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THE EXACT SEQUENCE OF A SHAPE FIBRATION

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Abstract. Using the definition of shape fibration for arbitrary topological spaces given in [5] we show when a restriction of shape fibration is again a shape fibration (Theorem 4.1) and when a shape fibration induces an isomorphism of homotopy pro-groups (Theorem 5.7) obtaining also the exact sequence of shape fibration (Theorem 5.9).

1. Introduction

The notion of a shape fibration for maps between compact metric spaces was introduced by S. Mardešić and T. M. Rushing in [11] and [12]. In [10] Mardešić has defined shape fibrations for maps between arbitrary topological spaces. In [5] the author has given an alternative definition of a shape fibration, which is equivalent to Mardešić's definition from [10]. Using some results from [5] and [10] we establish in the present paper the following two facts concerning shape fibrations $p : E \to B$, which are closed maps of a topological space E to a normal space B.

(*i*) If $B_0 \subseteq B$ is a closed subset of B, then the restriction of p to $E_0 = p^{-1}(B_0)$ is also a shape fibration whenever E_0 and B_0 are P-embedded in E and B respectively (Theorem 4.1).

(*ii*) If $e \in E$, $b = p(e)$ and $F = p^{-1}(b)$ is P-embedded in E, then p induces an isomorphism of the homotopy pro-groups

$$\mathbf{p}_*: \text{pro-}\pi_n(E, F, e) \to \text{pro-}\pi_n(B, b)$$

(Theorem 5.7).

As a corollary of (*ii*) one obtains the exact sequence of a shape fibration (Theorem 5.9).

These results generalize the corresponding results for compact metric spaces from [11] and [12]. The paper can be viewed as a continuation of papers [5] and [10].

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2. On resolution of spaces and maps

In this section we recall the definitions of a resolution of a space and of a resolution of a map [10], and we establish some facts needed in the sequel.

2.1. *Definition* ([10]). A map of systems $\mathbf{q} = (q_\lambda): E \to \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ is a *resolution* of the space E provided the following conditions are fulfilled:

($R1$) Let P be a polyhedron, $\mathscr{V}$ an open covering of P and $f : E \to P$ a map. Then there is a $\lambda \in \Lambda$ and a map $f_\lambda : E_\lambda \to P$ such that $f_\lambda q_\lambda$ and f are $\mathscr{V}$-near, which we denote by $(f_\lambda q_\lambda, f) \leqslant \mathscr{V}$.

($R2$) Let P be a polyhedron and $\mathscr{V}$ an open covering of P. Then there is an open covering $\mathscr{V}'$ of P with the following property. Whenever $f, f' : E_\lambda \to P$ are maps satisfying $(fq_\lambda, f'q_\lambda) \leqslant \mathscr{V}'$, then there is a $\lambda' \geqslant \lambda$ such that $(fq_{\lambda\lambda'}, f' q_{\lambda\lambda'}) \leqslant \mathscr{V}$.

If all E_λ's are polyhedra (ANR's), then $\mathbf{q} : E \to \mathbf{E}$ is called a polyhedral (ANR) resolution.

2.2. *Definition.* Let $p : E \to B$ be a map. A *resolution* of p is a triple $(\mathbf{q}, \mathbf{r}, \mathbf{p})$, which consists of resolutions $\mathbf{q} : E \to \mathbf{E}$ and $\mathbf{r} : B \to \mathbf{B} = (B_\mu, r_{\mu\mu'}, M)$ of the spaces E and B respectively and of a map of systems $\mathbf{p} = (p_\mu, \pi) : \mathbf{E} \to \mathbf{B}$ satisfying $\mathbf{p}\,\mathbf{q} = \mathbf{r}p$, i. e. $p_\mu\, q_{\pi(\mu)} = r_\mu\, p$, $\mu \in M$.

If a map $\mathbf{p} = (p_\lambda, 1_\Lambda) : \mathbf{E} \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ is a level map [5], then $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is called a *level-resolution.* In this case $\mathbf{p}\,\mathbf{q} = \mathbf{r}\,p$ is equivalent to $p_\lambda\, q_\lambda = r_\lambda\, p$, $\lambda \in \Lambda$.

It was shown in [10] that $\mathbf{q} : E \to \mathbf{E}$ is a resolution of E if it satisfies the following conditions:

($B1$) For each normal covering $\mathscr{U}$ of E there is a $\lambda \in \Lambda$ and a normal covering $\mathscr{U}_\lambda$ of E_λ such that $q_\lambda^{-1}(\mathscr{U}_\lambda)$ refines $\mathscr{U}$, which is denoted by $q_\lambda^{-1}(\mathscr{U}_\lambda) \geqslant \mathscr{U}$.

($B2$) For each $\lambda \in \Lambda$ and each open neighborhood U of $\mathrm{Cl}(q_\lambda(E))$ in E_λ there is a $\lambda' \geqslant \lambda$ such that $q_{\lambda\lambda'}(E_{\lambda'}) \subseteq U$.

Conversely, if all E_λ are normal, then every resolution $\mathbf{q} : E \to \mathbf{E}$ has properties ($B1$) and ($B2$) ([10]), Theorem 6). In particular, every polyhedral resolution has properties ($B1$) and ($B2$).

In the sequel we will use a special type of polyhedral resolutions, which we will call *canonical resolutions.* These are polyhedral resolutions $\mathbf{r} = (r_\mu) : B \to \mathbf{B} = (B_\mu, r_{\mu\mu'}, M)$ such that M is a cofinite directed set, each B_μ is the nerv $|N(\gamma_\mu)|$ of a normal covering γ_μ of B and $r_{\mu\mu'} : B_{\mu'} \to B_\mu$, $\mu \leqslant \mu'$, is a simplical map such that $r_{\mu\mu'}(V') = V$ implies $V' \subseteq V$, where $V' \in \gamma_{\mu'}$ and $V \in \gamma_\mu$. Moreover, $r_\mu : B \to B_\mu$ is the canonical map given by a locally finite partition of unity $(\Psi_V, V \in \gamma_\mu)$ subordinated to γ_μ, i. e.

$$r_\mu(x) = \sum_V \Psi_V(x)\, V, \qquad x \in B.$$

2.3. THEOREM. (*i*) *Every topological space B admits a canonical resolution.*

(*ii*) *If $\mathbf{r} : B \to \mathbf{B}$ is a canonical resolution of B, then every map $p : E \to B$ of topological spaces admits a polyhedral resolution $(\mathbf{q}, \mathbf{r}, \mathbf{p})$.*

A proof is obtained by obvious modifications of the proof of Theorem 11, [10].

The following lemma is needed in the sequel.

2.4. LEMMA. *Let B be a normal space and $\mathbf{r} = (r_\lambda) : B \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ a polyhedral resolution of B. Let $B_0 \subset B$ be a closed subset and let $\mathbf{r}_0 = (r_\lambda \mid B_0) : B_0 \to \mathbf{B}_0 = (B_{0\lambda}, r_{\lambda\lambda'} \mid B_{0\lambda'}, \Lambda)$ be a resolution of B_0 such that every $B_{0\lambda}$ is a closed subset of B_λ. Then for every open neighborhood V of B_0 in B and for every $\lambda \in \Lambda$ there is a $\lambda' \geqslant \lambda$ and an open neighborhood $V_{\lambda'}$ of $B_{0\lambda'}$ in $B_{\lambda'}$ such that*

$$r_{\lambda'}^{-1}(V_{\lambda'}) \subseteq V.$$

Proof. $\mathscr{U} = \{V, B \setminus B_0\}$ is a normal covering of B. Since $\mathbf{r}$ is a polyhedral resolution, it has the property $(B1)$. Consequently, there is a $\mu \in \Lambda$ and there is an open covering $\mathscr{U}_\mu$ of B_μ such that $r_\mu^{-1}(\mathscr{U}_\mu)$ refines $\mathscr{U}$. Let $\nu \in \Lambda$, $\nu \geqslant \lambda, \mu$. Then $\mathscr{U}_\nu = r_{\mu\nu}^{-1}(\mathscr{U}_\mu)$ is an open covering of B_ν such that $r_\nu^{-1}(U_\nu)$ refines $\mathscr{U}$. It follows that for each $U \in \mathscr{U}_\nu$

$$U \cap \mathrm{Cl}(r_\nu(B_0)) \neq \emptyset \Leftrightarrow U \cap r_\nu(B_0) \neq \emptyset \Rightarrow r_\nu^{-1}(U) \subseteq V \tag{1}$$

Let us put

$$V_\nu = \cup \{U \in \mathscr{U}_\nu \mid U \cap \mathrm{Cl}(r_\nu(B_0)) \neq \emptyset\}$$

