpha, \beta, \chi_\psi\chi^{-1}) \times \zeta(-l_i, l_i, \chi_\psi\chi_i)$ reduces for some $i = 1, \dots, t$. Replace $\zeta(-l_i, l_i, \chi_\psi\chi_i)$ with $\zeta(-l_i, l_i, \chi_\psi\chi_i^{-1})$. Now $\zeta(-\beta, \alpha, \chi_\psi\chi) \times \zeta(-l_i, l_i, \chi_\psi\chi_i^{-1})$ reduces and we are in case (i).

□

Theorem 5.2 (Zelevinsky Classification)**.** *Let σ be a genuine irreducible unramified representation of $\widetilde{Sp(n,F)}$. Then, either σ is negative, or there exist $k \in \mathbb{Z}_{>0}$, and a sequence $\chi_1, \dots, \chi_k$ of unramified unitary characters of $F^\times$, and there exist real numbers α_i, β_i, such that $\alpha_i+\beta_i \in \mathbb{Z}_{\geq 0}$ and $-\beta_i+\alpha_i > 0$, for $i = 1, \dots, k$ and there exists a genuine irreducible unramified negative representation σ_{neg} of the metaplectic group, such that $\sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}$. Data $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1), \dots, \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k)$ are unique up to permutation, while σ_{neg} is unique up to isomorphism. Moreover*

$$\sigma \cong \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}.$$

Proof. Suppose that σ is not negative and take an embedding as in Theorem 3.6

$$\sigma \hookrightarrow \zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}, \tag{5.1}$$

where $\chi_1, \dots, \chi_k$ are unramified unitary characters of $F^\times$, α_i, β_i are real numbers such that $\alpha_i+\beta_i \in \mathbb{Z}_{\geq 0}$, $-\beta_i+\alpha_i > 0$, $i = 1, \dots, k$, $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k)$ is irreducible and σ_{neg} is a genuine irreducible unramified negative representation. Note that we can permute Zelevinsky segment representations. Also, $\zeta(-\beta_i, \alpha_i, \chi_\psi\chi_i) \rtimes \sigma_{neg}$ is irreducible for every i, or else, by Proposition 4.5 and Lemma 5.1, its unique irreducible subrepresentation ρ_i is not unramified, resulting with

$$\begin{aligned}
\sigma \hookrightarrow &\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \cdots \times \zeta(-\beta_{i-1}, \alpha_{i-1}, \chi_\psi\chi_{i-1}) \times \\
&\zeta(-\beta_{i+1}, \alpha_{i+1}, \chi_\psi\chi_{i+1}) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \rho_i.
\end{aligned}$$

a contradiction to Lemma 3.3. Now we do the following process. We start with $\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1)$. Permute it until it is next to σ_{neg}, replace with $\zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1})$ and pull back to its place. To keep the embedding of σ, for this last action, note that $\zeta(-\beta_i, \alpha_i, \chi_\psi\chi_i) \times \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1})$ is irreducible for $i > 1$. Indeed, if i is the largest such that $\zeta(-\beta_i, \alpha_i, \chi_\psi\chi_i) \times \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1})$ reduces and π its unique irreducible subrepresentation, then it is not unramified by Lemma 2.1. Now

$$\sigma \hookrightarrow \zeta(-\beta_2, \alpha_2, \chi_\psi\chi_2) \times \cdots \times \zeta(-\beta_{i-1}, \alpha_{i-1}, \chi_\psi\chi_{i-1}) \times \pi \times \zeta(-\beta_{i+1}, \alpha_{i+1}, \chi_\psi\chi_{i+1}) \times \ldots \\ \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg},$$

a contradiction to Lemma 3.3. Thus

$$\sigma \hookrightarrow \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \times \zeta(-\beta_2, \alpha_2, \chi_\psi\chi_2) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}$$

We continue the process with $\zeta(-\beta_2, \alpha_2, \chi_\psi\chi_2), \ldots, \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k)$. In the end

$$\sigma \hookrightarrow \zeta(-\alpha_1, \beta_1, \chi_\psi\chi_1^{-1}) \times \zeta(-\alpha_2, \beta_2, \chi_\psi\chi_2^{-1}) \times \cdots \times \zeta(-\alpha_k, \beta_k, \chi_\psi\chi_k^{-1}) \rtimes \sigma_{neg}.$$

Lemma 3.1. of [15] implies that σ is a quotient of

$$\zeta(-\beta_1, \alpha_1, \chi_\psi\chi_1) \times \zeta(-\beta_2, \alpha_2, \chi_\psi\chi_2) \times \cdots \times \zeta(-\beta_k, \alpha_k, \chi_\psi\chi_k) \rtimes \sigma_{neg}.$$

Thus, Theorem 3.6, together with (5.1) gives the result. □

Appendix A. Reducibility of $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \omega_0$

The goal of this appendix is to prove a criterion for reducibility of the induced representation $\zeta(-\beta, \alpha, \chi_\psi\chi) \rtimes \omega_0$ where χ is a unitary unramified character of $F^\times$.

A.1. **The basic case of $n = 1$.** We first consider the reducibility of the induced representation $\chi_\psi\chi\nu^\alpha \rtimes \omega_0$. It is settled by the following theorem of [23] (see also [7]).

Theorem A.1. *Let χ be a unitary character of $F^\times$ and $\alpha \in \mathbb{R}$. Representation $\chi_\psi\chi\nu^\alpha \rtimes \omega_0$ reduces if and only if $\chi^2 = 1_{F^\times}$ and $\alpha \in \{\pm 1/2\}$. If $\alpha = -1/2$ and χ is unramified such that $\chi^2 = 1_{F^\times}$, then the unique unramified irreducible subquotient is a subrepresentation.*

A.2. **Lemmas of Tadić.** The following lemmas directly generalize from [22]. We include their statements to fix the notation.

