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ON STABILITY OF CONTROLLED SYSTEMS IN BANACH SPACES

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Abstract. In this paper we study stability properties for linear systems, the evolution of which can be described by a semigroup of class C_0 on a Banach space. Generalizations of a theorem of Datko and of Perron's criterion for linear controlled systems in Banach spaces are obtained.

1. Introduction

The aim of this paper is to study the stability properties for linear systems, the evolution of which can be described by a semigroup of class C_0 on a Banach space.

We define a new concept of internal stability ((p, q) stability) and give a sufficient condition for the exponential stability of a large class of such C_0 semigroups. We extend the bounded input, bounded output criteria of Perron for the case of a linear system

$$x(t, u) = \int_0^t T(t-s) Bu(s)\, ds,$$

where $T(t)$ is a C_0 semigroup on a Banach space X. A generalization of a well-known theorem of Lyapunov to linear controlled systems in Banach spaces is also obtained.

2. Stability of C_0 semigroups

Let X be a Banach space and let $T(t)$ be a C_0 (strongly continuous at the origin) semigroup of bounded operators on X.

Definition 2.1. The C_0 semigroup $T(t)$ is

(*i*) *exponentially stable* if there exist two positive numbers $N \geqslant 1$ and ν such that

$$\|T(t)\| \leqslant Ne^{-\nu t} \text{ for all } t \geqslant 0;$$

(*ii*) *stable* if there is $N > 0$ such that

$$\|T(t)\| \leqslant N \text{ for every } t \geqslant 0;$$

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(*iii*) *asymptotically stable* if

$$\lim_{t\to\infty} \|T(t)\| = 0;$$

(*iv*) L^p *stable* (where $1 \leqslant p < \infty$) if for each $x \in X$ there exists $N > 0$ such that

$$\int_0^\infty \|T(t)\,x\|^p\,dt \leqslant N\,\|x\|^p, \text{ for all } x \in X;$$

(*v*) (p, q) *stable* (where $1 \leqslant p, q \leqslant \infty$ if there exists $N > 0$ such that

$$\Big(\int_{t+\delta}^\infty \|T(s)\,x\|^q\,ds\Big)^{1/q} \leqslant N\delta^{\frac{1}{p}-2} \cdot \int_t^{t+\delta} \|T(s)\,x\|\,ds, \text{ if } q < \infty$$

and

$$\operatorname*{ess\,sup}_{s \geqslant t+\delta} \|T(s)\,x\| \leqslant N \cdot \delta^{\frac{1}{p}-2} \cdot \int_t^{t+\delta} \|T(s)\,x\|\,ds, \text{ if } q = \infty,$$

for all $t \geqslant 0$, $\delta > 0$ and $x \in X$.

LEMMA 2.1. *If $T(t)$ is a C_0-semigroup then there exist $M \geqslant 1$, $\omega > 0$ such that*

(*i*) $\|T(t)\| \leqslant Me^{\omega t}$ *for all* $t \geqslant 0$;

(*ii*) $\|T(t)\,x\| \leqslant Me^{\omega\delta}\,\|T(s)\,x\|$ *for all* $\delta > 0$ *and* $0 \leqslant s \leqslant t \leqslant s + \delta$;

(*iii*) $\delta\,\|T(t)\,x\| \leqslant Me^{\omega\delta} \cdot \int_{t-\delta}^{t} \|T(s)\,x\|\,ds$ *for any* $\delta > 0$ *and* $t \geqslant \delta$;

(*iv*) $\int_t^{t+\delta} \|T(s)\,x\|\,ds \leqslant M\,\delta\,e^{\omega\delta}\,\|T(t)\,x\|$ *for all* $\delta > 0$ *and* $t \geqslant 0$.

Proof. It is well known (see [1], pp. 165—166) that if

$$\omega > \overline{\lim_{t\to\infty}}\, \frac{\ln\|T(t)\|}{t} = \inf_{t>0} \frac{\ln\|T(t)\|}{t} = \omega_0 < \infty$$

then there exists $M \geqslant 1$ such that (*i*) holds.

The relations (*ii*) — (*iv*) follow immediately from (*i*) and the semigroup property.

