COMPOSITION SERIES OF A CLASS OF INDUCED REPRESENTATIONS, A CASE OF ONE HALF CUSPIDAL REDUCIBILITY

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Abstract. In this paper we determine the composition series of the induced representation $\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma$ where $a,b,c\in\frac{1}{2}(2\mathbb{Z}+1)$ such that $\frac{1}{2}\leq a<b<c$, ρ is an irreducible cuspidal unitary representation of a general linear group and σ is an irreducible cuspidal representation of a classical group.

Introduction

In this paper we determine the composition series of a class of standard representations in terms of Mœglin-Tadić classification of discrete series ([4],[5]). Interesting on its own, this result should also prove valuable for extending results about Jacquet modules of segment type representations obtained in [3].

To describe our results we introduce some notation. Fix a local non-archimedean field F of characteristic different than two. Let ρ be an irreducible cuspidal unitary representation of $GL(m_\rho,F)$ (this defines m_ρ) and $x,y\in\mathbb{R}$, such that $y-x+1\in\mathbb{Z}_{\geq 0}$. The set $[\nu^x\rho,\nu^y\rho]=\{\nu^x\rho,...,\nu^y\rho\}$ is called segment. The parabolically induced representation $\nu^y\rho\times\cdots\times\nu^x\rho$ has a unique irreducible subrepresentation, it is essentially square integrable and we denote it by $\delta([\nu^x\rho,\nu^y\rho])$. Also we denote $e([\nu^x\rho,\nu^y\rho])=e(\delta([\nu^x\rho,\nu^y\rho])=\frac{x+y}{2}$. If δ is an essentially square integrable representation of $GL(m_\delta,F)$, there exists a segment Δ such that $\delta=\delta(\Delta)$.

Let G_n be a symplectic or (full) orthogonal group having split rank n. Given a sequence of segments $\Delta_1,...,\Delta_k$, $e(\Delta_i)>0,\ i=1,...,k$ and an irreducible tempered representation τ of some $G_{n'}$ we denote by $Lang(\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\tau)$ the unique irreducible quotient, called the Langlands quotient, of parabolically induced representation $\delta(\Delta_{\varphi(1)})\times\cdots\times\delta(\Delta_{\varphi(k)})\rtimes\tau$ where φ is a permutation of the set $\{1,...,k\}$ such that $e(\Delta_{\varphi(1)})\geq\cdots\geq e(\Delta_{\varphi(k)})$. These induced representations are called standard representations and are important because by the Langlands classification every irreducible representation of G_n can be described as a Langlands quotient. Further if τ is a discrete series representation then by the Mœglin-Tadić classification of discrete series it is described by an admissible triple $(\mathrm{Jord},\tau_{cusp},\epsilon)$. Here Jord is a set Jordan blocks, τ_{cusp} a partial cuspidal support and ϵ a function from a subset of $\mathrm{Jord}\cup(\mathrm{Jord}\times\mathrm{Jord})$ into $\{\pm 1\}$. Results of Muić about reducibility of the generalized principal series $\delta([\nu^x\rho,\nu^y\rho])\rtimes\tau$ ([7],[6]) are stated case by case

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depending on Jord and x and y where the case $x=\frac{1}{2}$ plays an important role. In our situation, we provide some additional information, see Proposition 2.4. These results are used to compute composition series of the induced representation

$$\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma$$

where $a,b,c\in\frac{1}{2}(2\mathbb{Z}+1)$ such that $\frac{1}{2}\leq a<b<c$, ρ is an irreducible unitary cuspidal representation of $GL(m_\rho,F)$ and σ an irreducible cuspidal representation of G_n such that $\nu^{\frac{1}{2}}\rho\rtimes\sigma$ reduces.

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1. Preliminaries

Let F be a local non-archimedean field of characteristic different than two. Groups that we consider are as follows. As in [5] we fix a tower of symplectic or orthogonal non-degenerate F vector spaces V_n, $n\geq 0$ where n is the Witt index. We denote by G_n the group of isometries of V_n. It has split rank n. Also we fix the set of standard parabolic subgroups in the usual way. Standard parabolic proper subgroups of G_n are in bijection with the set of ordered partitions of positive integers $m\leq n$. Given positive integers $n_1,...,n_k$ such that $m=n_1+\cdots+n_k\leq n$ the corresponding standard parabolic subgroup P_s, $s=(n_1,...,n_k)$ has the Levi factor M_s isomorphic to