Lemma A.2. *Let $\widetilde{P_0}$, $\widetilde{P'}$, $\widetilde{P''}$ and $\widetilde{P'''}$ be parabolic subgroups of $\widetilde{Sp(n,F)}$ with Levi factors $\widetilde{M_0}$, $\widetilde{M'}$, $\widetilde{M''}$ and $\widetilde{M'''}$, such that $\widetilde{P'} \subseteq \widetilde{P''}$ and $\widetilde{P'} \subseteq \widetilde{P'''}$. Let σ_0 be an irreducible representation of $\widetilde{M_0}$ such that*

$$r_{\widetilde{M'}}^{\widetilde{Sp(n,F)}}(\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0)) \neq 0.$$

Assume that there exists an irreducible subquotient τ'' of $r_{\widetilde{M''}}^{\widetilde{Sp(n,F)}}(\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0))$ such that for every irreducible subquotient τ''' of $r_{\widetilde{M'''}}^{\widetilde{Sp(n,F)}}(\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0))$ we have:

$$s.s.\left(r_{\widetilde{M'}}^{\widetilde{M''}}(\tau'')\right) + s.s.\left(r_{\widetilde{M'}}^{\widetilde{M'''}}(\tau''')\right) \nleq s.s.\left(r_{\widetilde{M'}}^{\widetilde{Sp(n,F)}}(\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}((\sigma_0))\right).$$

Then, induced representation $\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0)$ is irreducible.

Lemma A.3. *Let $\widetilde{P_0} = \widetilde{M_0}N_0$ be a parabolic subgroup of $\widetilde{Sp(n,F)}$ and σ_0 an irreducible unitarizable representation of $\widetilde{M_0}$.*

Let $\widetilde{P'} = \widetilde{M'}N'$ and $\widetilde{P''} = \widetilde{M''}N''$ be parabolic subgroups of $\widetilde{Sp(n,F)}$ such that $\widetilde{P'} \subseteq \widetilde{P_0}$ and $\widetilde{P'} \subseteq \widetilde{P''}$. Assume that there exists an irreducible subquotient τ'' of $r_{\widetilde{M''}}^{\widetilde{Sp(n,F)}}(\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0))$ of multiplicity one. Let τ_0

be an irreducible subquotient of $r_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0))$ *and* σ' *an irreducible representation of* $\widetilde{M}'$. *Assume that the following three assertions hold:*

(i) $\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0) \hookrightarrow \mathrm{Ind}_{\widetilde{M'}}^{\widetilde{Sp(n,F)}}(\sigma')$

(ii) *If* τ_0' *is an irreducible subquotient of* $r_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0))$ *that is not isomorphic to* τ_0, *then* σ' *is not a subquotient of* $r_{\widetilde{M'}}^{\widetilde{M_0}}(\tau_0')$.

(iii) *There exists an irreducible subquotient* ρ' *of* $r_{\widetilde{M'}}^{\widetilde{M_0}}(\tau_0)$ *such that* ρ' *has the same multiplicity in* $r_{\widetilde{M'}}^{\widetilde{M''}}(\tau'')$ *and* $r_{\widetilde{M'}}^{\widetilde{Sp(n,F)}}(\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0))$.

Then $\mathrm{Ind}_{\widetilde{M_0}}^{\widetilde{Sp(n,F)}}(\sigma_0)$ *is irreducible.*

A.3. **The incomplete reducibility criterion.** Now we prove the irreducibility under certain conditions. The following two lemmas solve a special case.

Lemma A.4. *Let* χ *be a unitary unramified character of* $F^\times$ *such that* $\chi^2 = 1$. *The representation* $\delta(0,1,\chi_\psi\chi) \rtimes \omega_0$ *is irreducible.*

Proof. The proof is the same as the proof of Lemma 6.2 and Proposition 6.3 of [22], except that instead of using the analogue of Lemma 6.1 of *loc. cit.*, that is, $\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \rtimes \omega_0$ is irreducible, it is enough to have that all irreducible subquotients of $\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \rtimes \omega_0$ are isomorphic. This is valid by Corollary 8.3 of [8] and the fact that $\chi \times \delta(-1,1,\chi) \rtimes \mathbf{1}$ as the representation of the split odd special orthogonal group is irreducible, which is a consequence of Lemma 6.1 of [22] and part *(iii)* of Theorem 3.3 of [18].

For the sake of completeness we write down the proof. Using (2.6), we have

$$s.s.r_{(4;0)}(\chi_\psi\chi \times \chi_\psi\chi \times \chi_\psi\nu\chi \times \chi_\psi\nu\chi \rtimes \omega_0) = 4 \sum_{(\epsilon_1,\epsilon_2)\in\{\pm1\}^2} \chi_\psi\chi \times \chi_\psi\chi \times \chi_\psi\nu^{\epsilon_1}\chi \times \chi_\psi\nu^{\epsilon_2}\chi \otimes \omega_0$$

$$s.s.r_{(4;0)}(\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \rtimes \omega_0) = 4\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \otimes \omega_0 + 4\chi_\psi\chi \times \chi_\psi\nu\chi \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0.$$

Thus, the multiplicity of $\delta(0,1,\chi_\psi\chi) \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0$ in $\mu^*(\chi_\psi\chi \times \chi_\psi\chi \times \chi_\psi\nu\chi \times \chi_\psi\nu\chi \rtimes \omega_0)$ is 4, the same as in $\mu^*(\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \rtimes \omega_0)$.