THEOREM 2.2. *Let $T(t)$ be a C_0 semigroup on the Banach space X. Then the following statements are equivalent*:

(*i*) $T(t)$ *is exponentially stable*;

(*ii*) $T(t)$ *is asymptotically stable*;

(*iii*) $T(t)$ *is* L^p *stable*;

(*iv*) *there exists* $N > 0$ *such that*

$$t\,\|T(t)\| \leqslant N \text{ for every } t \geqslant 0;$$

(*v*) *there exists a function* $V : X \to R_+$ *with the properties*:

(*v′*) $\lim\limits_{t\to\infty} V(T(t)x) = 0$ *for every* $x \in X$;

(*v″*) $\dfrac{dt}{d} V(T(t)x) = -\|T(t)x\|^2$ *for each* $x \in X$;

(*v‴*) *there is* $M > 0$ *such that*

$$V(x) \leqslant M \cdot \|x^2\| \text{ for all } x \in X.$$

Proof. See [2].

THEOREM 2.3. *If* $T(t)$ *is* (p, q) *stable with* $(p, q) \neq (1, \infty)$ *then there exists a function* $\eta : R_+ \to R_+$ *with*

$$\lim_{t\to\infty} \eta(t) = 0$$

and such that for all $\delta_0 > 0$ *and* $\delta \geqslant \delta_0$ *we have*

$$\int_t^{t+\delta} \|T(s)x\|\,ds \leqslant \eta(\delta_0) \cdot \int_{t_0}^{t_0+\delta} \|T(s)x\|\,ds$$

for all $t_0 \geqslant 0$, $t \geqslant t_0 + \delta_0$ *and* $x \in X$.

Proof. Let $\delta \geqslant \delta_0 > 0$ and let n be a positive integer such that $n\delta_0 \leqslant \delta < (n+1)\delta_0$.

If we denote by $\delta_1 = \delta/n$ then from $t \geqslant t_0 + \delta_0$ and $s = t_0 + k\delta_1$, $k = 0, 1, \ldots, n-1$, by (p, q) stability of $T(t)$ and Hölder's inequality we have

$$\int_{s+t-t_0}^{s+t-t_0+\delta_1} \|T(\tau)x\|\,d\tau \leqslant \delta_1^{1/q'} \cdot \|T(\cdot)x\|_{L^q[s+\delta_0,\infty)} \leqslant (2\delta_0)^{1/q'} \cdot N \cdot \delta_0^{\frac{1}{p}-2}.$$

$$\int_s^{s+\delta_0} \|T(\tau)x\|\,d\tau \leqslant \eta(\delta_0) \cdot \int_s^{s+\delta_1} \|T(\tau)x\|\,d\tau,$$

where

$$\eta(\delta_0) = N(2\delta_0)^{1/q'} \cdot \delta_0^{\frac{1}{p}-2}.$$

Taking $s = t_0 + k\delta_1$, $k = 0, 1, 2, \ldots, n-1$ and adding we obtain

$$\int_t^{t+\delta} \|T(\tau)x\|\,d\tau = \int_t^{t+n\delta_1} \|T(\tau)x\|\,d\tau \leqslant \eta(\delta_0) \cdot \int_{t_0}^{t_0+\delta} \|T(\tau)x\|\,d\tau$$

and the theorem is proved.

LEMMA 2.4. *Let $f : R_+ \to R_+$ be a function with the property that there is $\delta > 0$ such that*

$$f(t+\delta) \geqslant 2f(t) \text{ for every } t \geqslant 0,$$

and

$$2f(t) \geqslant f(t_0) \text{ for fll } t_0 \geqslant 0 \text{ and } t \in [t_0, t_0+\delta].$$

Then there exists $\nu > 0$ such that

$$4f(t) \geqslant e^{\nu(t-t_0)} f(t_0) \quad \text{for all } t \geqslant t_0 \geqslant 0.$$

The proof is immediate ([4]). Indeed, if $\nu = \dfrac{\ln 2}{\delta}$ and n is the positive integer with

$$n\delta \leqslant t - t_0 < (n+1)\,\delta$$

then

$$4f(t) \geqslant 2f(t_0 + n\delta) \geqslant 2^{n+1} f(t_0) = e^{\nu(n+1)\delta} f(t_0) \geqslant e^{\nu(t-t_0)} f(t_0).$$

THEOREM 2.5. *If $T(t)$ is (p, q) stable with $(p, q) \neq (1, \infty)$ then there exists $\nu > 0$ such that for every $\delta > 0$ there is $N > 0$ with*

$$\int_t^{t+\delta} \|T(s)\,x\|\, ds \leqslant N e^{-\nu(t-t_0)} \|T(t_0)\,x\|$$

for all $t \geqslant t_0 \geqslant o$ and $x \in X$.