Clearly, V_ν is an open set in B_ν and $\mathrm{Cl}(r_\nu(B_0)) \subseteq V_\nu$. Moreover, by (1), one has

$$r_\nu^{-1}(V_\nu) \subseteq V. \tag{2}$$

The set $V_\nu \cap B_{0\nu}$ is an open neighborhood of $\mathrm{Cl}(r_\nu(B_0))$ in $B_{0\nu}$. Hence, by property $(B2)$ of $\mathbf{r}_0$, there is a $\lambda' \geqslant \nu$ such that $r_{\nu\lambda'}(B_{0\lambda}) \subseteq \subseteq V_\nu \cap B_{0\nu} \subseteq V_\nu$, i. e. $B_{0\lambda'} \subseteq r_{\nu\lambda'}^{-1}(V_\nu)$. Using normality of $B_{\lambda'}$ one can find an open set $V_{\lambda'}$ in $B_{\lambda'}$ such that $B_{0\lambda'} \subseteq V_{\lambda'} \subseteq \mathrm{Cl}(V_{\lambda'}) \subseteq \subseteq r_{\nu\lambda'}^{-1}(V_\nu)$. Then $V_{\lambda'}$ is the desired neighborhood of $B_{0\lambda'}$ because, by (2),

$$r_{\lambda'}^{-1}(V_{\lambda'}) \subseteq r_{\lambda'}^{-1} r_{\nu\lambda'}^{-1}(V_\nu) = r_\nu^{-1}(V_\nu) \subseteq V. \tag{3}$$

2.5. THEOREM. *Let $p : E \to B$ be a closed map of a topological space E into a normal space B, let B_0 be a closed subset of B and let $E_0 = p^{-1}(B_0)$ be P-embedded in E. Furthermore, let $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ be a polyhedral level-resolution of p and let $\mathbf{r}_0 = (r_\lambda \mid B_0) : B_0 \to \mathbf{B}_0 = (B_{0\lambda}, r_{\lambda\lambda'} \mid B_{0\lambda'}, \Lambda)$ be a resolution of B_0 such that each $B_{0\lambda}$ is a closed subset of B_λ. Then $\mathbf{q}_0 = (q_{0\lambda}) : E_0 \to \mathbf{E}_0 = (E_{0\lambda}, q_{\lambda\lambda'} \mid E_{0\lambda'}, \Lambda)$ is a resolution of E_0, where $q_{0\lambda} = q_\lambda \mid E_0$ and*

$$E_{0\lambda} = p_\lambda^{-1}(B_{0\lambda}), \qquad \lambda \in \Lambda. \tag{4}$$

Recall that $E_0 \subseteq E$ is P-embedded in E provided every normal covering $\mathscr{U}_0$ of E_0 admits a normal covering $\mathscr{U}$ of E such that $\mathscr{U} \mid E_0 = \{U \cap E_0 \mid U \in \mathscr{U}\}$ refines $\mathscr{U}$ ([1], Theorem 14.7, p. 178).

In order to prove Theorem 2.5 we need the following proposition.

2.6. PROPOSITION. *Let $p : E \to B$ be a closed map of topological spaces, let $B_0 \subseteq B$ be a closed subset, $E_0 = p^{-1}(B_0)$ and let U be an open neighborhood of E_0 in E. Then there is an open neighborhood V of B_0 in B such that $p^{-1}(V) \subseteq U$.*

Proof of 2.6. Since p is a closed mapping and $E \setminus U$ is a closed set in E, it follows that $V = B \setminus p(E \setminus U)$ is an open neighborhood of B_0 in B having the required property $p^{-1}(V) \subseteq U$.

Proof of Theorem 2.5. $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is a level-resolution of p and hence

$$p_\lambda q_\lambda = r_\lambda p, \qquad \lambda \in \Lambda. \tag{5}$$

Since $\mathbf{B}_0$ is an inverse system, one also has

$$r_{\lambda\lambda'}(B_{0\lambda'}) \subseteq B_{0\lambda}, \qquad \lambda \leqslant \lambda'. \tag{6}$$

It readily follows that

$$q_{\lambda\lambda'}(E_{0\lambda'}) \subseteq E_{0\lambda}, \qquad \lambda \leqslant \lambda' \tag{7}$$

$$\mathrm{Cl}\,(q_\lambda(E_0)) \subseteq E_{0\lambda}, \qquad \lambda \in \Lambda. \tag{8}$$

In order to show that $\mathbf{q}_0 : E_0 \to \mathbf{E}_0$ is a resolution of E_0, it sufficies to verify the conditions $(B1)$ and $(B2)$ for $\mathbf{q}_0$.

Condition $(B1)$. Let $\mathscr{U}_0$ be a normal covering of E_0. Since E_0 is P-embedded in E, there is a normal covering $\mathscr{U}$ of E such that $\mathscr{U} \mid E_0$ refines $\mathscr{U}_0$. The polyhedral resolution $\mathbf{q} : E \to \mathbf{E}$ has the property $(B1)$ and therefore there is a $\lambda \in \Lambda$ and an open covering $\mathscr{U}_\lambda$ of E_λ such that $q_\lambda^{-1}(\mathscr{U}_\lambda)$ refines $\mathscr{U}$. Then $\mathscr{U}_{0\lambda} = \mathscr{U}_\lambda \mid E_{0\lambda}$ is a normal covering of $E_{0\lambda}$ and $q_{0\lambda}^{-1}(\mathscr{U}_{0\lambda})$ refines $\mathscr{U} \mid E_0$ and thus also refines $\mathscr{U}_0$.

Condition $(B2)$. Let $\lambda \in \Lambda$ and let $U_{0\lambda}$ be an open neighborhood of $\mathrm{Cl}\,(q_\lambda(E_0))$ in $E_{0\lambda}$. Then there is an open set U_λ in E_λ such that

$$U_\lambda \cap E_{0\lambda} = U_{0\lambda}. \tag{9}$$

By normality of E_λ, there is also an open set U'_λ in E_λ such that

$$\mathrm{Cl}\,(q_\lambda(E_0)) \subseteq U'_\lambda \subseteq \mathrm{Cl}\,(U'_\lambda) \subseteq U_\lambda. \tag{10}$$

We put

$$U = q_\lambda^{-1}(U'_\lambda) \tag{11}$$

Clearly, U is an open neighborhood of $E_0 = p^{-1}(B_0)$ in E. Hence, by proposition 2.6, there is an open neighborhood V of B_0 in B such that $p^{-1}(V) \subseteq U$, and therefore

$$p(E \setminus U) \subseteq B \setminus V. \tag{12}$$

Using Lemma 2.4 we can find a $\lambda' \geqslant \lambda$ and an open neighborhood $V_{\lambda'}$ of $B_{0\lambda'}$ in $B_{\lambda'}$ such that $r_{\lambda'}^{-1}(V_{\lambda'}) \subseteq V$, which implies

$$r_{\lambda'}(B \setminus V) \subseteq B_{\lambda'} \setminus V_{\lambda'}. \tag{13}$$

Since $U = q_{\lambda}^{-1}(U'_{\lambda}) = q_{\lambda'}^{-1}\, q_{\lambda\lambda'}^{-1}(U'_{\lambda})$, it follows that $q_{\lambda'}(U) \subseteq q_{\lambda\lambda'}^{-1}(U'_{\lambda})$, which together with (10) implies

$$\operatorname{Cl}(q_{\lambda'}(U)) \subseteq q_{\lambda\lambda'}^{-1}(U_{\lambda}). \tag{14}$$

Furthermore, by (5), (12) and (13), we have $p_{\lambda'}\, q_{\lambda'}(E \setminus U) = r_{\lambda'}\, p(E \setminus U) \subseteq r_{\lambda'}(B \setminus V) \subseteq B_{\lambda'} \setminus V_{\lambda'}$, which implies

$$\operatorname{Cl} q_{\lambda'}(E \setminus U) \subseteq p_{\lambda'}^{-1}(B_{\lambda'} \setminus V_{\lambda'}) \subseteq p_{\lambda'}^{-1}(B_{\lambda'} \setminus B_{0\lambda'}) = E_{\lambda'} \setminus E_{0\lambda'}. \tag{15}$$

By normality of $E_{\lambda'}$, there is an open set $U_{\lambda'}$ in $E_{\lambda'}$ such that

$$\operatorname{Cl}(q_{\lambda'}(E \setminus U)) \subseteq U_{\lambda'} \subseteq \operatorname{Cl}(U_{\lambda'}) \subseteq E_{\lambda'} \setminus E_{0\lambda'}. \tag{16}$$

Now (14) and (16) imply

$$\operatorname{Cl}(q_{\lambda'}(E)) \subseteq q_{\lambda\lambda'}^{-1}(U_{\lambda}) \cup U_{\lambda'}.$$

Using property ($B2$) for $\mathbf{q}$, we can find a $\lambda'' \geqslant \lambda'$ such that

$$q_{\lambda'\lambda''}(E_{\lambda''}) \subseteq q_{\lambda\lambda'}^{-1}(U_{\lambda}) \cup U_{\lambda'}. \tag{17}$$

Finally, (7), (17), (16) and (9) imply

$$\begin{aligned} q_{\lambda\lambda''}(E_{0\lambda''}) = q_{\lambda\lambda'}\, q_{\lambda'\lambda''}(E_{0\lambda''}) &\subseteq q_{\lambda\lambda'}(E_{0\lambda'} \cap q_{\lambda'\lambda''}(E_{\lambda''})) \subseteq \\ &\subseteq q_{\lambda\lambda'}(E_{0\lambda'} \cap q_{\lambda\lambda'}^{-1}(U_{\lambda})) \cup q_{\lambda\lambda'}(E_{0\lambda'} \cap U_{\lambda'}) \subseteq \\ &\subseteq q_{\lambda\lambda'}(E_{0\lambda'}) \cap U_{\lambda} \subseteq E_{0\lambda} \cap U_{\lambda} = U_{0\lambda}. \end{aligned}$$