Now, recall that, by Theorem A.1, $\chi_\psi\chi \rtimes \omega_0$ and $\chi_\psi\nu\chi \rtimes \omega_0$ are irreducible. By (2.6), we have

$$\begin{aligned}\mu^*(\delta(0,1,\chi_\psi\chi) \rtimes \omega_0) =& \chi_\psi\mathbf{1} \otimes \delta(0,1,\chi_\psi\chi) \rtimes \omega_0 + [\chi_\psi\nu\chi \otimes \chi_\psi\chi \rtimes \omega_0 + \chi_\psi\chi \otimes \chi_\psi\nu\chi \rtimes \omega_0]+ \\ &+ [\delta(0,1,\chi_\psi\chi) \otimes \omega_0 + \chi_\psi\chi \times \chi_\psi\nu\chi \otimes \omega_0 + \delta(-1,0,\chi_\psi\chi) \otimes \omega_0].\end{aligned}$$

Suppose that $\delta(0,1,\chi_\psi\chi) \rtimes \omega_0$ reduces. Then, there exists an irreducible subquotient π, such that $r_{(1;1)}(\pi) = \chi_\psi\nu\chi \otimes \chi_\psi\chi \rtimes \omega_0$. Using (2.6) and Lemma 2.1, we see

$$\mu^*(\pi) = \chi_\psi\mathbf{1} \otimes \pi + \chi_\psi\nu\chi \otimes \chi_\psi\chi \rtimes \omega_0 + 2\delta(0,1,\chi_\psi\chi) \otimes \omega_0.$$

Using (2.6), we have

$$\begin{aligned}s.s.r_{(4;0)}(\delta(0,1,\chi_\psi\chi) \rtimes \pi) =& 2\delta(0,1,\chi_\psi\chi) \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0 + 2\chi_\psi\chi \times \chi_\psi\nu\chi \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0 + \\ &+ 2\delta(-1,0,\chi_\psi\chi) \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0.\end{aligned}$$

Thus $s.s.r_{(4;0)}(\delta(0,1,\chi_\psi\chi) \rtimes \pi) \geq 4\delta(0,1,\chi_\psi\chi) \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0$. As we proved that the multiplicity of $\delta(0,1,\chi_\psi\chi) \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0$ in $\mu^*(\chi_\psi\chi \times \chi_\psi\chi \times \chi_\psi\nu\chi \times \chi_\psi\nu\chi \rtimes \omega_0)$ is 4, the same as in $\mu^*(\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \rtimes \omega_0)$, and by the above argument all irreducible subquotients of $\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \rtimes \omega_0$ are isomorphic, we have $s.s.(\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \rtimes \omega_0) \le s.s.(\delta(0,1,\chi_\psi\chi) \rtimes \pi)$. Writing down $s.s.r_{(4,0)}(\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \rtimes \omega_0) \le s.s.r_{(4,0)}(\delta(0,1,\chi_\psi\chi) \rtimes \pi)$, we get

$$4\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \otimes \omega_0 + 4\chi_\psi\chi \times \chi_\psi\nu\chi \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0 \le$$
$$2\delta(0,1,\chi_\psi\chi) \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0 + 2\chi_\psi\chi \times \chi_\psi\nu\chi \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0 + 2\delta(-1,0,\chi_\psi\chi) \times \delta(0,1,\chi_\psi\chi) \otimes \omega_0.$$

But, using Lemma 2.1 and (2.3) and (2.4), we see that $\chi_\psi\chi \times \delta(-1,1,\chi_\psi\chi) \otimes \omega_0$ appears only two times on the righthand side, a contradiction. We have proved that $\delta(0,1,\chi_\psi\chi) \rtimes \omega_0$ does not reduce. □

Lemma A.5. *Let χ be a unitary unramified character of $F^\times$ such that $\chi^2 = 1$. The representation $\zeta(0,1,\chi_\psi\chi) \rtimes \omega_0$ is irreducible.*

Proof. Using formulas (2.5) and (2.6), we have

$$s.s.r_{(1;1)}(\zeta(0,1,\chi_\psi\chi) \rtimes \omega_0) = \chi_\psi\nu^{-1}\chi \otimes \chi_\psi\chi \rtimes \omega_0 + \chi_\psi\chi \otimes \chi_\psi\nu^1\chi \rtimes \omega_0 \tag{A.1}$$

$$s.s.r_{(1;1)}(\delta(0,1,\chi_\psi\chi) \rtimes \omega_0) = \chi_\psi\nu^1\chi \otimes \chi_\psi\chi \rtimes \omega_0 + \chi_\psi\chi \otimes \chi_\psi\nu^1\chi \rtimes \omega_0 \tag{A.2}$$

all summands being irreducible by Theorem A.1. Since $\delta(0,1,\chi_\psi\chi) \rtimes \omega_0$ is irreducible by Lemma A.4, we see that it is not isomorphic to any irreducible subquotient of $\zeta(0,1,\chi_\psi\chi) \rtimes \omega_0$. Now observe

$$\zeta(0,1,\chi_\psi\chi) \rtimes \omega_0 \hookrightarrow \chi_\psi\chi \times \chi_\psi\nu^1\chi \rtimes \omega_0 \cong \chi_\psi\chi \times \chi_\psi\nu^{-1}\chi \rtimes \omega_0 \to \chi_\psi\nu^{-1}\chi \times \chi_\psi\chi \rtimes \omega_0$$

where the kernel of the last map, induced from $\chi_\psi\chi \times \chi_\psi\nu^{-1}\chi \to \chi_\psi\nu^{-1}\chi \times \chi_\psi\chi$, is $\delta(-1,0,\chi_\psi\chi) \rtimes \omega_0 \cong \delta(0,1,\chi_\psi\chi) \rtimes \omega_0$. Thus,

$$\zeta(0,1,\chi_\psi\chi) \rtimes \omega_0 \hookrightarrow \chi_\psi\nu^{-1}\chi \times \chi_\psi\chi \rtimes \omega_0$$