Proof. Let $\delta > 0$, $x \in X$ and let δ_0 be sufficiently large such that

$$\eta(\delta_0) < \frac{1}{2}.$$

Let n be a positive integer such that $n\delta > 4\,\delta_0$ and let us consider the function $f : R_+ \to R_+$ defined by

$$f(t) = \Big(\int_t^{t+n\delta} \|T(s)\,x\|\, ds\Big)^{-1}.$$

Then by preceding theorem we obtain

$$\frac{1}{f(t_0+\delta_0)} \leqslant \frac{\eta(\delta_0)}{f(t_0)} \leqslant \frac{1}{2f(t_0)}$$

and hence $f(t_0+\delta_0) \geqslant 2f(t_0)$ for every $t_0 \geqslant 0$.
If $t \in [t_0, t_0+\delta_0]$ then

$$\frac{1}{f(t)} = \int_t^{t_0+\delta_0} \|T(s)\,x\|\, ds + \int_{t_0+\delta_0}^{t+n\delta} \|T(s)\,x\|\, ds \leqslant \frac{1}{f(t_0)} +$$

$$+\int_{t_0+\delta_0}^{t_0+\delta_0+n\delta} \|T(s)\,x\|\, ds \leqslant \frac{1}{f(t_0)} + \frac{\eta(\delta_0)}{f(t_0)} \leqslant \frac{2}{f(t_0)},$$

which implies that

$$2f(t) \geqslant f(t_0) \text{ for all } t_0 \geqslant 0 \text{ and } t \in [t_0, t_0+\delta].$$

From Lemma 2.4. we obtain that there exists $\nu > 0$ such that

$$4f(t) \geqslant f(t_0) \cdot e^{\nu(t-t_0)} \text{ for all } t \geqslant t_0 \geqslant 0.$$

By preceding inequality and Lemma 2.1. we conclude that

$$\int_t^{t+\delta} \|T(s)\,x\|\, ds = \frac{1}{f(t)} \leqslant 4e^{-\nu(t-t_0)} \cdot \int_{t_0}^{t_0+n\delta} \|T(s)\,x\|\, ds \leqslant$$

$$\leqslant 4Mn\delta\, e^{n\delta\omega}\, e^{-\nu(t-t_0)}\, \|T(t_0)\,x\|$$

for all $t \geqslant t_0 \geqslant 0$.

3. (L^p, L^q) stability of controlled system

Let $T(t)$ be a C_0-semigroup on a separable Banach space X. Consider the linear control system described by the following integral model

$$(T, B, \mathscr{U}_p) \quad x(t, u) = \int_0^t T(t-s)\, Bu(s)\, ds,$$

where $u \in \mathscr{U}_p = L^p(R_+, U)\,(1 \leqslant p \leqslant \infty)$, $B \in L(U, X)$ (the Banach space of bounded linear operators from the Banach space U to X).

Here $\mathscr{U}_p$ is the Banach space of all U-valued, strongly measurable functions f defined a. e. on $R_+ = [0, \infty)$ such that

$$\|f\|_p = \left(\int_0^\infty \|f(s)\|^p\, ds\right)^{1/p} < \infty, \text{ if } p < \infty$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{s \geqslant 0} \|f(s)\| < \infty, \text{ if } p = \infty.$$

We also denote

$$\mathscr{H}_p = L^p(R_+, X),\ \mathscr{U}_p(\delta) = L^p([0, \delta], U), \text{ where } \delta > 0$$

and

$$p' = \begin{cases} \infty, & \text{if } p = 1 \\ 1, & \text{if } p = \infty \\ \dfrac{p}{p-1}, & \text{if } 1 < p < \infty. \end{cases}$$

Definition 3.1. We say that $(T, B, \mathcal{U}_p)$ is *controlled* if there exists $\delta > 0$ such that for every $x \in X$ there is $u \in \mathcal{U}_p(\delta)$ with $x(\delta, u) = x$.