3. Approximate homotopy liftings and shape fibrations

3.1. *Definition* ([5]). Let $\mathbf{p} = (p_{\lambda}, 1_{\Lambda}) : \mathbf{E} = (E_{\lambda}, q_{\lambda\lambda'}, \Lambda) \to \mathbf{B} = (B_{\lambda}, r_{\lambda\lambda'}, \Lambda)$ be a level map of systems. We say that $\mathbf{p}$ has the aproximate homotopy lifting property ($AHLP$) with respect to a class of spaces $\mathscr{X}$ provided for each $\lambda \in \Lambda$ and for arbitrary normal coverings $\mathscr{U}$ and $\mathscr{V}$ of E_{λ} and B_{λ} respectively, there is a $\lambda' \geqslant \lambda$ and a normal covering $\mathscr{V}'$ of $B_{\lambda'}$ with the following property. Whenever $X \in \mathscr{X}$ and $h : X \to E_{\lambda'}$, $H : X \times I \to B_{\lambda'}$ are maps satisfying

$$(p_{\lambda'} h, H_0) \leqslant \mathscr{V}' \tag{1}$$

then there is a homotopy $\widetilde{H} : X \times I \to E_\lambda$ such that

$$(q_{\lambda\lambda'} h, \widetilde{H}_0) \leqslant \mathscr{U} \tag{2}$$

$$(p_\lambda \widetilde{H}, r_{\lambda\lambda'} H) \leqslant \mathscr{V}. \tag{3}$$

We call λ a lifting index and $\mathscr{V}'$ a lifting mesh for $\lambda, \mathscr{U}$ and $\mathscr{V}$.

3.2. THEOREM. *Let* $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ *be a level map of systems having AHLP with respect to the class of all paracompact spaces X. If all* E_λ *are polyhedra, then* $\mathbf{p}$ *has the stronger homotopy lifting property obtained from Def.* 3.1. *by replacing* (2) *by* $q_{\lambda\lambda'} h = \widetilde{H}_0$.

In the proof we need the following two propositions.

3.3. PROPOSITION. *Let P be a polyhedron and* $\mathscr{U}$ *an open covering of P. Then there is an open covering* $\mathscr{V}$ *of P, which refines* $\mathscr{U}$ *and has the property that any two* $\mathscr{V}$*-near maps* $f, g : X \to P$ *from an arbitrary topological space X into P are* $\mathscr{U}$*-homotopic.*

Proof. Let K be a triangulation of P so fine that the covering $\overline{\{\mathrm{St}\,(v, K) \mid v \in K^\circ\}}$ refines $\mathscr{U}$ (K° denotes the set of vertices of K). We claim that $\mathscr{V} = \{\mathrm{St}\,(v, K) \mid v \in K^\circ\}$ has the desired property. Indeed, let $f, g : H \to P = |K|$ be $\mathscr{V}$-near maps. Then there is a map $h : X \to P$ such that f and h and also h and g are contiguous maps (see the proof of [2], Theorem 2.2). This means that each $x \in X$ admits simplexes $\sigma_x, \sigma'_x \in K$ such that $f(x), h(x) \in \sigma_x$, $h(x), g(x) \in \sigma'_x$. Let

$$H(x, t) = \begin{cases} H_1(x, t), & 0 \leqslant t \leqslant \dfrac{1}{2} \\ H_2(x, t), & \dfrac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

where

$$H_1(x, t) = (1 - 2t) f(x) + 2th(x)$$

$$H_2(x, t) = (2 - 2t) h(x) + (2t - 1) g(x)$$

Clearly, H connects f to g. Moreover, for each $x \in X$ $H(\{x\} \times I) \subseteq \sigma_x \cup \sigma'_x \subseteq \overline{\mathrm{St}\,(v, K)}$ for any vertex v of $\sigma_x \cap \sigma'_x$. Since $\overline{\{\mathrm{St}\,(v, K)} \mid v \in K^\circ\}$ refines $\mathscr{U}$ there is a $U \in \mathscr{U}$ such that $H(\{x\} \times I) \subseteq U$.

3.4. PROPOSITION. *Let X be a paracompact space and $\mathscr{U}$ an open covering of $X \times I$. Then there is a map $\varphi : X \to (0, 1]$ such that each $x \in X$ admits a $U \in \mathscr{U}$ with $\{x\} \times [0, \varphi(x)] \subseteq U$.*

Proof. For $x \in X$ let $U_x \in \mathscr{U}$ be such that $(x, 0) \in U_x$. Then there is an open neighborhood V_x of x in X and a number $t_x \in (0, 1]$ such that $V_x \times [0, t_x] \subseteq U_x$. Clearly, $\mathscr{V} = \{V_x \mid x \in X\}$ is an open covering of X. Let $\mathscr{V}'$ be a locally finite open refinement of $\mathscr{V}$. For $V' \in \mathscr{V}'$ choose a point $x \in X$ such that $V' \subseteq V_x$. Then put $t_{V'} = t_x$. Let $(\Psi_{V'}, V' \in \mathscr{V}')$ be a partition of unity subordinated to the covering $\mathscr{V}'$. Then the desired mapping $\varphi : X \to (0, 1]$ is given by

$$\varphi(x) = \operatorname{Max}\{t_{V'} \Psi_{V'}(x) \mid V' \in \mathscr{V}'\}.$$

Indeed, for each $x \in X$ there is a $V' \in \mathscr{V}'$ such that $\varphi(x) = t_{V'} \Psi_{V'}(x)$. Since $\varphi(x) > 0$, we have $x \in V'$. Moreover, there is an $x' \in X$ such that $t_{V'} = t_{x'}$ and $V' \subseteq V_{x'}$. Consequently,

$$\{x\} \times [0, \varphi(x)] \subseteq V' \times [0, t_{V'}] \subseteq V_{x'} \times [0, t_{x'}] \subseteq U_{x'}.$$

Proof of Theorem 3.2. Let $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ be a level map of systems having the *AHLP* with respect to all paracompact spaces. Let $\lambda \in \Lambda$ and let $\mathscr{V}$ be a normal covering of B_λ. Choose a star-refinement $\mathscr{V}^*$ of $\mathscr{V}$ and let $\mathscr{U}$ be an open covering of E_λ which refines $p_\lambda^{-1}(\mathscr{V}^*)$ and is so fine that any two $\mathscr{U}$-near maps into E_λ are $p_\lambda^{-1}(\mathscr{V}^*)$-homotopic (Proposition 3.3). Let $\lambda' \geqslant \lambda$ be a lifting index and let a normal covering $\mathscr{V}'$ of $B_{\lambda'}$ be a lifting mesh for λ, $\mathscr{U}$, and $\mathscr{V}^*$. If $h : X \to E_{\lambda'}$ and $H : X \times I \to B_{\lambda'}$ are maps satisfying $(p_{\lambda'} h, H_0) \leqslant \mathscr{V}'$, then there is a homotopy $\widetilde{H}' : X \times I \to E_\lambda$ satisfying

$$(p_\lambda \widetilde{H}', r_{\lambda\lambda'} H) \leqslant \mathscr{V}^* \tag{4}$$

and $(q_{\lambda\lambda'} h, \widetilde{H}'_0) \leqslant \mathscr{U}$. By the choice of $\mathscr{U}$ it follows that there is a $p_\lambda^{-1}(\mathscr{V}^*)$-homotopy $\widetilde{H}'' : X \times I \to E_\lambda$ satisfying

$$\widetilde{H}''_0 = q_{\lambda\lambda'} h, \quad \widetilde{H}''_1 = \widetilde{H}'_0. \tag{5}$$

Then $p_\lambda \widetilde{H}'' : X \times I \to B_\lambda$ is a $\mathscr{V}^*$-homotopy. By (4) each $(x, t) \in$ $\in X \times I$ admits a $V^*_{(x,t)} \in \mathscr{V}^*$ such that $p_\lambda \widetilde{H}'(x, t)$, $r_{\lambda\lambda'} H(x, t) \in$ $\in V^*_{(x,t)}$. Consequently, there is an open neighborhood $U_{(x,t)}$ of (x, t) in $X \times I$ such that $p_\lambda \widetilde{H}'(U_{(x,t)}) \subseteq V^*_{(x,t)}$ and $r_{\lambda\lambda'} H(U_{(x,t)}) \subseteq V^*_{(x,t)}$. Hence $\mathscr{W} = \{U_{(x,t)} \mid (x, t) \in X \times I\}$ is an open covering of $X \times I$ such that for every $U \in \mathscr{W}$ there is a $V^* \in \mathscr{V}^*$ satisfying $p_\lambda \widetilde{H}'(U) \subseteq V^*$ and $r_{\lambda\lambda'} H(U) \subseteq V^*$. Using Proposition 3.4, one can find a map $\varphi : X \to (0, 1]$ such that each $x \in X$ admits a $V^* \in \mathscr{V}^*$ such that

$$p_\lambda \widetilde{H}'(\{x\} \times [0, \varphi(x)]) \subseteq V^*, \; r_{\lambda\lambda'} H(\{x\} \times [0, \varphi(x)]) \subseteq V^*. \tag{6}$$

Let us define $\widetilde{H} : X \times I \to E_\lambda$ by

$$\widetilde{H}(x, t) = \begin{cases} \widetilde{H}''\left(x, \dfrac{2t}{\varphi(x)}\right), & 0 \leqslant t \leqslant \dfrac{\varphi(x)}{2} \\ \widetilde{H}'(x, 2t - \varphi(x)), & \dfrac{\varphi(x)}{2} \leqslant t \leqslant \varphi(x) \\ \widetilde{H}'(x, t), & \varphi(x) \leqslant t \leqslant 1 \end{cases} \tag{7}$$

Using (7), (5), (4) and (6) one readily shows that $\widetilde{H}_0 = q_{\lambda\lambda'} h$ and $(p_\lambda \widetilde{H}, r_{\lambda\lambda'} H) \leqslant \mathscr{V}$.