Let π be the unique irreducible subrepresentation of $\zeta(0,1,\chi_\psi\chi) \rtimes \omega_0$. Frobenius reciprocity implies that $s.s.r_{(1;1)}(\pi) \ge \chi_\psi\nu^{-1}\chi \otimes \chi_\psi\chi \rtimes \omega_0$. Also, from $\zeta(0,1,\chi_\psi\chi) \rtimes \omega_0 \hookrightarrow \chi_\psi\chi \times \chi_\psi\nu^1\chi \rtimes \omega_0$, we get $s.s.r_{(1;1)}(\pi) \ge \chi_\psi\chi \otimes \chi_\psi\nu^1\chi \rtimes \omega_0$. Thus,

$$s.s.r_{(1;1)}(\pi) \ge \chi_\psi\nu^{-1}\chi \otimes \chi_\psi\chi \rtimes \omega_0 + \chi_\psi\chi \otimes \chi_\psi\nu^1\chi \rtimes \omega_0$$

Comparing to (A.1), we see that $\pi = \zeta(0,1,\chi_\psi\chi) \rtimes \omega_0$, showing irreducibility. □

Theorem A.6. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta + 1 \in \mathbb{Z}_{>0}$, and let χ be a unitary unramified character of $F^\times$. Suppose $\chi^2 \ne 1_{F^\times}$ or $-\beta \notin 1/2 - \mathbb{Z}_{\ge 0}$ or $\alpha \notin -1/2 + \mathbb{Z}_{\ge 0}$. Then the representation $\zeta(-\beta,\alpha,\chi_\psi\chi) \rtimes \omega_0$ is irreducible.*

Proof. We prove the two cases of the theorem separately. The third case, $\alpha \notin -1/2 + \mathbb{Z}_{\ge 0}$, follows from Case 2 below using relation (2.7).

Case 1: $\chi^2 \ne 1_{F^\times}$.

Let β be a real number. We show, by induction on $n \in \mathbb{Z}_{\ge 0}$, irreducibility of the representation $\sigma := \zeta(-\beta,-\beta+n,\chi_\psi\chi) \rtimes \omega_0$. Theorem A.1 provides the induction basis, so assume $n \ge 1$. Using (2.5), we have

$$\mu^*(\sigma) = \sum_{i=0}^{n+1}\sum_{j=0}^{i} \zeta(\beta-n,\beta-i,\chi_\psi\chi^{-1}) \times \zeta(-\beta,j-\beta-1,\chi_\psi\chi) \otimes \zeta(j-\beta,i-\beta-1,\chi_\psi\chi) \rtimes \omega_0.$$

Let M', M'', M''' be Levi subgroups of $Sp(n+1,F)$ that correspond to $(1,\dots,1;0)$, $(n+1;0)$, $(1;n)$, resp.

$$\begin{aligned} &\text{s.s.}(r^{\widetilde{Sp(n+1,F)}}_{\widetilde{M'''}}(\sigma)) = \\ &\nu^{\beta-n}\chi_\psi\chi^{-1} \otimes \zeta(-\beta,-\beta+n-1,\chi_\psi\chi) \rtimes \omega_0 + \nu^{-\beta}\chi_\psi\chi \otimes \zeta(-\beta+1,-\beta+n,\chi_\psi\chi) \rtimes \omega_0. \end{aligned} \tag{A.3}$$

$$\text{s.s.}(r^{\widetilde{Sp(n+1,F)}}_{\widetilde{M''}}(\sigma)) = \sum_{i=0}^{n+1} \zeta(\beta-n,\beta-i,\chi_\psi\chi^{-1}) \times \zeta(-\beta,i-\beta-1,\chi_\psi\chi) \otimes \omega_0. \tag{A.4}$$

Take $\tau'' = \nu^{\beta-n}\chi_\psi\chi^{-1} \times \zeta(-\beta, -\beta+n-1, \chi_\psi\chi) \otimes \omega_0$, which is the irreducible summand on the right hand side of (A.4) for $i = n$. We want to apply Lemma A.2. By induction hypothesis, summands on the right hand side of (A.3) are irreducible. First take $\tau''' = \nu^{\beta-n}\chi_\psi\chi^{-1} \otimes \zeta(-\beta, -\beta+n-1, \chi_\psi\chi) \rtimes \omega_0$. Assume that

$$(1 \otimes \text{s.s.}r_{(1,\dots,1;0)})(\tau''') + (\text{s.s.}r_{(1,\dots,1)} \otimes 1)(\tau'') \leq \text{s.s.}\left(r_{\widetilde{M'}}^{\widetilde{Sp(n+1,F)}}(\sigma)\right). \tag{A.5}$$

From (A.3) we have $(\text{s.s.}r_{(1,\dots,1)}\otimes 1)(\tau'') \leq \nu^{-\beta}\chi_\psi\chi \otimes \text{s.s.}(r_{(1,\dots,1)}(\zeta(-\beta+1, -\beta+n, \chi_\psi\chi) \rtimes \omega_0)$, a contradiction. Take now $\tau''' = \nu^{-\beta}\chi_\psi\chi \otimes \zeta(-\beta+1, -\beta+n, \chi_\psi\chi) \rtimes \omega_0$. If (A.5) was valid, then (A.3) would imply $(\text{s.s.}r_{(1,\dots,1)} \otimes 1)(\tau'') \leq \nu^{\beta-n}\chi_\psi\chi^{-1} \otimes \text{s.s.}\left(r_{(1,\dots,1;0)}(\zeta(-\beta, -\beta+n-1, \chi_\psi\chi) \rtimes \omega_0)\right)$. Again contradiction, and Lemma A.2 shows that σ is irreducible.