$$GL(n_1,F)\times\cdots\times GL(n_k,F)\times G_{n-m}.$$

Further, if δ_i is a smooth representation of $GL(n_i,F)$, $i=1,...,k$ and τ a smooth representation of G_{n-m}, denote by $\pi=\delta_1\otimes\cdots\otimes\delta_k\otimes\tau$ the representation of M_s and by

$$\delta_1\times\cdots\times\delta_k\rtimes\tau=\mathrm{Ind}_{M_s}^{G_n}(\pi)$$

the representation induced from π using normalized parabolic induction. If σ is a smooth representation of G_n we denote by $\mathrm{r}_s(\sigma)=\mathrm{r}_{M_s}(\sigma)=\mathrm{Jacq}_{M_s}^{G_n}(\sigma)$ the normalized Jacquet module of σ. We have the Frobenius reciprocity

$$\mathrm{Hom}_{G_n}(\sigma,\mathrm{Ind}_{M_s}^{G_n}(\pi))=\mathrm{Hom}_{M_s}(\mathrm{Jacq}_{M_s}^{G_n}(\sigma),\pi).$$

Let ρ be an an irreducible cuspidal unitary representation of $GL(m_\rho,F)$ (this defines m_ρ) and $x,y\in\mathbb{R}$, such that $y-x+1\in\mathbb{Z}_{\geq 0}$. The set $[\nu^x\rho,\nu^y\rho]=\{\nu^x\rho,...,\nu^y\rho\}$ is called segment. The induced representation $\nu^y\rho\times\cdots\times\nu^x\rho$ has the unique irreducible subrepresentation, it is essentially square integrable, and we denote it by $\delta([\nu^x\rho,\nu^y\rho])$. We also denote $e([\nu^x\rho,\nu^y\rho])=e(\delta([\nu^x\rho,\nu^y\rho])=\frac{x+y}{2}$. For $y-x+1\in\mathbb{Z}_{<0}$ define $[\nu^x\rho,\nu^y\rho]=\emptyset$ and $\delta(\emptyset)$ is the irreducible representation of the trivial group. Let $\Delta=[\nu^x\rho,\nu^y\rho]$ and $\widetilde{\Delta}=[\nu^{-y}\widetilde{\rho},\nu^{-x}\widetilde{\rho}]$ where $\widetilde{\rho}$ denotes the contragredient of ρ. We have $\delta(\Delta)\widetilde{\ }=\delta(\widetilde{\Delta})$. By [10] if δ is an essentially square integrable representation of $GL(m_\delta,F)$, there exists a segment Δ such that $\delta=\delta(\Delta)$. If Δ' and Δ'' are segments such that $\Delta''\subseteq\Delta'$ then $\delta(\Delta')\times\delta(\Delta'')$ is irreducible and $\delta(\Delta')\times\delta(\Delta'')\cong\delta(\Delta'')\times\delta(\Delta')$.

Given a sequence of segments $\Delta_1,...,\Delta_k$, $e(\Delta_i)>0$, $i=1,...,k$ and an irreducible tempered representation τ of some $G_{n'}$ we denote by $Lang(\delta(\Delta_1)\times$

$\cdots \times \delta(\Delta_k) \rtimes \tau)$ the unique irreducible quotient, called the Langlands quotient, of $\delta(\Delta_{\varphi(1)}) \times \cdots \times \delta(\Delta_{\varphi(k)}) \rtimes \tau$ where φ is a permutation of the set $\{1, ..., k\}$ such that $e(\Delta_{\varphi(1)}) \geq \cdots \geq e(\Delta_{\varphi(k)})$. It appears with multiplicity one in the induced representation and is the unique irreducible subrepresentation of $\delta(\widetilde{\Delta}_{\varphi(1)}) \times \cdots \times \delta(\widetilde{\Delta}_{\varphi(k)}) \rtimes \tau$. By the Langlands classification every irreducible representation of G_n can be written as a Langlands quotient.

If σ is a discrete series representation of G_n then by the Mœglin-Tadić classification of discrete series ([4],[5]) it is described by an admissible triple $(\mathrm{Jord}, \sigma_{cusp}, \epsilon)$. We note that the classification, written under a natural hypothesis, is now unconditional, see page 3160 of [2]. Here Jord is a set of pairs (a, ρ) where ρ is an irreducible self-dual cuspidal representation of $GL(m_\rho, F)$, a is a positive integer of parity depending on ρ and $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$ is irreducible. We write $\mathrm{Jord}_\rho = \{a : (a, \rho) \in \mathrm{Jord}\}$ and for $a \in \mathrm{Jord}_\rho$ let a_- be the largest element of Jord_ρ strictly less than a, if such exists. Next, σ_{cusp} is the unique irreducible cuspidal representation of some $G_{n'}$ such that there exists an irreducible representation π of $GL(m_\pi, F)$ such that $\sigma \hookrightarrow \pi \rtimes \sigma_{cusp}$. It is called the partial cuspidal support of σ. Finally, ϵ is a function from a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ into $\{\pm 1\}$. It is defined on a pair $(a, \rho), (a', \rho') \in \mathrm{Jord}$ if and only if $\rho \cong \rho'$ and $a \neq a'$. In such case we formally denote the value on the pair by $\epsilon(a, \rho)\epsilon(a', \rho)^{-1}$ and it is equal to the product of $\epsilon(a, \rho)$ and $\epsilon(a', \rho)^{-1}$ it they are defined. Suppose that $(a, \rho) \in \mathrm{Jord}$ and a_- is defined. Then