3.5. *Definition.* A map of topological spaces $p : E \to B$ is called a *shape fibration* provided there is a polyhedral level-resolution $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ *of* p such that the level map of systems $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ has the AHLP with respect to the class of all topological spaces.

By [10], Theorem 4, if p is a shape fibration and $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is an arbitrary polyhedral resolution of p, then $\mathbf{p}$ has the *AHLP* with respect to all topological spaces. In [5], Theorem 5.3 it was shown that Definition 3.5 is equivalent to the definition of a shape fibration given by Mardešić in [10]. In particular, one can always assume that the index set Λ of the inverse systems $\mathbf{E}$ and $\mathbf{B}$ is cofinite.

4. Restrictions of a shape fibration

The main result of this section is the following theorem.

4.1. THEOREM. *Let* $p : E \to B$ *be a shape fibration, which is a closed map of a topological space* E *to a normal space* B. *If* $B_0 \subseteq B$ *is a closed subset of* B *and if* B_0 *and* $E_0 = p^{-1}(B_0)$ *are P-embedded in* B *and* E *respectively, then* $p_0 = p \mid E_0 : E_0 \to B_0$ *is also a shape fibration.*

Proof. Let $\mathbf{r} : (B, B_0) \to (\mathbf{B}, \mathbf{Q})$ be a polyhedral resolution of a pair of spaces (B, B_0) ([13], I, § 6.5). Since B_0 is P-embedded in B, the induced morphisms $\mathbf{r} : B \to \mathbf{B}$ and $\mathbf{r}_1 : B_0 \to \mathbf{Q}$ are polyhedral resolutions of B and B_0 respectively ([13], I § 6, Theorem 11). By construction of the resolution $\mathbf{r} : (B, B_0) \to (\mathbf{B}, \mathbf{Q})$ ([13], I § 6, Theorem 10), $\mathbf{r} : B \to \mathbf{B}$ is a canonical resolution of B in the sense of 2. Let $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ be a polyhedral resolution of $p : E \to B$ given by Theorem 2.3 (*ii*). By [5], Lemma 4.6 and Remark 4.7 we can assume that $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is a polyhedral level-resolution of p. Consequently, $\mathbf{q} = (q_\lambda) : E \to \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$, $\mathbf{r} = (r_\lambda) : B \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ are polyhedral resolutions of E and B respectively, and $\mathbf{p} = (p_\lambda, 1_\Lambda) : \mathbf{E} \to \mathbf{B}$ is a level map of systems such that

$$p_\lambda q_\lambda = r_\lambda p, \quad \lambda \in \Lambda. \tag{1}$$

Furthermore, by the construction given in [13], I § 6, Theorem 10, each Q_λ is a closed polyhedral neighborhood of $\mathrm{Cl}(r_\lambda(B_0))$ in B_λ and

$$r_{\lambda\lambda'}(Q_{\lambda'}) \subseteq \mathrm{Int}\, Q_\lambda, \quad \lambda < \lambda'. \tag{2}$$

Using the induction on the number of predecessors of $\lambda \in \Lambda$ (Λ is assumed to be cofinite), one can assign to each λ a closed polyhedral neighborhood C_λ of Q_λ in B_λ such that

$$r_{\lambda\lambda'}(C_{\lambda'}) \subseteq \mathrm{Int}\, Q_\lambda, \quad \lambda < \lambda'. \tag{3}$$

Indeed, let Λ_k be the set of all $\lambda \in \Lambda$ with exactly k predecessors different from λ. If $\lambda \in \Lambda_0$, we take for C_λ an arbitrary closed polyhedral neighborhood of Q_λ in B_λ. Now assume that we have already defined C_λ satisfying (3) for all $\lambda \in \bigcup_{j=0}^{k-1} \Lambda_j$. Let $\lambda \in \Lambda_k$ and let $\lambda_1, \lambda_2, \ldots$ $\ldots, \lambda_k < \lambda$ be all predecessors of λ different from λ. Then $\lambda_i \in \bigcup_{j=0}^{k-1} \Lambda_j$, $i = 1, 2, \ldots, k$, and the closed polyhedral neighborhoods C_{λ_i} have already been constructed. By (2), $r_{\lambda_i\lambda}^{-1}(\mathrm{Int}\, Q_{\lambda_i})$, $i = 1, 2, \ldots, k$, are open neighborhoods of Q_λ in B_λ. Hence, the same is true for $\bigcap_{i=1}^{k} r_{\lambda_i\lambda}^{-1}(\mathrm{Int}\, Q_{\lambda_i})$. Therefore, there exists a closed polyhedral neighborhood C_λ of Q_λ in B_λ such that $C_\lambda \subseteq \bigcap_{i=1}^{k} r_{\lambda_i\lambda}^{-1}(\mathrm{Int}\, Q_{\lambda_i})$. Clearly, C_λ satisfies (3).

By (3), $\mathbf{C} = (C_\lambda, r_{\lambda\lambda'} | C_{\lambda'}, \Lambda)$ is an inverse system of polyhedra. Let $\mathbf{r}_2 : B_0 \to \mathbf{C}$ be given by $r_{2\lambda} = r_\lambda | B_0 : B_0 \to C_\lambda$. We claim that $\mathbf{r}_2$ is a resolution of B_0. It suffices to verify the properties $(B1)$ and $(B2)$ for $\mathbf{r}_2$.

$(B1)$ Let $\mathcal{U}_0$ be a normal covering of B_0. Since B_0 is P-embdded in B, there is a normal covering $\mathcal{U}$ of B such that $\mathcal{U} | B_0$ refines $\mathcal{U}_0$. Since $\mathbf{r} : B \to \mathbf{B}$ satisfies $(B1)$, there is a $\lambda \in \Lambda$ and an open covering $\mathcal{U}_\lambda$ of B_λ such that $r_\lambda^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$. Then $\mathcal{U}_{0\lambda} = \mathcal{U}_\lambda | C_\lambda$ is an open covering of C_λ and $r_{2\lambda}^{-1}(\mathcal{U}_{0\lambda})$ refines $\mathcal{U}_0$.

$(B2)$ Let U be an open neighborhood of $\mathrm{Cl}(r_\lambda(B_0))$ in C_λ. Then $U \cap Q_\lambda$ is an open neighborhood of $\mathrm{Cl}(r_\lambda(B_0))$ in Q_λ. Since $\mathbf{r}_1 : B_0 \to$ $\to \mathbf{Q}$ has the property $(B2)$, there is a $\lambda' \geqslant \lambda$ satisfying $r_{\lambda\lambda'}(Q_{\lambda'}) \subseteq$ $\subseteq U \cap Q_\lambda$. Then by (3), $\lambda'' \geqslant \lambda'$ implies $r_{\lambda\lambda''}(C_{\lambda''}) \subseteq r_{\lambda\lambda'}(\mathrm{Int}\, Q_{\lambda'}) \subseteq U$.

Again, by induction on the number of predecessors of $\lambda \in \Lambda$ different from λ, one can assign to each λ a closed polyhedral neighborhood $B_{0\lambda}$ of C_λ in B_λ in such a way that

$$r_{\lambda\lambda'}(B_{0\lambda'}) \subseteq \mathrm{Int}\, Q_\lambda, \quad \lambda < \lambda' \tag{4}$$

and that

$$\mathbf{r}_0 = (r \mid B_0) : B_0 \to \mathbf{B}_0 = (B_{0\lambda}, r_{\lambda\lambda'} \mid B_{0\lambda'}, \Lambda) \tag{5}$$

is a resolution of B_0.

We now put $P_\lambda = p_\lambda^{-1}(C_\lambda)$ and remark that (3) implies

$$q_{\lambda\lambda'}(P_{\lambda'}) \subseteq \operatorname{Int} P_\lambda, \quad \lambda < \lambda'. \tag{6}$$

Since $\operatorname{Cl}(r_\lambda(B_0)) \subseteq C_\lambda$ it follows by Theorem 2.5 that

$$\mathbf{q}_1 = (q_\lambda \mid E_0) : E_0 \to \mathbf{P} = (P_\lambda, q_{\lambda\lambda'} \mid P_{\lambda'}, \Lambda) \tag{7}$$

is a resolution of E_0.

Arguing as above by induction on the number of predecessors of λ different from λ, one can now assign to each $\lambda \in \Lambda$ a closed polyhedral neighborhood $E_{0\lambda}$ of P_λ in E_λ so that

$$q_{\lambda\lambda}(E_{0\lambda'}) \subseteq \operatorname{Int} P_\lambda, \quad \lambda < \lambda' \tag{8}$$

$$E_{0\lambda} \subseteq p_\lambda^{-1}(\operatorname{Int} B_{0\lambda}), \quad \lambda \in \Lambda \tag{9}$$

$$\mathbf{q}_0 = (q_\lambda \mid E_0) : E_0 \to \mathbf{E}_0 = (E_{0\lambda}, q_{\lambda\lambda'} \mid E_{0\lambda'}, \Lambda) \tag{10}$$

is a polyhedral resolution of E_0.