Let $C_\delta : \mathcal{U}_p(\delta) \to X$ be the linear operator defined by

$$C_\delta(u) = x(\delta, u).$$

It is easy to see that the adjoint

$$C_\delta^* : X^* \to \mathcal{U}_p(\delta)^*$$

is defined by

$$(C_\delta^* x^*)(s) = B^* T(\delta - s)^* x^*, \; s \in [0, \delta].$$

THEOREM 3.1. *The following statements are equivalent*:

(*i*) $(T, B, \mathcal{U}_p)$ *is controlled*;

(*ii*) *there exists* $\delta > 0$ *such that*

$$C_\delta(\mathcal{U}_p(\delta)) = X;$$

(*iii*) *there are* $\delta > 0$, $m > 0$ *such that*

$$\|C_\delta^* x^*\|_{L^{p'}([0,\delta], U^*)} \geqslant m \cdot \|x^*\|$$

for all $x^* \in X^*$;

Proof. See [7].

Particularly we obtain the following

COROLLARY 3.2. $(T, B, \mathcal{U}_2)$ *is controlled if and only if there exist* $\delta > 0$ *and* $m > 0$ *such that*

$$W_\delta x^* \overset{d}{=} \int_0^\delta \|B^* T^*(s) x^*\|^2 \, ds \geqslant m \|x^*\|^2$$

for all $x^* \in X^*$.

Remark 3.1. It is easy to see that if $(T, B, \mathcal{U}_p)$ is controlled then there exist $\delta > 0$, $m > 0$ such that for every $x \in X$ there is $u \in \mathcal{U}_p(\delta)$ such that $x(\delta, u) = x$ and

$$\|u\|_{L^p[0,\delta]} \leqslant m \|x\|.$$

(see [7]).

Now let us note three assumptions which will be used at various times.

Assumption 1. We say that $(T, B, \mathcal{U}_p)$ satisfies the Assumption 1 if the range of B is of second category in X.

Assumption 2. The system $(T, B, \mathcal{U}_p)$ satisfies the Assumption 2 if it is controlled.

Assumption 3. $(T, B, \mathscr{U}_p)$ satisfies the Assumption 3 if

$$T(t)\, x \neq 0 \text{ for all } t \geqslant 0 \text{ and } x \in X,\ x \neq 0.$$

Remark 3.2. According to the more refined version of the open--mapping theorem ([3]) it follows that if $(T, B, \mathscr{U}_p)$ satisfies the Assumption 1 then there exist an operator $B^+ : X \to V$ and $b > 0$ such that

$$BB^+ x = x \text{ and } \|B^+ x\| \leqslant b\, \|x\|$$

for every $x \in X$.

It is easy to verify that if $(T, B, \mathscr{U}_p)$ satisfies the Assumption 1 then it also satisfies the Assumption 2.

Definition 3.2. The system $(T, B, \mathscr{U}_p)$ il said to be (L^p, L^q) stable (where $1 \leqslant p, q \leqslant \infty$) if the linear operator Λ defined by

$$\Lambda u = x(.\,, u),$$

is a bounded operator from $\mathscr{U}_p$ to $\mathscr{H}_q$.

THEOREM 3.3. *If the system $(T, B, \mathscr{U}_p)$ is controlled and (L^p, L^q) stable,then $T(t)$ is*

(*i*) *uniformly stable, if $q = \infty$;*

(*ii*) *exponentially stable if $1 \leqslant q < \infty$.*

Proof. Let $x \in X$. From Remark 3.1. it follows that there exist $\delta, m > 0$ and $u \in \mathscr{U}_p(\delta)$ such that $x(\delta, u) = x$ and

$$\|u\|_{L^p[0,\delta]} \leqslant m\, \|x\|.$$

Let

$$v(s) = \begin{cases} u(s), & s \in [0, \delta] \\ 0, & s > \delta. \end{cases}$$

Clearly $v \in \mathscr{U}_p$, $\|v\|_p \leqslant m \cdot \|x\|$ and $x(t, v) = T(t - \delta)\, x$ for $t \geqslant \delta$. From (L^p, L^q) stability of $(T, B, \mathscr{U}_p)$ we have that there is $N > 0$ such that

$$\|\Lambda v\|_q \leqslant N\, \|v\|_p \leqslant m \cdot N \cdot \|x\|$$

and hence (*i*) follows immediately.

If $1 \leqslant q < \infty$ then

$$\int_0^\infty \|T(t)\, x\|^q\, dt = \int_\delta^\infty \|T(t - \delta)\, x\|^q\, dt \leqslant \|\Lambda v\|_q^q \leqslant (mN)^q\, \|x\|^q$$

for all $x \in X$.

By Theorem 2.2. it follows that the semigroup $T(t)$ is exponentially stable.