$$\epsilon(a, \rho)\epsilon(a_-, \rho)^{-1} = 1 \Leftrightarrow \text{there exists a representation } \pi' \text{of some } G_{n_{\pi'}}$$
$$\text{such that } \sigma \hookrightarrow \delta([\nu^{(a_-+1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'.$$

If $(a, \rho) \in \mathrm{Jord}$ and a is even then $\epsilon(a, \rho)$ is defined. Additionally, if $a = \min(\mathrm{Jord}_\rho)$ then

$$\epsilon(a, \rho) = 1 \Leftrightarrow \text{there exists a representation } \pi'' \text{of some } G_{n_{\pi''}}$$
$$\text{such that } \sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi''.$$

Now we recall the Tadić formula for computing Jacquet modules. Let $R(G_n)$ be the Grothendieck group of the category of smooth representations of G_n of finite length. It is the free Abelian group generated by classes of irreducible representations of G_n. If σ is a smooth finite length representation of G_n denote by s.s.(σ) the semisimplification of σ, that is the sum of classes of composition series of σ. Put $R(G) = \oplus_{n \geq 0} R(G_n)$. For $\pi_1, \pi_2 \in R(G)$ we define $\pi_1 \leq \pi_2$ if $\pi_2 - \pi_1$ is a linear combination of classes of irreducible representations with non-negative coefficients. Similarly we have $R(GL) = \oplus_{n \geq 0} R(GL(n, F))$. We have the map $\mu^* : R(G) \to R(GL) \otimes R(G)$ defined by

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} \mathrm{s.s.}(r_{(k)}(\sigma)),\ \sigma \in R(G_n).$$

The following result derives from Theorems 5.4 and 6.5 of [9], see also section 1. in [5]. They are based on Geometrical Lemma (2.11 of [1]).

Theorem 1.1. *Let σ be a smooth representation of a finite length of G_n, ρ an irreducible unitary cuspidal representation of $GL(m_\rho, F)$ and $x, y \in \mathbb{R}$, such that*

$y-x+1\in\mathbb{Z}_{\geq 0}$. *Then*

$$\mu^*(\delta([\nu^x\rho,\nu^y\rho])\rtimes\sigma)=\sum_{\delta'\otimes\sigma'\leq\mu^*(\sigma)}\sum_{i=0}^{y-x+1}\sum_{j=0}^{i}\delta([\nu^{i-y}\widetilde{\rho},\nu^{-x}\widetilde{\rho}])\times\delta([\nu^{y+1-j}\rho,\nu^y\rho])\times\delta'\otimes\delta([\nu^{y+1-i}\rho,\nu^{y-j}\rho])\rtimes\sigma' \tag{1.1}$$

where $\delta'\otimes\sigma'$ *denotes an irreducible subquotient in the appropriate Jacquet module.*

We also note that in the apropriate Grothendieck group

$$\delta([\nu^x\rho,\nu^y\rho])\rtimes\sigma=\delta([\nu^{-y}\widetilde{\rho},\nu^{-x}\widetilde{\rho}])\rtimes\sigma. \tag{1.2}$$

2. Basic reducibilities

In this section we fix the notation and prepare some reducibility results. Let ρ be an irreducible unitary cuspidal representation of $GL(m_\rho,F)$ and σ an irreducible cuspidal representation of G_n such that $\nu^{\frac{1}{2}}\rho\rtimes\sigma$ reduces. By Proposition 2.4 of [8] ρ is self-dual. Let $a,b,c\in\frac{1}{2}(2\mathbb{Z}+1)$ such that $\frac{1}{2}\leq a<b<c$.

The following result is Theorem 2.3 from [6] proved using Jacquet module computation.