Now (1), (5), (9) and (10) imply that $(\mathbf{q}_0, \mathbf{r}_0, \mathbf{p}_0)$ is a polyhedral level-resolution of $p_0 : E_0 \to B_0$, where $\mathbf{p}_0 : \mathbf{E}_0 \to \mathbf{B}_0$ is a level-map of systems given by the maps $p_{0\lambda} = p_\lambda \mid E_{0\lambda} : E_{0\lambda} \to B_{0\lambda}$. The theorem will be proved if we show that $\mathbf{p}_0 : \mathbf{E}_0 \to \mathbf{B}_0$ has the *AHLP* with respect to the class of all topological spaces.

Let $\lambda \in \Lambda$ and let $\mathscr{U}_0, \mathscr{V}_0$ be open coverings of $E_{0\lambda}$ and $B_{0\lambda}$ respectively. Then for each $U \in \mathscr{U}_0$ and each $V \in \mathscr{V}_0$ there are open sets U' in E_λ and V' in B_λ such that $U' \cap E_{0\lambda} = U$ and $V' \cap B_{0\lambda} = V$. Clearly, $\mathscr{U} = \{E \setminus E_{0\lambda}, U' \mid U \in \mathscr{U}_0\}$ and $\mathscr{V} = \{B \setminus B_0, V' \mid V \in \mathscr{V}_0\}$ are open coverings of E_λ and B_λ respectively, satisfying $(\mathscr{U} \setminus \{E_\lambda \setminus E_{0\lambda}\}) \mid E_{0\lambda} = \mathscr{U}_{0\lambda}$ and $(\mathscr{V} \setminus \{B \setminus B_{0\lambda}\}) \mid B_{0\lambda} = \mathscr{V}_0$. Let $\mathscr{V}' = \{\operatorname{Int} C_\lambda, B_\lambda \setminus Q_\lambda\}$ and let $\mathscr{W}$ be an open covering of B_λ such that $\mathscr{W}$ refines both $\mathscr{V}$ and $\mathscr{V}'$.

Since $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is a polyhedral level-resolution of the shape fibration p we conclude that $\mathbf{p}$ has the *AHLP* with respect to the class of all topological spaces. Consequently, there is a $\lambda' \geqslant \lambda$ and an open covering $\mathscr{W}'$ of $B_{\lambda'}$ such that λ' is a lifting index and $\mathscr{W}'$ is a lifting mesh for λ, $\mathscr{U}$ and $\mathscr{W}$ with respect to $\mathbf{p}$. We claim that λ' is a lifting index and $\mathscr{W}'_0 = \mathscr{W}' \mid B_{0\lambda'}$ is a lifting mesh for λ, $\mathscr{U}_0$ and $\mathscr{V}_0$ with respect to $\mathbf{p}_0$. Indeed, let X be a topological space and let $h: X \to E_{0\lambda'}$, $H : X \times I \to B_{0\lambda'}$ be mappings satisfying

$$(p_{0\lambda'} h, H_0) \leqslant \mathscr{W}'_0.$$

Let $i : E_{0\lambda'} \to E_{\lambda'}$ and $j : B_{0\lambda'} \to B_{\lambda'}$ be the inclusion maps. Then $ih : X \to E_{\lambda'}$ and $jH : X \times I \to B_{\lambda'}$ are mappings satisfying

$$(p_{\lambda'} ih, j H_0) \leqslant \mathscr{W}'.$$

By the choice of λ' and $\mathscr{W}'$ it follows the existence of a homotopy $\widetilde{H} : X \times I \to E_\lambda$ such that

$$(q_{\lambda\lambda'}\, ih, \widetilde{H}_0) \leqslant \mathscr{U} \tag{11}$$

and

$$(p_\lambda \widetilde{H}, r_{\lambda\lambda'}\, jH) \leqslant \mathscr{W}. \tag{12}$$

Since $\mathscr{W}$ refines $\mathscr{V}'$, (12) implies

$$(p_\lambda \widetilde{H}, r_{\lambda\lambda'}\, jH) \leqslant \mathscr{V}'. \tag{12'}$$

(12′) implies that for each $(x, t) \in X \times I$ either $\{p_\lambda \widetilde{H}(x,t), r_{\lambda\lambda'}\, jH(x,t)\} \subseteq \operatorname{Int} C_\lambda$ or $\{p_\lambda \widetilde{H}(x,t), r_{\lambda\lambda'}\, jH(x,t)\} \subseteq B_\lambda \setminus Q_\lambda$. Since, by (4), $r_{\lambda\lambda'}\, jH(x,t) \in r_{\lambda\lambda'}(B_{0\lambda'}) \subseteq Q_\lambda$, we conclude that $p_\lambda \widetilde{H}(x,t) \subseteq \operatorname{Int} C_\lambda$. Consequently, $\widetilde{H}$ maps $X \times I$ into $p_\lambda^{-1}(C_\lambda) = P_\lambda \subset E_{0\lambda}$. Now, since $q_{\lambda\lambda'}\, ih(X) \subseteq E_{0\lambda}$, (11) implies $\widetilde{H}_0(X) \subseteq E_{0\lambda'}$ i. e. $q_{\lambda\lambda'}\, h(X) \cap (E_\lambda \setminus E_{0\lambda}) = \emptyset$ and $\widetilde{H}_0(X) \cap (E_\lambda \setminus E_{0\lambda}) = \emptyset$. Therefore,

$$(q_{\lambda\lambda'}\, h, \widetilde{H}_0) \leqslant \mathscr{U}_0.$$

Since $\mathscr{W}$ refines $\mathscr{V}$, (12) implies $(p_\lambda \widetilde{H}, r_{\lambda\lambda'}\, jH) \leqslant \mathscr{V}$, or $(p_{0\lambda} \widetilde{H}, r_{\lambda\lambda'} H) \leqslant \mathscr{V}$ because $\widetilde{H}(X \times I) \subseteq E_{0\lambda}$. Since $p_{0\lambda} \widetilde{H}(X \times I) \cap (B_\lambda \setminus B_{0\lambda}) = \emptyset$ and $r_{\lambda\lambda'} H(X \times I) \cap (B_\lambda \setminus B_{0\lambda}) = \emptyset$ it follows that

$$(p_{0\lambda} \widetilde{H}, r_{\lambda\lambda'}\, H) \leqslant \mathscr{V}_0.$$

4.2. COROLLARY. *Let $p : E \to B$ be a shape fibration, which is a closed map, let B_0 be a closed subset of B and let $E_0 = p^{-1}(B_0)$. If E and B are (a) paracompact, (b) collectionwise normal or (c) pseudocompact normal spaces, then $p_0 = p \mid E_0 : E_0 \to B_0$ is also a shape fibration.*

Corollary 4.3 follows immediately from Theorem 4.1 because every closed subset of a space satisfying either one of the conditions (*a*), (*b*) or (*c*) is P-embedded in that space (for (*a*) see [1], Theorem 15.11 and Corollary 17.5, for (*b*) see [1], Corollary 15.7 and for (*c*) see [1], Theorem 15.4).

Since every closed set of a compact Hausdorff space is P-embedded in that space ([18], p. 372) and since every map of compact Hausdorff spaces is closed, Theorem 4.1. also implies the following corollary.

4.3. COROLLARY. *Let $p : E \to B$ be a shape fibration of compact Hausdorff spaces and let B_0 be a closed subset of B, $E_0 = p^{-1}(B_0)$. Then $p_0 = p \mid E_0 : E_0 \to B_0$ is also a shape fibration.*

Notice that Corollary 4.3 is a generalization of Proposition 4 of [11].

5. The exact sequence of a shape fibration

The purpose of this section is to show that every shape fibration induces a certain exact sequence of homotopy pro-groups. This fact is obtained as a corollary of the main result of this paper, which says that a shape fibration $p : E \to B$, which is a closed map of a topological space E into a normal space B, induces an isomorphism of homotopy pro-groups (Theorem 5.7). In the proof we will need the following two facts from [6].

5.1. If Y is an ANR and $\mathscr{U}$ is a given open covering of Y, then there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that any two $\mathscr{V}$-near maps $f, g : X \to Y$ defined on an arbitrary space X are $\mathscr{U}$-homotopic, which we denote by $f \simeq_{\mathscr{U}} g$ ([6], Theorem 1.1, p. 111).

5.2. If Y is an ANR and $\mathscr{U}$ is a given open covering of Y, then there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that for any two $\mathscr{V}$-near maps $f, g : X \to Y$ defined on a metrizable space X and for any $\mathscr{V}$-homotopy $F : A \times I \to Y$ defined on a closed subspace A of X with $F_0 = f \mid A$ and $F_1 = g \mid A$, there exists a $\mathscr{U}$-homotopy $H : X \times I \to Y$ such that $H_0 = f$, $H_0 = g$ and $H \mid A \times I = F$ ([6], Theorem 1.2, p. 112).

By a triple of topological spaces (Y, Y_1, Y_0) we mean a topological space Y and two closed subsets $Y_0 \subseteq Y_1 \subseteq Y$.