THEOREM 3.4. *If* $(T, B, \mathcal{U}_p)$ *satisfies the Assumption* 1 *and is* (L^p, L^∞) *stable with* $p > 1$ *then it is exponentially stable.*

Proof. Let $x \in X$ with $\|x\| = 1$. If there exists $t_0 > 0$ such that $T(t_0)\,x = 0$ then

$$T(t)\,x = T(t - t_0)\,T(t_0)\,x = 0 \text{ for all } t \geqslant t_0$$

and the conclusion is obvious.

Suppose that $T(t)\,x \neq 0$ for all $t \geqslant 0$. For every $t > 0$ let u_t be the input

$$u_t(s) = \begin{cases} \dfrac{B^+\,T(s)\,x}{\|T(s)\,x\|}, & s \leqslant t, \\ 0 \quad , & s > t, \end{cases}$$

where $B^+ : U \to X$ is the operator defined in Remark 3.2.
Then

$$x(t, u_t) = f(t)\,T(t)\,x, \text{ where } f(t) = \int_0^t \frac{ds}{\|T(s)\,x\|}.$$

By (L^p, L^∞) stability of $(T, B, \mathcal{U}_p)$ it follows that there exists $M_1 > 0$ such that

$$\frac{f(t)}{f'(t)} = f(t)\,\|T(t)\,x\| \leqslant M_1\,b\,t^{1/p} \text{ for all } t \geqslant 0. \tag{3.1}$$

Let $t \geqslant 1$. By integration we obtain that

$$f(t) \geqslant f(1)\,e^{2\nu}(t^{1/p'-1}), \text{ where } \nu = \frac{p'}{2M_1 b}. \tag{3.2}$$

Let $M, \omega > 0$ such that

$$\|T(t)\| \leqslant Me^{\omega t} \text{ for all } t \geqslant 0.$$

Then

$$f(1) = \int_0^1 \frac{ds}{\|T(s)\,x\|} \geqslant \frac{1}{M}\int_0^1 e^{-\omega s}ds \geqslant \frac{e^{-\omega}}{M}.$$

From (3.1) and (3.2), it follows that

$$\|T(t)\,x\| \leqslant \frac{M_1\,bt^{1/p}}{f(t)} \leqslant \frac{M_1\,bt^{1/p}}{f(1)} \leqslant \frac{M_1\,bt^{1/p}}{f(1)} \cdot e^{2\nu(1-t^{1/p'})} =$$

$$= M_2\,t^{1/p}\,e^{-2\nu t^{1/p'}},$$

where $M_2 = bMM_1\,e^{(\omega+2\nu)}$. If we denote by

$$K = \sup_{t \geqslant 1} t^{1/p} e^{-\nu t(2t^{-1/p}-1)} \leqslant \sup_{t \geqslant 1} t^{1/p} e^{-\nu t} \leqslant \sup_{s \geqslant 0} \frac{s}{e^{\nu_s p}} < \infty$$

and

$$N = \max\{KM_2, \sup_{t \in [0,1]} e^{\nu t} \|T(t)\|\}$$

then we obtain that

$$\|T(t)x\| \leqslant Ne^{-\nu t} \|x\| \text{ for all } x \in X \text{ and } t \geqslant 0.$$

The theorem is proved.

7. The main results

The purpose of this section is to establish the relationships between the stability concepts introduced in the preceding sections.

A technical lemma which will be used in the sequel is

LEMMA 4.1. *If $T(t)$ is exponentially stable then there exists $\nu > 0$ such that for every $p \in [1, \infty)$ there is $M_p > 0$ with*

$$\|(\Lambda u)(t)\|^q \leqslant M_p^q \|u\|_q^{q-p} \cdot \int_0^t e^{-\nu q(t-s)} \|u(s)\|^p ds$$

for all $t \geqslant 0$ and $q \in [p, \infty)$.