Theorem 2.1.

i) The induced representation $\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma$ *is of length two. Besides its Langlands quotient it has the unique irreducible subrepresentation, discrete series* σ_1. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma=\sigma_1+Lang(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma).$$

Here $Jord(\sigma_1)=\{(2a+1,\rho)\},\epsilon_{\sigma_1}(2a+1,\rho)=1$.

ii) The induced representation $\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma$ *is of length three. Besides its Langlands quotient it has two nonisomorphic irreducible subrepresentation* σ_2 *and* σ_3. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma=\sigma_2+\sigma_3+Lang(\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma).$$

Here $Jord(\sigma_2)=Jord(\sigma_3)=\{(2b+1,\rho),(2c+1,\rho)\}$,
$\epsilon_{\sigma_2}(2b+1,\rho)=\epsilon_{\sigma_2}(2c+1,\rho)=1,\epsilon_{\sigma_3}(2b+1,\rho)=\epsilon_{\sigma_3}(2c+1,\rho)=-1$.

The next proposition follows from Theorem 2.1 of [6].

Proposition 2.2. *The induced representation* $\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma_1$ *is of length three. Besides its Langlands quotient it has two nonisomorphic irreducible subrepresentations, discrete series* σ_4 *and* σ_5. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma_1=\sigma_4+\sigma_5+Lang(\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma_1).$$

Here $Jord(\sigma_4)=Jord(\sigma_5)=\{(2a+1,\rho),(2b+1,\rho),(2c+1,\rho)\}$,
$\epsilon_{\sigma_4}(2a+1,\rho)=\epsilon_{\sigma_4}(2b+1,\rho)=\epsilon_{\sigma_4}(2c+1,\rho)=1$,
$\epsilon_{\sigma_5}(2a+1,\rho)=1,\epsilon_{\sigma_5}(2b+1,\rho)=\epsilon_{\sigma_5}(2c+1,\rho)=-1$.

We have

Proposition 2.3. *The representation* $\delta([\nu^{-b}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma$ *has two irreducible subrepresentations* σ_4 *and* σ_5 *and they appear with multiplicity one.*

Proof. By Theorem 2.1 and Proposition 2.2 we have

$$\sigma_4 \oplus \sigma_5 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma.$$

To see that there are no other irreducible subrepresentations let

$$\pi \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$$

be an irreducible subrepresentation. Frobenius reciprocity implies $\mu^*(\pi) \geq \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma$. We show that $\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma$ appears with multiplicity two in $\mu^*(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)$. Looking for possible occurrences, formula (1.1) implies that there exist $i, j, k, l \in \mathbb{Z}$ such that $0 \leq l \leq k \leq a + \frac{1}{2}$, $0 \leq j \leq i \leq b + c + 1$ and

$$\begin{aligned} \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) &\leq \delta([\nu^{k-a}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{a+1-l}\rho, \nu^a\rho]) \\ &\quad \times \delta([\nu^{i-c}\rho, \nu^b\rho]) \times \delta([\nu^{c+1-j}\rho, \nu^c\rho]), \\ \sigma &\leq \delta([\nu^{a+1-k}\rho, \nu^{a-l}\rho]) \times \delta([\nu^{c+1-i}\rho, \nu^{c-j}\rho]) \rtimes \sigma. \end{aligned}$$

Comparing cuspidal support in the first equation equation we see $i - c = -b$ or $c + 1 - j = -b$. The second inequality implies $k = l$ and $i = j$. So we have $i = j = c - b$ or $i = j = c + b + 1$. Now $k = l = a + \frac{1}{2}$. This showed that there are at most two irreducible subrepresentations in $\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma$, so there are no others than σ_4 and σ_5. ☐

Now we prove

Proposition 2.4. *In the appropriate Grothendieck group we have*

$$\begin{aligned} \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 &= \sigma_4 + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2), \\ \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3 &= \sigma_5 + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3). \end{aligned}$$

Proof. By Lemma 6.1 of [7] the induced representations on the left side of equations reduce. The proof of that lemma claims that all irreducible subquotients of the induced representations other than belonging Langlands quotients are discrete series. The argument as in proof of Theorem 2.1 of [6] implies that they are all subrepresentations.

Let π_4 be a discrete series subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2$ and π_5 a discrete series subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3$. By Theorem 2.1 $\sigma_2 \oplus \sigma_3 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma$ so we have

$$\begin{aligned} \pi_4 \oplus \pi_5 &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3 \\ &\cong \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes (\sigma_2 \oplus \sigma_3) \\ &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma. \end{aligned} \tag{2.1}$$

By Proposition 2.3 π_4 and π_5 are not isomorphic and we have

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 = \pi_4 + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2), \tag{2.2}$$

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3 = \pi_5 + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3) \tag{2.3}$$

where $\{\pi_4, \pi_5\} = \{\sigma_4, \sigma_5\}$.