Case 2: $\chi^2 = 1_{F^\times}$ and $-\beta \notin 1/2 - \mathbb{Z}_{\geq 0}$.

First we prove irreducibility of σ in the special case $\sigma := \zeta(-n, n, \chi_\psi\chi) \rtimes \omega_0$ for $n \in \mathbb{Z}_{\geq 0}$. Theorem A.1 covers the case $n = 0$. Let $n \geq 1$. Using (2.5), we have

$$\mu^*(\sigma) = \sum_{i=0}^{2n+1}\sum_{j=0}^{i} \zeta(-n, n-i, \chi_\psi) \times \zeta(-n, -n+j-1, \chi_\psi\chi) \otimes \zeta(j-n, -n+i-1, \chi_\psi\chi) \rtimes \omega_0.$$

$$\begin{aligned} &\text{s.s.}(r_{(2n;1)}(\sigma)) = \zeta(-n, -1, \chi_\psi\chi) \times \zeta(-n, -1, \chi_\psi\chi) \otimes \chi_\psi\chi \rtimes \omega_0 + \\ &2\sum_{k=1}^{n} \zeta(-n, -k-1, \chi_\psi\chi) \times \zeta(-n, k-1, \chi_\psi\chi) \otimes \nu^k\chi_\psi\chi \rtimes \omega_0. \end{aligned} \tag{A.6}$$

All summands in (A.6) are irreducible. Let $\tau'' = \zeta(-n, -1, \chi_\psi\chi) \times \zeta(-n, -1, \chi_\psi\chi) \otimes \chi_\psi\chi \rtimes \omega_0$. It appears with multiplicity one in (A.6). Moreover,

$$\text{s.s.}(r_{(2n+1;0)}(\sigma)) = 2\sum_{i=0}^{n} \zeta(-n, n-i, \chi_\psi\chi) \times \zeta(-n, -n+i-1, \chi_\psi\chi) \otimes \omega_0, \tag{A.7}$$

and all summands in (A.7) are irreducible. Let $\tau_0 = \zeta(-n, 0, \chi_\psi\chi) \times \zeta(-n, -1, \chi_\psi\chi) \otimes \omega_0$. Since $\nu^n\chi_\psi\chi \rtimes \omega_0$ is irreducible, and for $i = 1, \dots, n-1$ and $n \geq 2$, $\nu^i\chi_\psi\chi \times \nu^{-n}\chi_\psi\chi$ is irreducible, we have $\nu^n\chi_\psi\chi \rtimes \omega_0 \cong \nu^{-n}\chi_\psi\chi \rtimes \omega_0$ and $\nu^i\chi_\psi\chi \times \nu^{-n}\chi_\psi\chi \cong \nu^{-n}\chi_\psi\chi \times \nu^i\chi_\psi\chi$ ($n \geq 2$). So for $n \geq 2$

$$\begin{aligned} &\zeta(-n, n, \chi_\psi\chi) \rtimes \omega_0 \hookrightarrow \nu^{-n}\chi_\psi\chi \times \nu^{-n+1}\chi_\psi\chi \times \cdots \times \nu^{n-1}\chi_\psi\chi \times \nu^n\chi_\psi\chi \rtimes \omega_0 \\ &\cong \nu^{-n}\chi_\psi\chi \times \nu^{-n+1}\chi_\psi\chi \times \cdots \times \nu^{n-1}\chi_\psi\chi \times \nu^{-n}\chi_\psi\chi \rtimes \omega_0 \\ &\cong \nu^{-n}\chi_\psi\chi \times \cdots \times \nu^{-1}\chi_\psi\chi \times \chi_\psi\chi \times \nu^{-n}\chi_\psi\chi \times \nu\chi_\psi\chi \times \cdots \times \nu^{n-1}\chi_\psi\chi \rtimes \omega_0 \\ &\dots \\ &\cong \nu^{-n}\chi_\psi\chi \times \cdots \times \nu^{-1}\chi_\psi\chi \times \chi_\psi\chi \times \nu^{-n}\chi_\psi\chi \times \nu^{-n+1}\chi_\psi\chi \times \cdots \times \nu^{-1}\chi_\psi\chi \rtimes \omega_0. \end{aligned}$$

Thus, for $n \geq 1$

$$\begin{aligned} &\zeta(-n, n, \chi_\psi\chi) \rtimes \omega_0 \hookrightarrow \\ &\nu^{-n}\chi_\psi\chi \times \cdots \times \nu^{-1}\chi_\psi\chi \times \chi_\psi\chi \times \nu^{-n}\chi_\psi\chi \times \nu^{-n+1}\chi_\psi\chi \times \cdots \times \nu^{-1}\chi_\psi\chi \rtimes \omega_0. \end{aligned} \tag{A.8}$$