Proof. Let $u \in L^p$ and $N, \nu > 0$ such that

$$\|T(t)\| \leqslant Ne^{-2\nu t} \text{ for all } t \geqslant 0.$$

Let $1 < p < q < \infty$ and

$$r = \frac{p(q-1)}{q-p}.$$

Then $r' = p'/q'$ and by Hölder's inequality we obtain

$$\|(\Lambda u)(t)\|^q \leqslant N^q \cdot \|B\|^q (\int_0^t e^{-2\nu(t-s)} \|u(s)\| ds)^q \leqslant N^q \|B\|^q \cdot$$

$$\cdot (\int_0^t e^{-q'(t-s)} \|u(s)\|^{q'\left(1-\frac{p}{q}\right)} ds)^{q-1} \cdot \int_0^t e^{-\nu q(t-s)} \cdot \|u(s)\|^p ds \leqslant$$

$$\leqslant N^q \|B\|^q \cdot (\int_0^t e^{-\nu q' r'(t-s)} ds)^{\frac{q-1}{r}} \cdot (\int_0^t \|u(s)\|^{rq'\left(1-\frac{p}{q}\right)} ds)^{\frac{q-1}{r}} \cdot$$

$$\cdot \int_0^t e^{-\nu q(t-s)} \|u(s)\|^p ds \leqslant N^q \|B\|^q (\int_0^t e^{-\nu p'(t-s)} ds)^{\frac{q-1}{r'}}.$$

$$\cdot \left(\int_0^t \|u(s)\|^p \, ds\right)^{\frac{q-1}{r}} \cdot \int_0^t e^{-\nu q(t-s)} \|u(s)\|^p \, ds \leqslant K_p^q \|u\|_p^{q-p} \cdot$$

$$\cdot \int_0^t e^{-\nu q(t-s)} \cdot \|u(s)\|^p \, ds,$$

where $K_p = N \|B\| (\nu p')^{-1/p'}$.

If $1 < p = q < \infty$ then

$$\|(\Lambda u)(t)\|^q \leqslant N^q \|B\|^q \left(\int_0^t e^{-\nu p'(t-s)} \, ds\right)^{q/p'} \cdot \int_0^t e^{-\nu p(t-s)} \|u(s)\|^p \, ds \leqslant$$

$$\leqslant K_p^q \cdot \int_0^t \mathrm{e}^{-\nu q(t-s)} \|u(s)\|^p \, ds.$$

If $1 = p < q < \infty$ then

$$\|(\Lambda u)(t)\|^q \leqslant N^q \|B\|^q \left(\int_0^t e^{-\nu q'(t-s)} \|u(s)\| \, ds\right)^{q/q'} \cdot \int_0^t e^{-\nu q(t-s)} \|u(s)\| \, ds \leqslant$$

$$\leqslant N^q \|B\|^q \cdot \|u\|_p^{q-1} \cdot \int_0^t e^{-\nu q(t-s)} \|u(s)\| \, ds.$$

Finally, if $p = q = 1$ then the inequality

$$\|(\Lambda u)(t)\| \leqslant N \cdot \|B\| \cdot \int_0^t e^{-\nu(t-s)} \|u(s)\| \, ds$$

is obvious.

Hence

$$\|(\Lambda u)(t)\|^q \leqslant M_p^q \|u\|_p^{q-p} \cdot \int_0^t e^{-\nu q(t-s)} \|u(s)\|^p \, ds,$$

where

$$M_p = \max \{N \|B\|, K_p\}.$$

THEOREM 4.2. *Let* $p, q \in [1, \infty]$ *with* $p \leqslant q$ *and* $(p, q) \neq \neq (1, \infty)$. *Suppose that* $(T, B, \mathscr{U}_p)$ *satisfy the Assumptions* 1 *and* 3. *Then following statements are equivalent*:

(*i*) *the semigroup* $T(t)$ *is* (p, q) *stable*;

(*ii*) *the semigroup* $T(t)$ *is exponentially stable*;

(*iii*) *the system* $(T, B, \mathscr{U}_p)$ *is* (L^p, L^q) *stable.*

Proof. (*i*) $\Rightarrow$ (*ii*). Let $x \in X$ and $\delta > 0$.

Firstly, we suppose that $T(t)\, x \neq 0$ for all $t > 0$. From Theorem 2.5. we obtain that

$$T(t)\,x = \frac{1}{\delta}\int\limits_{t-\delta}^{t}\|T(t)\,x\|\,ds \leqslant \frac{Me^{\omega\delta}}{\delta}\cdot\int\limits_{t-\delta}^{t}\|T(s)\,x\|\,ds \leqslant \frac{MNe^{\omega\delta}}{\delta}\,e^{-\nu t}\,\|x\|$$

for all $t \geqslant \delta$.