We now prove that $\pi_4 = \sigma_4$ and $\pi_5 = \sigma_5$. It is enough to see that $\epsilon_{\pi_4}(2a + 1, \rho)\epsilon_{\pi_4}(2b + 1, \rho)^{-1} = 1$. Since $\epsilon_{\sigma_2}(2b + 1, \rho) = 1$ and $\min(Jord_\rho(\sigma_2)) = 2b + 1 \in$

$2\mathbb{Z}$ there exists an irreducible representation τ of $G_{n+(c+\frac{1}{2})m_\rho}$ such that $\sigma_2 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \tau$. Now we have

$$\begin{aligned}\pi_4 \hookrightarrow &\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \tau \cong \\ &\delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \tau \hookrightarrow \\ &\delta([\nu^{a+1}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \tau.\end{aligned}$$

By Lemma 3.2 of [5] there exists an irreducible representation τ' of $G_{n+(2a+c+\frac{3}{2})m_\rho}$ such that

$$\pi_4 \hookrightarrow \delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \tau'.$$

Now $\epsilon_{\pi_4}(2a+1,\rho)\epsilon_{\pi_4}(2b+1,\rho)^{-1} = 1$ As we proved that $\pi_4 = \sigma_4$ and $\pi_5 = \sigma_5$ equations (2.2) and (2.3) give the claim of the proposition. □

3. The main theorem

Theorem 3.1. *The induced representation $\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$ is of length six, and it has two non-isomorphic irreducible subrepresentations. They are discrete series. In the appropriate Grothendieck group we have*

$$\begin{aligned}&\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma = \\ &\quad \sigma_4 + \sigma_5 + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3) \\ &\quad + Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1) + Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma).\end{aligned}$$

Moreover

$$\begin{aligned}&Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) \oplus Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3) \oplus Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1) \hookrightarrow \\ &\qquad (\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5).\end{aligned}$$

Proof. Suppose that $-b+c \geq \frac{1}{2}+a$. Otherwise we have similar proof. We look at the composition of some intertwining operators

$$\begin{aligned}\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma &\to \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma \\ &\to \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \sigma \\ &\to \delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma \\ &\to \delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma.\end{aligned}$$

Since $\frac{1}{2} \leq a < b < c$ the first and the third map are isomorphisms. By Theorem 2.1 the kernel of the second map is in the appropriate Grothendieck group $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 + \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3$. By Proposition 2.4 this equals to

$$\sigma_4 + \sigma_5 + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3).$$

By Theorem 2.1 the kernel of the last map is in the appropriate Grothendieck group $\delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \sigma_1 = \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1$ by (1.2), which is by the Proposition 2.2 equal to

$$\sigma_4 + \sigma_5 + Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1).$$

The image of the composition is

$$Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma).$$

We see that σ_4 and σ_5 appear in two kernels, but by Proposition 2.3 they appear with multiplicity one in the induced representation, so we proved the first formula of the theorem.

To prove the second formula of the theorem, observe that by Theorem 2.1 and Propositions 2.2 and 2.3 we have

$$\sigma_4 \oplus \sigma_5 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma \quad \text{and}$$

$$Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1) \hookrightarrow (\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5). \tag{3.1}$$

Additionally, Proposition 2.4 and (2.1) imply

$$\sigma_4 \oplus \sigma_5 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$$

and

$$\begin{aligned} &Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) \oplus Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3) \hookrightarrow \\ &\qquad (\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5) \end{aligned} \tag{3.2}$$

Now equations (3.1) and (3.2) prove the second formula of the theorem. □

4. Consequences

We have the following result

Corollary 4.1. *In the appropriate Grothendieck group we have*

$$\begin{aligned} &\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma) = \\ &\qquad Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1) + Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma), \\ &\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma) = Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma) \\ &\qquad + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) + Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3). \end{aligned}$$

Except $Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)$ *all irreducible subquotients of induced representations on the left hand side appear as subrepresentations.*

Proof. Using the exactness of the parabolic induction, Theorem 2.1, Proposition 2.4 and (2.1) and Theorem 3.1 we have

$$\begin{aligned} &\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes Lang(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma) \cong \\ &(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma)/(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes (\sigma_2 \oplus \sigma_3)) \cong \\ &(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3). \end{aligned}$$

Comparing this with the result of the main theorem gives the first formula of the corollary. Similarly, for the second formula use Proposition 2.2 and observe that

$$\begin{aligned} &\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes Lang(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma) \cong \\ &(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1). \end{aligned}$$

□

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