Let $\sigma' = \nu^{-n}\chi_\psi\chi \otimes \cdots \otimes \nu^{-1}\chi_\psi\chi \otimes \chi_\psi\chi \otimes \nu^{-n}\chi_\psi\chi \otimes \nu^{-n+1}\chi_\psi\chi \otimes \cdots \otimes \nu^{-1}\chi_\psi\chi \otimes \omega_0$. We use Lemma A.3 to prove irreducibility of $\sigma = \zeta(-n, n, \chi_\psi\chi) \rtimes \omega_0$. Let $\sigma_0 = \zeta(-n, n, \chi_\psi\chi) \otimes \omega_0$. From (A.8), we have condition (i) of Lemma A.3 For (ii), note that σ' is not a subquotient of

$$r_{(1,\dots,1)}(\zeta(-n, n-i, \chi_\psi\chi) \times \zeta(-n, -n+i-1, \chi_\psi\chi)) \otimes \omega_0,\ i = 0, \dots, n-1,$$

because every irreducible subquotient of $\zeta(-n, n-i, \chi_\psi\chi) \times \zeta(-n, -n+i-1, \chi_\psi\chi)$ has $\nu\chi\chi_\psi$ in its cuspidal support. Now, from (A.7), we have condition (ii) of Lemma A.3, for τ_0 defined after formula (A.7). That leaves us with condition (iii). Since

$$\tau_0 = \zeta(-n, 0, \chi_\psi\chi) \times \zeta(-n, -1, \chi_\psi\chi) \otimes \omega_0 \cong \zeta(-n, -1, \chi_\psi\chi) \times \zeta(-n, 0, \chi_\psi\chi) \otimes \omega_0,$$

we have $\tau_0 \hookrightarrow \nu^{-n}\chi_\psi\chi \times \cdots \times \nu^{-1}\chi_\psi\chi \times \nu^{-n}\chi_\psi\chi \times \cdots \times \chi_\psi\chi \otimes \omega_0$. Let

$$\rho' = \nu^{-n}\chi_\psi\chi \otimes \cdots \otimes \nu^{-1}\chi_\psi\chi \otimes \nu^{-n}\chi_\psi\chi \otimes \cdots \otimes \chi_\psi\chi \otimes \omega_0 \in s.s.(r_{(1,\ldots,1;0)}(\tau_0)).$$

From (A.6), we see that it is enough to show that ρ' is not a subquotient of

$$\zeta(-n,-k-1,\chi_\psi\chi) \times \zeta(-n,k-1,\chi_\psi\chi) \otimes \nu^k\chi_\psi\chi \rtimes \omega_0,\ 1 \leq k \leq n.$$

That is clear, because $r_{(1;0)}(\nu^k\chi_\psi\chi \rtimes \omega_0) = \nu^{-k}\chi_\psi\chi \otimes \omega_0 + \nu^k\chi_\psi\chi \otimes \omega_0,\ 1 \leq k \leq n$. So, by Lemma A.3, representation $\sigma = \zeta(-n,n,\chi_\psi\chi) \rtimes \omega_0$ is irreducible.

We consider now the general case $\sigma := \zeta(-\beta,-\beta+n,\chi_\psi\chi) \rtimes \omega_0$ for $-\beta \notin 1/2 - \mathbb{Z}_{\geq 0}$. As in Case 1, we show irreducibility of σ by induction on $n \in \mathbb{Z}_{\geq 0}$. Because of Theorem A.1, we may assume $n \geq 1$. Using (2.5), we obtain

$$\mu^*(\sigma) = \sum_{i=0}^{n+1}\sum_{j=0}^{i} \zeta(\beta-n,\beta-i,\chi_\psi\chi) \times \zeta(-\beta,j-\beta-1,\chi_\psi\chi) \otimes \zeta(j-\beta,i-\beta-1,\chi_\psi\chi) \rtimes \omega_0.$$

Let M', M'', M''' be Levi subgroups of $Sp(n+1,F)$ that correspond to $(1,\ldots,1;0)$, $(n+1;0)$, $(1;n)$, resp.

$$\begin{aligned}&\text{s.s.}(r_{\widetilde{M'''}}^{\widetilde{Sp(n,F)}}(\sigma)) = \\ &\nu^{\beta-n}\chi_\psi\chi \otimes \zeta(-\beta,-\beta+n-1,\chi_\psi\chi) \rtimes \omega_0 + \nu^{-\beta}\chi_\psi\chi \otimes \zeta(-\beta+1,-\beta+n,\chi_\psi\chi) \rtimes \omega_0\end{aligned} \tag{A.9}$$

$$\text{s.s.}(r_{\widetilde{M''}}^{\widetilde{Sp(n,F)}}(\sigma)) = \sum_{i=0}^{n+1} \zeta(\beta-n,\beta-i,\chi_\psi\chi) \times \zeta(-\beta,i-\beta-1,\chi_\psi\chi) \otimes \omega_0 \tag{A.10}$$

By the induction hypothesis, all summands in (A.9) are irreducible. We have two cases:

a) $\beta \neq 0$. Because of (2.7), taking contragredient if necessary, we may assume $(-\beta+(-\beta+n))/2 > 0$, as it is equal to zero only for $\beta \in \mathbb{Z}_{>0}$ and $n = 2\beta$, which is settled above. Thus, the $i=1$ summand $\tau'' = \zeta(\beta-n,\beta-1,\chi_\psi\chi) \times \nu^{-\beta} \otimes \omega_0$ in (A.10) is irreducible. We use Lemma A.2. First, let

$$\tau''' = \nu^{\beta-n}\chi_\psi\chi \otimes \zeta(-\beta,-\beta+n-1,\chi_\psi\chi) \rtimes \omega_0 \text{ and assume}$$

$$\text{s.s.}\left(r_{\widetilde{M'}}^{\widetilde{Sp(n,F)}}(\sigma)\right) \geq (1 \otimes \text{s.s.}r_{(1,\ldots,1;0)})(\tau''') + (\text{s.s.}r_{(1,\ldots,1)} \otimes 1)(\tau''). \tag{A.11}$$