5.3. LEMMA. *Let (Y, Y_1, Y_0) be a triple of ANR-spaces, i. e. $Y, Y_1, Y_0 \in ANR$, and let $\mathscr{U}$ be an open covering of Y. Then there exists an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that any two $\mathscr{V}$-near maps of metrizable triples $f, g : (X, X_1, X_0) \to (Y, Y_1, Y_0)$ are $\mathscr{U}$-homotopic maps of triples.*

Proof. Let $\mathscr{S}$ be an open refinement of $\mathscr{U}$ such that for any two $\mathscr{S}$-near maps $f, g : X \to Y$ and any $\mathscr{S}$-homotopy $F : X_1 \times I \to Y$ with $F_0 = f \mid X_1$ and $F_1 = g \mid X_1$, there exists a $\mathscr{U}$-homotopy $H : X \times I \to Y$ such that $H_0 = f$, $H_1 = g$ and $H \mid H_1 \times I = F$ (5.2). We put $\mathscr{S}_1 = \mathscr{S} \mid Y_1$. Let $\mathscr{L}$ be an open refinement of $\mathscr{S}_1$ such that for any two $\mathscr{L}$-near maps $f_1, g_1 : X_1 \to Y_1$ and any $\mathscr{L}$-homotopy $G : X_0 \times I \to Y_1$ with $G_0 = f_1 \mid X_0$, $G_1 = g_1 \mid X_0$, there exists an $\mathscr{S}_1$-homotopy $F' : H_1 \times I \to Y_1$ such that $F'_0 = f_1, F'_1 = g_1$ and $F' \mid X_0 \times I = G$ (5.2). We now put $\mathscr{P} = \mathscr{L} \mid Y_0$. Let $\mathscr{P}'$ be an open refinement of $\mathscr{P}$ with the property that any two P'-near maps into Y_0 are $\mathscr{P}$-homotopic (5.1).

For each $P \in \mathscr{P}'$ there is an open set V_P in Y such that $V_P \cap Y_0 = P$. Then $\mathscr{V}' = \{Y \setminus Y_0, V_P, \mid P \in \mathscr{P}'\}$ is an open covering of Y and $\mathscr{V}' \mid Y_0$ refines $\mathscr{P}'$. Similarly, there is an open covering $\mathscr{V}''$ of Y such that $\mathscr{V}'' \mid Y_1$ refines $\mathscr{L}$. Let $\mathscr{V}$ be an open covering of Y which refines $\mathscr{V}', \mathscr{V}''$ nad $\mathscr{S}$. Then $\mathscr{V}$ also refines $\mathscr{U}$, because $\mathscr{S}$ refines $\mathscr{U}$.

We claim that the covering $\mathscr{V}$ has the required property. Indeed, let $f, g : (X, X_1, X_0) \to (Y, Y_1, Y_0)$ be $\mathscr{V}$-near maps. Then the maps $f \mid X_0, g \mid X_0 : X_0 \to Y_0$ are $\mathscr{V} | Y_0$-near, and therefore also $\mathscr{P}'$-near. By the choice of $\mathscr{P}'$ there is a $\mathscr{P}$-homotopy $G : X_0 \times I \to Y_0$ with $G_0 = f \mid X_0, G_1 = g \mid X_0$. Since $\mathscr{P}$ refines $\mathscr{L}$ we conclude that G is also an $\mathscr{L}$-homotopy. From $(f \mid X_1, g \mid X_1) \leqslant \mathscr{V} \mid Y_1$ it follows $(f |X_1, g| X_1) \leqslant \mathscr{L}$, because $\mathscr{V} \mid Y_1$ refines $\mathscr{L}$. By the choice of $\mathscr{L}$ there is an $\mathscr{S}_1$-homotopy $F' : X_1 \times I \to Y_1$ with $F'_0 = f |X_1, F'_1 | X_1 = g \mid X_1$ and $F' \mid X_0 \times I = G$. Furthermore, F' is an $\mathscr{S}$-homotopy, because $\mathscr{S}_1$ refines $\mathscr{S}$. $(f, g) \leqslant \mathscr{V}$ imply $(f, g) \leqslant \mathscr{S}$, because $\mathscr{V}$ refines $\mathscr{S}$. By the choice of $\mathscr{S}$ it follows that there is a $\mathscr{U}$-homotopy $H : X \times I \to Y$ with $H_0 = f, H_1 = g$ and $H \mid X_1 \times I = F'$. H is a homotopy of triples, because $H(X_1 \times I) = F'(X_1 \times I) \subseteq Y_1$ and $H(X_0 \times I) = F'(X_0 \times I) = G(X_0 \times I) \subseteq Y_0$.

5.4. LEMMA. *Let (P, P_1, P_0) be a triple of polyhedra and let $\mathscr{U}$ be an open covering of P. Then there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that for any metrizable triple (X, X_1, X_0), any two $\mathscr{V}$-near maps of triples $f, g : (X, X_1, X)) \to (P, P_1, P_0)$ are $\mathscr{U}$-homotopic as maps of triples.*

Proof. Let Q be the polyhedron P endowed with the metric topology. We define Q_1 and Q_0 analogously. Then (Q, Q_1, Q_0) is a triple of ANR-spaces [8] and the identity map $i : (P, P_1, P_0) \to (Q, Q_1, Q_0)$ is a homotopy equivalence of triples ([8], Theorem 2.2) with a homotopy inverse $j : (Q, Q_1, Q_0) \to (P, P_1, P_0)$. Let $\mathscr{U}'$ be a star-refinement of $\mathscr{U}$ and let (K, K_1, K_0) be a triangulation of (P, P_1, P_0) so fine that the star-covering $\mathscr{K} = \{\mathrm{St}(v, K) \mid v \in K^\circ\}$ of $P = |K|$ refines $\mathscr{U}'$ ([17], p. 125—126). Since each star is an open set with respect to the metric topology, we conclude that $\mathscr{K}$ is also an open covering of Q. The fact that (Q, Q_1, Q_0) is a triple of ANR-spaces implies the existence of an open covering $\mathscr{V}$ of Q which refines $\mathscr{K}$ and has the property from Lemma 5.3 for maps from (X, X_1, X_0) into (Q, Q_1, Q_0) (Lemma 5.3). The continuity of $i : P \to Q$ implies that $\mathscr{V}$ is also an open covering of P. We claim that $\mathscr{V}$ has the required property.

Let $f, g : (X, X_1, X_0) \to (P, P_1, P_0)$ be two $\mathscr{V}$-near maps. Then if and ig are two $\mathscr{V}$-near maps from (X, X_1, X_0) into (Q, Q_1, Q_0). Consequently, by the choice of the covering $\mathscr{V}$, there is a $\mathscr{K}$-homotopy of triples $H : (X \times I, X_1 \times I, X_0 \times I) \to (Q, Q_1, Q_0)$ with $H_0 = if, H_1 = ig$. Also $jH : (X \times I, X_1 \times I, X_0 \times I) \to (P, P_1, P_0)$ is a $\mathscr{K}$-homotopy of triples, because j and 1_P are contiguous with respect to K. Furthermore,

$$jH : jif \simeq_{\mathscr{K}} jig \tag{1}$$

Since $ji \simeq_{\mathscr{K}} 1_P$ as a homotopy of triples, we have also

$$f \simeq_{\mathscr{K}} jif, \tag{2}$$

$$g \simeq_{\mathscr{K}} jig. \tag{3}$$

(2), (1) are (3) imply

$$f \simeq_{\mathscr{K}} ijf \simeq_{\mathscr{K}} jig \simeq_{\mathscr{K}} g.$$

Since $\mathscr{K}$ refines $\mathscr{U}'$ it follows that

$$f \simeq_{\mathscr{U}'} jif \simeq_{\mathscr{U}'} jig \simeq_{\mathscr{U}'} g. \tag{4}$$

Finally, (4) implies $f \simeq_{\mathscr{U}} g$, because $\mathscr{U}'$ is a star-refinement of $\mathscr{U}$. The last homotopy is a homotopy of triples, because such are all the homotopies in (4).

The notion of a resolution of triples $\mathbf{q} : (E, E_1, E_0) \to (\mathbf{E}, \mathbf{E}_1, \mathbf{E}_0)$ can be defined just like the notion of a resolution of pairs defined in [13]. If we look at the proofs of all the facts used in the proof of Theorem 8, I, § 6 in [13] we see that they remain valid provided we replace everywhere pairs by triples. In particular, the following analogues of Theorem 8 of [13] I § 6 holds.

5.5. PROPOSITION. *Let* $\mathbf{q} : (E, E_1, E_0) \to (\mathbf{E}, \mathbf{E}_1, \mathbf{E}_0)$ *be a resolution of* (E, E_1, E_0). *Then the corresponding inverse system* $[(\mathbf{E}, \mathbf{E}_1, \mathbf{E}_0)]$ *in* $H\,Top^3$ *is associated with* (E, E_1, E_0) *(in the sense of Morita* [15]) *via* $[\mathbf{q}] : (E, E_1, E_0) \to [(\mathbf{E}, \mathbf{E}_1, \mathbf{E}_0)]$.

By a slight modification of Lemma 5 and Theorem 9 of [13], § 6, we also obtain the following fact.