Let

$$N_1 = \max\frac{MNe^{\omega\delta}}{\delta},\ \sup_{t\in[0,\delta]} e^{-t\nu}\,\|T(t)\|\}.$$

Then

$$\|T(t)\,x\| \leqslant N_1\,e^{-\nu t}\,\|x\| \text{ for all } t \geqslant 0.$$

The case when there exists $t_0 > 0$ such that $T(t_0)\,x = 0$ is obvious, because then

$$T(t)\,x = 0 \text{ for all } t \geqslant t_0.$$

$(ii) \Rightarrow (iii)$. Let $u \in \mathcal{U}_p$ and $N, \nu > 0$ such that

$$\|T(t)\| \leqslant Ne^{-2\nu t} \text{ for all } t \geqslant 0.$$

Firstly, we suppose that $q = \infty$ and $p \geqslant 1$. Then

$$\|\Lambda u\|_\infty \leqslant N\,\|B\|\operatorname*{ess\,sup}_{t\geqslant 0}\int\limits_0^t e^{-\nu(t-s)}\,\|u(s)\|\,ds \leqslant M_p\,\|u\|_p,$$

where

$$M_p = \begin{cases} N\|B\|, & \text{if } p = 1\\ \dfrac{N\|B\|}{2\nu p'}, & \text{if } p > 1.\end{cases}$$

Hence $(T, B, \mathcal{U}_p)$ is (L^p, L^∞) stable for every $p \geqslant 1$.

Let now $1 \leqslant p \leqslant q < \infty$ and

$$v(t,\tau) = \begin{cases} u(t-\tau), & 0 \leqslant \tau \leqslant t\\ 0\quad, & 0 \leqslant t < \sigma.\end{cases}$$

Then from Lemma 4.1. we have

$$\|\Lambda u\|_q^q \leqslant M_p^q\,\|u\|_p^{q-p}\cdot\int\limits_0^\infty\Big(\int\limits_0^t e^{-\nu q\tau}\,\|u(t-\tau)\|^p\,d\tau\Big)\,dt =$$

$$= M_p^q\,\|u\|_p^{q-p}\cdot\int\limits_0^\infty e^{-\nu q\tau}\,\Big(\int\limits_0^\infty\|v(t,\tau)\|^p\,dt\Big)\,d\tau = M_p^q\cdot\|u\|_p^{q-p}\cdot$$

$$\cdot\int\limits_0^\infty e^{-\nu q\tau}\,d\tau\cdot\int\limits_0^\infty\|u(s)\|^p\,ds = \frac{M_p^q\,\|u\|_p^q}{\nu_q},$$

and hence $(T, B, \mathcal{U}_p)$ is (L^p, L^q) stable.

(iii) ⇒ *(i)*. Let $t > 0$, $\delta > 0$ and $x \in X$, $x \neq 0$. Let $u_t(.)$ be the input function defined by

$$u_t(s) = \begin{cases} \dfrac{B^+ T(s) x}{\|T(s) x\|}, & \text{if } s \in [t, t+\delta] \\ 0 \quad , & \text{if } s \notin [t, t+\delta]. \end{cases}$$

Clearly $u_t \in \mathcal{U}_p$, $\|u_t\|_p \leqslant b\, \delta^{1/p}$ and

$$x(s, u_t) = f(t)\, T(s)\, x, \text{ for every } s \geqslant t + \delta$$

where

$$f(t) = \int_t^{t+\delta} \frac{ds}{\|T(s) x\|}.$$

From (L^p, L^q) stability of $(T, B, \mathcal{U}_p)$ it follows that there exists $M > 0$ such that

$$f(t)\, \|T(.)\, x\|_{L^q[t+\delta, \infty]} \leqslant \|x(., u_t)\|_q \leqslant Mb\, \delta^{1/p}.$$

By Schwarz's inequality we have

$$\delta^2 \leqslant f(t) \cdot \int_t^{t+\delta} \|T(s)\, x\|\, ds$$

and hence

$$\|T(.)\, x\|_{L^q[t+\delta, \infty]} \leqslant \frac{\|x(., u_t)\|_q}{f(t)} \leqslant Mb\, \delta^{\frac{1}{p} - 2} \int_t^{t+\delta} \|T(s)\, x\|\, ds.$$

The theorem is proved.

Remark 4.1. From the proof of the preceding theorem it follows that if the Assumption 3 holds then the equivalence *(i)* ⇔ *(ii)* is true.

Remark 4.2. The equivalence *(ii)* ⇔ *(iii)* is true:

1° if $1 \leqslant p \leqslant q < \infty$ and the Assumption 2 holds;

2° if $q = \infty$ and the Assumption 1 holds.