From (A.9), we have a contradiction

$$(\text{s.s.}r_{(1,\ldots,1)} \otimes 1)(\tau'') \leq \nu^{-\beta}\chi_\psi\chi \otimes \text{s.s.}\left(r_{(1,\ldots,1)}(\zeta(-\beta+1,-\beta+n,\chi_\psi\chi) \rtimes \omega_0)\right).$$

Take now $\tau''' = \nu^{-\beta}\chi_\psi\chi \otimes \zeta(-\beta+1,-\beta+n,\chi_\psi\chi) \rtimes \omega_0$ and assume (A.11). Then (A.9) implies

$$(\text{s.s.}r_{(1,\ldots,1)} \otimes 1)(\tau'') \leq \nu^{\beta-n}\chi_\psi\chi \otimes \text{s.s.}\left(r_{(1,\ldots,1;0)}(\zeta(-\beta,-\beta+n-1,\chi_\psi\chi) \rtimes \omega_0)\right),$$

a contradiction. So by Lemma A.2, $\sigma = \zeta(-\beta,-\beta+n,\chi_\psi\chi) \rtimes \omega_0$ is irreducible.

b) $\beta = 0$. Since we already proved in Lemma A.5 that $\zeta(0,1,\chi_\psi\chi) \rtimes \omega_0$ is irreducible, we assume $n \geq 2$. Take the $i=2$ summand

$$\tau'' = \zeta(-n,-2,\chi_\psi\chi) \times \zeta(0,1,\chi_\psi\chi) \otimes \omega_0,$$

in (A.10). It is irreducible. Now (A.9) becomes

$$\text{s.s.}(r_{\widetilde{M'''}}^{\widetilde{Sp(n,F)}}(\sigma)) = \nu^{-n}\chi_\psi\chi \otimes \zeta(0,n-1,\chi_\psi\chi) \rtimes \omega_0 + \chi_\psi\chi \otimes \zeta(1,n,\chi_\psi\chi) \rtimes \omega_0. \tag{A.12}$$

We want to use Lemma A.2. First, let $\tau''' = \nu^{-n}\chi_\psi\chi \otimes \zeta(0,n-1,\chi_\psi\chi) \rtimes \omega_0$. Assume that

$$(1 \otimes \text{s.s.}r_{(1,\ldots,1;0)})(\tau''') + (\text{s.s.}r_{(1,\ldots,1)} \otimes 1)(\tau'') \leq \text{s.s.}\left(r_{\widetilde{M'}}^{\widetilde{Sp(n,F)}}(\zeta(0,n,\chi_\psi\chi) \rtimes \omega_0)\right). \tag{A.13}$$

But (A.2) implies a contradiction $(\mathrm{s.s.}r_{(1,\ldots,1)}\otimes 1)(\tau'')\leq \chi_\psi\chi\otimes \mathrm{s.s.}r_{(1,\ldots,1;0)}(\zeta(1,n,\chi_\psi\chi)\rtimes\omega_0)$. Now take $\tau'''=\chi_\psi\chi\otimes\zeta(1,n,\chi_\psi\chi)\rtimes\omega_0$ and assume (A.13). Relation (A.2) implies

$$(\mathrm{s.s.}r_{(1,\ldots,1)}\otimes 1)(\tau'')\leq \nu^{-n}\chi_\psi\chi\otimes \mathrm{s.s.}r_{(1,\ldots,1;0)}(\zeta(0,n-1,\chi_\psi\chi)\rtimes\omega_0),$$

a contradiction. So $\zeta(-\beta,-\beta+n,\chi_\psi\chi)\rtimes\omega_0$ is irreducible, by Lemma A.2.

□

A.4. **The complete reducibility criterion.** Now we finally prove the main theorem of this section.

Theorem A.7. *Let $\alpha,\beta\in\mathbb{R}$ be such that $\alpha+\beta+1\in\mathbb{Z}_{>0}$, and let χ be a unitary unramified character of $F^\times$. Representation $\zeta(-\beta,\alpha,\chi_\psi\chi)\rtimes\omega_0$ reduces if and only if $\chi_\psi\chi\nu^i\rtimes\omega_0$ reduces for some i such that $-\beta\leq i\leq\alpha$ and $\alpha-i\in\mathbb{Z}$.*

Proof. From Theorem A.1, we know that Theorem A.6 covers all the cases in which $\chi_\psi\chi\nu^i\rtimes\omega_0$ is irreducible for all i such that $-\beta\leq i\leq\alpha$ and $\alpha-i\in\mathbb{Z}$. It remains to check the cases in which at least one of these $\chi_\psi\chi\nu^i\rtimes\omega_0$ reduces. That is, $\chi^2=1_{F^\times}$ and $-\beta\in 1/2-\mathbb{Z}_{\geq 0}$ and $\alpha\in -1/2+\mathbb{Z}_{\geq 0}$. In other words, we must show that $\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\rtimes\omega_0$ reduces for $k,l\in\mathbb{Z}_{\geq 0}$.