5.6. PROPOSITION. *Let* $\mathbf{q} : (E, E_1, E_0) \to (\mathbf{E}, \mathbf{E}_1, \mathbf{E}_0)$ *be a morphism in pro-*Top^3 *and let* $\mathbf{q} : E \to \mathbf{E}$, $q_1 = \mathbf{q} \mid E_1 : E_1 \to \mathbf{E}_1$ *and* $\mathbf{q}_0 = \mathbf{q} \mid E_0 : E_0 \to \mathbf{E}_0$ *be the induced morphisms in pro-Top. If* $\mathbf{q} : E \to \mathbf{E}$ *is a resolution of* E *and* $\mathbf{q}_1, \mathbf{q}_0$ *have property (B2), then* $\mathbf{q} : (E, E_1, E_0) \to (\mathbf{E}, \mathbf{E}_1, \mathbf{E}_0)$ *is a resolution of the triple* (E, E_1, E_0).

We are now able to prove the main result of this paper.

5.7. THEOREM. *Let* $p : E \to B$ *be a shape fibration which is a closed map of a topological space* E *into a normal space* B. *If* $e \in E$, $b = p(e)$, $F = p^{-1}(b)$ *and if* F *is P-embedded in* E, *then* p *induces an isomorphism of the homotopy pro-groups*

$$\mathbf{p}_* : \text{pro-}\pi_n(E, F, e) \to \text{pro-}\pi_n(B, b).$$

Proof. The proof is patterned after the proof of Theorem 2 of [12].

(*i*) Let $\mathbf{r} : (B, \{b\}) \to (\mathbf{B}, \mathbf{Q})$ be a polyhedral resolution of the pair $(B, \{b\})$. Since $\{b\}$ is P-embedded in B we obtain (as in the proof of Theorem 4.1) a polyhedral level-resolution $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ of $p : E \to B$ with Λ cofinite and a resolution $\mathbf{r}_1 = \mathbf{r} \mid \{b\} : \{b\} \to \mathbf{Q}$ of $\{b\}$. Then, $\mathbf{q} = (q_\lambda) : E \to \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ and $\mathbf{r} = (r_\lambda) : B \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$

are polyhedral resolutions of E and B respectively; $\mathbf{p} = (p_\lambda, 1_\Lambda) : \mathbf{E} \to \mathbf{B}$ is a level map of systems such that $p_\lambda q_\lambda = r_\lambda p$ for each $\lambda \in \Lambda$ and $\mathbf{r}_1 = (r_\lambda | \{b_\lambda\} : \{b\} \to \mathbf{Q} = (Q_\lambda, r_{\lambda\lambda'} | Q_{\lambda\lambda'}, \Lambda)$ is such a resolution that every Q_λ is a closed polyhedral neighborhood of $r_\lambda(b) = b_\lambda$ in B_λ with

$$r_{\lambda\lambda'}(Q_{\lambda'}) \subseteq \operatorname{Int} Q_\lambda, \quad \lambda < \lambda'. \tag{5}$$

Let $e_\lambda = q_\lambda(e)$, $\lambda \in \Lambda$. As in the proof of Theorem 4.1 one can assign (by induction on the number of predecessors of λ) to each $\lambda \in \Lambda$ a closed polyhedral neighborhood C_λ of Q_λ in B_λ such that

$$r_{\lambda\lambda'}(C_{\lambda'}) \subseteq \operatorname{Int} Q_\lambda, \quad \lambda < \lambda' \tag{6}$$

and that $\mathbf{r}_2 = (r_\lambda | \{b\}) : \{b\} \to \mathbf{C} = (C, r_{\lambda\lambda'} | C_{\lambda'}, \Lambda)$ is a polyhedral resolution of $\{b\}$. Again, as in the proof of Theorem 4.1 one constructs neighborhoods D_λ of C_λ in B_λ such that

$$r_{\lambda\lambda'}(D_{\lambda'}) \subseteq \operatorname{Int} Q_\lambda, \quad \lambda < \lambda' \tag{7}$$

and that $\mathbf{r}_2 = (r_\lambda | \{b\}) : \{b\} \to \mathbf{D} = (D_\lambda, r_{\lambda\lambda'} | D_{\lambda'}, \Lambda)$ is a polyhedral resolution of $\{b\}$. As in the proof of Theorem 4.1 we put $P_\lambda = p^{-1}(C_\lambda)$ and see that $\mathbf{q}_1 = (q_\lambda | F) : F \to \mathbf{P} = (P_\lambda, q_{\lambda\lambda'} | P_{\lambda'}, \Lambda)$ is a resolution of $F = p^{-1}(b)$. We then construct closed polyhedral neighborhoods F_λ of P_λ in E_λ such that

$$q_{\lambda\lambda'}(F_{\lambda'}) \subseteq \operatorname{Int} P_\lambda, \quad \lambda < \lambda' \tag{8}$$

$$F_\lambda \subseteq p_\lambda^{-1}(\operatorname{Int} D_\lambda), \quad \lambda \in \Lambda \tag{9}$$

and such that $\mathbf{q}_0 : E \to \mathbf{F} = (F_\lambda, q_{\lambda\lambda'} | F_{\lambda'}, \Lambda)$ is a polyhedral resolution of F.

By (9) we conclude that for each $\lambda \in \Lambda$, $p_\lambda : (E_\lambda, F_\lambda, e_\lambda) \to (B_\lambda, D_\lambda, b_\lambda)$. Therefore, for each $\lambda \in \Lambda$, p_λ induces a homomorphism $p_{\lambda *} : \pi_n(E_\lambda, F_\lambda, e_\lambda) \to \pi_n(B_\lambda, D_\lambda, b_\lambda)$. Furthermore, by Proposition 5.6, we conclude that $\mathbf{q} : (E, F, e) \to (\mathbf{E}, \mathbf{F}, \mathbf{e})$ is a resolution of the triple (E, F, e), and thus, by Proposition 5.5, the inverse system $[(\mathbf{E}, \mathbf{F}, \mathbf{e})]$ in $H\,Top^3$ is associated with (E, F, e). Similarly, we conclude that $[(\mathbf{B}, \mathbf{D}, \mathbf{b})]$ is associated with (B, b). Therefore, the homomorphisms $p_{\lambda *}$ induce a morphism of homotopy pro-groups $\mathbf{p}_* :$ pro-$\pi_n(E, F, e) \to$ pro-$\pi_n(B, b)$ ([14], p. 318).

(*ii*) In order to show that $\mathbf{p}_*$ is an isomorphism, it is sufficient, by Morita's lemma ([16], Theorem 1.1), to show that for each $\lambda \in \Lambda$ there is a $\mu \in \Lambda$, $\mu \geqslant \lambda$, and a homomorphism $g : \pi_n(B_\mu, D_\mu, b_\mu) \to (E_\lambda, F_\lambda, e_\lambda)$ such that the following diagram commutes

$$\begin{tikzcd}[column sep=huge, row sep=large]
\pi_n(E_\lambda, F_\lambda, e_\lambda) \arrow[d, "p_{\lambda *}"'] & \pi_n(E_\mu, F_\mu, e_\mu) \arrow[l, "q_{\lambda\mu *}"'] \arrow[d, "p_{\mu *}"] \\
\pi_n(B_\lambda, D_\lambda, b_\lambda) & \pi_n(B_\mu, D_\mu, b_\mu) \arrow[l, "r_{\lambda\mu *}"] \arrow[ul, dashed, "g"']
\end{tikzcd} \tag{10}$$

Since $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is a polyhedral resolution of the shape fibration $p : E \to B$, we can assume that $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ has the *AHLP* with respect to all topological spaces. Furthermore, since each E_λ is a polyhedron, $\mathbf{p}$ has the stronger lifting property in the sense of Theorem 3.2 with respect to all paracompact spaces.

Let $\lambda \in \Lambda$ and let $\mathscr{V}_\lambda = \{\text{Int}\, C_\lambda, B_\lambda \setminus Q_\lambda\}$. Let $\lambda' \geqslant \lambda$ be a lifting index for λ, $\mathscr{V}_\lambda$ and let $\mathscr{V}'_{\lambda'}$ be an open covering of $B_{\lambda'}$, which is a lifting mesh for λ, $\mathscr{V}_\lambda$. By Lemma 5.4, there is a refinement $\mathscr{V}''_{\lambda'}$ of $\mathscr{V}'_{\lambda'}$ such that any two $\mathscr{V}''_{\lambda'}$-near maps of triples from $(I^n, \partial I^n, J^{n-1})$ into $(B_{\lambda'}, D_{\lambda'}, b_{\lambda'})$ are $\mathscr{V}'_{\lambda'}$-homotopic as maps of triples, where $J^{n-1} = (\partial I^{n-1} \times I) \cup \cup (I^{n-1} \times 1)$. Let $\mathscr{V}_{\lambda'} = \{\text{Int}\, C_{\lambda'}, B_{\lambda'} \setminus Q_{\lambda'}\}$ and let $\mathscr{W}_{\lambda'}$ be an open covering of $B_{\lambda'}$, which refines both the coverings $\mathscr{V}_{\lambda'}$ and $\mathscr{V}''_{\lambda'}$. Then $\mathscr{W}_{\lambda'}$ refines also $\mathscr{V}'_{\lambda'}$ and so $\mathscr{W}_{\lambda'}$ is a lifting mesh for λ and $\mathscr{V}_\lambda$. Finally, let $\mu \in \Lambda$, $\mu \geqslant \lambda'$, be a lifting index and let the open covering $\mathscr{V}_\mu$ of B_μ be a lifting mesh for λ' and $\mathscr{W}_{\lambda'}$.