The case when $T(t) = \exp(At)$, where $A \in L(X)$ and Y is a Hilbert space is contained in [6].

Remark 4.3. The equivalence of *(ii)* and *(iii)* in the Assumption 2 for $p = q = \infty$ is an open question.

THEOREM 4.3. *Suppose that* $(T, B, \mathcal{U}_2)$ *is controlled. Then* $T(t)$ *is exponentially stable if and only if there exists a function* V : : $X^* \to R_+$ *with the properties*

(i) $\lim\limits_{t \to \infty} V(T(t)^* x^*) = 0$ *for all* $x^* \in X^*$;

(*ii*) $\frac{d}{dt} V(T(t)^* x^*) = - \|B^* T(t)^* x^*\|^2$ *for every* $x^* \in X^*$;

(*iii*) *there exists* $M > 0$ *such that*

$$V(x^*) \leqslant M \|x^*\|^2 \text{ for any } x^* \in X^*.$$

Proof. If $T(t)$ is exponentially stable then from Theorem 2.2. it is easy to verify that the function $V : X^* \to R$, defined by

$$V(x^*) = \int_0^\infty \|B^* T(s)^* x^*\|^2 \, ds$$

has the properties (*i*) — (*iii*).

From Corollary 3.2. there exist $\delta > 0$, $m > 0$ such that

$$W_\delta x^* \geqslant m \|x^*\|^2 \text{ for all } x^* \in X^*.$$

Then

$$V(T(t)^* x^*) - V(x^*) = \int_0^t \frac{d}{ds} V(T(s)^* x^*) \, ds =$$

$$= - \int_0^t \|B^* T(s)^* x^*\|^2 \, ds$$

and hence

$$\int_0^t \|B^* T(s)^* x^*\|^2 \, ds = V(x^*) - V(T(t)^* x^*) \leqslant V(x^*) \leqslant M \|x^*\|^2,$$

which implies that

$$\int_0^\infty \|B^* T(s)^* x^*\|^2 \, ds \leqslant M \|x^*\|^2 \text{ for all } x^* \in X^* \tag{4.1}$$

and

$$V(x^*) - V(T(\delta)^* x^*) \geqslant m \|x^*\|^2 \text{ for every } x^* \in X^*. \tag{4.2}$$

Then

$$\|T(s)^* x^*\|^2 \leqslant \frac{1}{m} (V(T(s)^* x^*) - V(T(\delta + s)^* x^*)) =$$

$$= - \frac{1}{m} \int_s^{s+\delta} \frac{d}{d\tau} V(T(\tau)^* x^*) \, d\tau = \frac{1}{m} \int_s^{s+\delta} \|B^* T(\tau)^* x^*\|^2 \, d\tau.$$

From (4.1) and (4.2) we have

$$\int_0^t \|T(s)^* x^*\|^2 ds \leqslant \frac{1}{m} \int_0^t \left(\int_s^{s+\delta} \|B^* T(\tau)^* x^*\|^2 d\tau\right) ds =$$

$$= \frac{1}{m} \int_0^t \left(\int_0^\delta \|B^* T^* (u + s) x^*\|^2 du\right) ds =$$

$$= \frac{1}{m} \int_0^\delta \left(\int_0^t \|B^* T(u + s)^* x^*\|^2 ds\right) du = \frac{1}{m} \int_0^\delta \left(\int_u^{u+t} \|B^* T(\tau)^* x^*\|^2 d\tau\right) du \leqslant$$

$$\leqslant \frac{1}{m} \int_0^\delta \left(\int_0^\infty \|B^* T(\tau)^* x^*\|^2 d\tau\right) du \leqslant \frac{\delta M}{m} \|x^*\|^2,$$

for all $t \geqslant 0$ and $x^* \in X^*$.

From Theorem 2.2. it follows that $T(t)^*$ and hence also $T(t)$ is an exponentially stable semigroup.

Remark 4.4. The preceding theorem is an extension of a Datko's theorem (see [2]). The case when X is a Hilbert space has been considered in [6].

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O STABILNOSTI UPRAVLJANIH SISTEMA U BANACHOVIM PROSTORIMA

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Sadržaj

U članku se studiraju svojstva stabilnosti linearnih sistema čija se evolucija može opisati pomoću polugrupe klase C_0 na Banachovom prostoru. Generalizirani su Datkov teorem i Perronov kriterij za linearno upravljane sisteme u Banachovim prostorima.