Because of (2.7) and Lemma 4.1, we may assume $|\frac{1}{2}-k|\leq\frac{1}{2}+l$ and $k\geq 1$, resp. Now

$$\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\rtimes\omega_0\hookrightarrow \nu^{\frac{1}{2}-k}\chi_\psi\chi\times\cdots\times\nu^{-\frac{1}{2}}\chi_\psi\chi\times\nu^{\frac{1}{2}}\chi_\psi\chi\times\cdots\times\nu^{\frac{1}{2}+l}\chi_\psi\chi\rtimes\omega_0.$$

Let $\sigma_{\frac{1}{2}+l}$ be the unramified irreducible subquotient of $\zeta(\frac{1}{2},\frac{1}{2}+l,\chi_\psi\chi)\rtimes\omega_0$, as in Lemma 4.1. Suppose that $\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\rtimes\omega_0$ is irreducible. Then, by Lemma 3.4,

$$\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\rtimes\omega_0\hookrightarrow\zeta(\frac{1}{2}-k,-\frac{1}{2},\chi_\psi\chi)\rtimes\sigma_{\frac{1}{2}+l}. \tag{A.14}$$

We will get a contradiction by proving that (A.14) does not hold. Using (2.5), we have

$$\begin{aligned}\mu^*(\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\rtimes\omega_0)=&\sum_{i=0}^{l+k+1}\sum_{j=0}^{i}\zeta(-\frac{1}{2}-l,k-\frac{1}{2}-i,\chi_\psi\chi)\times\zeta(\frac{1}{2}-k,\frac{1}{2}-k+j-1,\chi_\psi\chi)\\&\otimes\zeta(j+\frac{1}{2}-k,i+\frac{1}{2}-k-1,\chi_\psi\chi)\rtimes\omega_0,\end{aligned}\tag{A.15}$$

$$\begin{aligned}\mathrm{s.s.}\left(r_{(l+k+1;0)}(\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\rtimes\omega_0)\right)=&\sum_{i=0}^{l+k+1}\zeta(-\frac{1}{2}-l,k-\frac{1}{2}-i,\chi_\psi\chi)\times\\&\times\zeta(\frac{1}{2}-k,-\frac{1}{2}-k+i,\chi_\psi\chi)\otimes\omega_0.\end{aligned}\tag{A.16}$$

By Lemma 4.1 and (2.5), we have:

$$\begin{aligned}\mu^*(\zeta(\frac{1}{2}-k,-\frac{1}{2},\chi_\psi\chi)\rtimes\sigma_{\frac{1}{2}+l})=&\sum_{s=0}^{l+1}\sum_{i=0}^{k}\sum_{j=0}^{i}\zeta(\frac{1}{2},-\frac{1}{2}+k-i,\chi_\psi\chi)\times\zeta(\frac{1}{2}-k,-k-\frac{1}{2}+j,\chi_\psi\chi)\times\\&\times\zeta(-\frac{1}{2}-l,-\frac{1}{2}-s,\chi_\psi\chi)\otimes\zeta(j+\frac{1}{2}-k,i-\frac{1}{2}-k,\chi_\psi\chi)\rtimes\sigma_{s-\frac{1}{2}},\end{aligned}\tag{A.17}$$

$$\begin{aligned}\mathrm{s.s.}\left(r_{(k+l+1;0)}(\zeta(\frac{1}{2}-k,-\frac{1}{2},\chi_\psi\chi)\rtimes\sigma_{\frac{1}{2}+l})\right)=&\sum_{i=0}^{k}\zeta(\frac{1}{2},-\frac{1}{2}+k-i,\chi_\psi\chi)\times\\&\times\zeta(\frac{1}{2}-k,-k-\frac{1}{2}+i,\chi_\psi\chi)\times\zeta(-\frac{1}{2}-l,-\frac{1}{2},\chi_\psi\chi)\otimes\omega_0.\end{aligned}\tag{A.18}$$

Now we have two possibilities. If $-\frac{1}{2}-l<\frac{1}{2}-k$, (A.18) implies that $\mu^*(\zeta(\frac{1}{2}-k,-\frac{1}{2},\chi_\psi\chi)\rtimes\sigma_{\frac{1}{2}+l})$ does not contain $\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\otimes\omega_0$. Since $\mu^*(\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\rtimes\omega_0)$ contains $\zeta(\frac{1}{2}-k,\frac{1}{2}+l,\chi_\psi\chi)\otimes\omega_0$, we have a contradiction with (A.14). Else, if $-\frac{1}{2} - l = \frac{1}{2} - k$, (A.18) implies that multiplicity of $\zeta(\frac{1}{2} - k, \frac{1}{2} + l, \chi_\psi \chi) \otimes \omega_0$ in $\mu^*(\zeta(\frac{1}{2} - k, -\frac{1}{2}, \chi_\psi \chi) \rtimes \sigma_{\frac{1}{2}+l})$ is one. Namely, $\zeta(\frac{1}{2} - k, -k - \frac{1}{2} + i, \chi_\psi \chi)$ must be $\chi_\psi \mathbf{1}$ ($i = 0$), because $\zeta(\frac{1}{2} - k, \frac{1}{2} + l, \chi_\psi \chi)$ in its cuspidal support has no repetition of $\nu^{\frac{1}{2}-k}\chi_\psi \chi$. But multipicity of $\zeta(\frac{1}{2} - k, \frac{1}{2} + l, \chi_\psi \chi) \otimes \omega_0$ in $\zeta(\frac{1}{2} - k, \frac{1}{2} + l, \chi_\psi \chi) \rtimes \omega_0$ is two, $i = 0$ and $i = k + l + 1$ in (A.16). Again we get contradiction with (A.14), so $\zeta(\frac{1}{2} - k, \frac{1}{2} + l, \chi_\psi \chi) \rtimes \omega_0$ reduces. $\square$

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