Let $\alpha \in \pi_n(B_\mu, D_\mu, b_\mu)$ be given by a map $\Phi : (I^n, \partial I^n, J^{n-1}) \to \to (B_\mu, D_\mu, b_\mu)$ and let $\varphi : J^{n-1} \to E_\mu$ be the constant map $\varphi(J^{n-1}) = = e_\mu$. Notice that $p_\mu \varphi = \Phi \mid J^{n-1}$, and therefore

$$(p_\mu \varphi, \Phi \mid J^{n-1}) \leqslant \mathscr{V}_\mu. \tag{11}$$

Since $(I^n, J^{n-1}) \approx (I^n, I^{n-1} \times 0)$, one can view φ as a map $I^{n-1} \times \times 0 \to E_\mu$ and Φ as a homotopy $I^{n-1} \times I \to B_\mu$ with the initial stage equal to $\Phi \mid J^{n-1}$. Therefore, by (11) and by the choice of μ and $\mathscr{V}_\mu$ there is a map $\tilde{\Phi} : I^n \to E_{\lambda'}$ such that

$$\tilde{\Phi} \mid J^{n-1} = q_{\lambda'\mu}\varphi = e_{\lambda'} \tag{12}$$

$$(p_{\lambda'}\tilde{\Phi}, r_{\lambda'\mu}\Phi) \leqslant \mathscr{W}_{\lambda'}. \tag{13}$$

Since $\mathscr{W}_{\lambda'}$ refines $\mathscr{V}_{\lambda'}$ (13) implies

$$(p_{\lambda'}\tilde{\Phi}, r_{\lambda'\mu}\Phi) \leqslant \mathscr{V}_{\lambda'} = \{\text{Int}\, C_{\lambda'}, B_{\lambda'} \setminus Q_{\lambda'}\}. \tag{13'}$$

By (7) we have $r_{\lambda'\mu}\Phi(\partial I^n) \subseteq r_{\lambda'\mu}(D_\mu) \subseteq Q_{\lambda'\mu}$, which implies $r_{\lambda'\mu}\Phi(\partial I^n) \cap \cap (B_{\lambda'} \setminus Q_{\lambda'}) = \emptyset$. Now (13') implies $p_{\lambda'}\tilde{\Phi}(\partial I^n) \subseteq C_{\lambda'}$, i. e. $\tilde{\Phi}(\partial I^n) \subseteq p_{\lambda'}^{-1}(C_{\lambda'}) = P_{\lambda'} \subseteq F_{\lambda'}$. Thus, we conclude, by (12) that

$\Phi : (I^n, \partial I^n, J^{n-1}) \to (E_{\lambda'}, F_{\lambda'}, e_{\lambda'})$. Therefore, $[\Phi] \in \pi_n (E_{\lambda'}, F_{\lambda'}, e_{\lambda'})$. We now define g by

$$g(a) = g([\Phi]) = [q_{\lambda\lambda'} \widetilde{\Phi}] = q_{\lambda\lambda'_*} [\widetilde{\Phi}]. \tag{14}$$

(*iii*) We will now show that g is independent of the choice of $\widetilde{\Phi}$ and Φ. Let $\Phi' : (I^n, \partial I^n, J^{n-1}) \to (B_\mu, D_\mu, b_\mu)$ be another representative of $a = [\Phi]$ and let $\widetilde{\Phi}'$ satisfy (12) and (13) with Φ, $\widetilde{\Phi}$ replaced by Φ', $\widetilde{\Phi}'$ respectively. Then $\Phi \simeq \Phi'$, and thus there is a homotopy

$$H : (I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (B_\mu, D_\mu, b_\mu)$$

such that $H_0 = \Phi$ and $H_1 = \Phi'$.

We now consider the map $h : (I^n \times 0) \cup (I^n \times 1) \cup (J^{n-1} \times \times I) \to E_{\lambda'}$ given by

$$h \mid I^n \times 0 = \widetilde{\Phi},\ h \mid I^n \times 1 = \widetilde{\Phi}',\ h \mid J^{n-1} \times I = e_{\lambda'}.$$

It is easy to see that h is continuous and that

$$(p_{\lambda'} h, r_{\lambda'\mu} H) \leqslant \mathscr{W}_{\lambda'}.$$

By the choice of λ' and $\mathscr{W}_{\lambda'}$, it follows the existence of a homotopy $\widetilde{H} : I^n \times I \to E_\lambda$ with

$$\widetilde{H} \mid I^n \times 0 = q_{\lambda\lambda'} h \mid I^n \times 0 = q_{\lambda\lambda'} \widetilde{\Phi} \tag{15}$$

$$\widetilde{H} \mid I^n \times 1 = q_{\lambda\lambda'} h \mid I^n \times 1 = q_{\lambda\lambda'} \widetilde{\Phi}' \tag{16}$$

$$\widetilde{H} \mid J^{n-1} \times I = q_{\lambda\lambda'} h \mid J^{n-1} \times I = e_\lambda \tag{17}$$

$$(p_\lambda \widetilde{H}, r_{\lambda\mu} H) \leqslant \mathscr{V}_\lambda = \{\mathrm{Int}\, C_\lambda, B_\lambda \setminus Q_\lambda\}. \tag{18}$$

Since $H(\partial I^n \times I) \subseteq D_\mu$ (7) implies $r_{\lambda\mu} H(\partial I^n \times I) \subseteq Q_\lambda$. Therefore, $r_{\lambda\mu} H(\partial I^n \times I) \cap (B_\lambda \setminus Q_\lambda) = \emptyset$. By (18) it follows that $p_\lambda H(\partial I^n \times I) \subseteq \mathrm{Int}\, C_{\lambda'}$ which implies that $H(\partial I^n \times I) \subseteq F_\lambda$. Thus, we conclude that $\widetilde{H} : (I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (E_\lambda, F_\lambda, e_\lambda)$. (15) and (16) imply

$$\widetilde{H} : q_{\lambda\lambda'} \widetilde{\Phi} \simeq q_{\lambda\lambda'} \widetilde{\Phi}'.$$

Consequently,

$$g([\Phi]) = [q_{\lambda\lambda'} \widetilde{\Phi}] = [q_{\lambda\lambda'} \widetilde{\Phi}'] = g([\Phi']).$$

(*iv*) We now show that g is a homomorphism of groups. Let $\alpha = \alpha' \alpha''$ and let $\alpha' = [\Phi']$, $\alpha'' = [\Phi'']$. Then $\alpha = [\Phi]$, where $\Phi : (I^n, \partial I^n, J^{n-1}) \to (B_\mu, D_\mu, b_\mu)$ is given by

$$\Phi(x, s, t) = \begin{cases} \Phi'(x, 2s, t), & 0 \leqslant s \leqslant \frac{1}{2} \\ \Phi''(x, 2s-1, t), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases} \tag{19}$$

where $x \in I^{n-2}$, $t \in I$. Notice that Φ', Φ'' induce $\widetilde{\Phi}', \widetilde{\Phi}'' : (I^n, \partial I^n, J^{n-1}) \to (E_{\lambda'}, F_{\lambda'}, e_{\lambda'})$ and the analogues of (12) and (13) hold. Let $\widetilde{\Phi} : (I^n, \partial I^n, J^{n-1}) \to (E_{\lambda'}, F_{\lambda'}, e_{\lambda'})$ be defined by

$$\widetilde{\Phi}(x, s, t) = \begin{cases} \widetilde{\Phi}'(x, 2s, t), & 0 \leqslant s \leqslant \frac{1}{2} \\ \widetilde{\Phi}''(x, 2s-1, t), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases} \tag{20}$$

where $x \in I^{n-2}$, $t \in I$. From (19), (20) and from (12), (13) applied to $\widetilde{\Phi}'$ and $\widetilde{\Phi}''$, one obtains (12) and (13) for $\widetilde{\Phi}$, which proves

$$g([\Phi]) = q_{\lambda\lambda' *}([\widetilde{\Phi}]).$$

However, by (20), $[\widetilde{\Phi}] = [\widetilde{\Phi}'][\widetilde{\Phi}'']$, and thus we obtain $g(\alpha' \alpha'') = g(\alpha) = g([\Phi]) = q_{\lambda\lambda' *}([\widetilde{\Phi}]) = q_{\lambda\lambda' *}([\widetilde{\Phi}'])\, q_{\lambda\lambda' *}([\widetilde{\Phi}'']) = g(\alpha')\, g(\alpha'')$. Let us establish the commutativity of diagram (10).

(*v*) First we show that

$$p_{\lambda *}\, g = r_{\lambda\mu *}.$$

If $\alpha = [\Phi] \in \pi_n(B_\mu, D_\mu, b_\mu)$, then

$$p_{\lambda *}\, g(\alpha) = p_{\lambda *}\, q_{\lambda\lambda' *}([\widetilde{\Phi}]) = [p_\lambda\, q_{\lambda\lambda'} \widetilde{\Phi}] \tag{21'}$$

$$r_{\lambda\mu *}(\alpha) = r_{\lambda\mu *}([\Phi]) = [r_{\lambda\mu} \Phi]. \tag{21''}$$

Since $\mathscr{W}_{\lambda'}$ refines $\mathscr{V}''_{\lambda'}$ (13) implies $(p_{\lambda'} \widetilde{\Phi}, r_{\lambda'\mu} \Phi) \leqslant \mathscr{V}''_{\lambda'}$. By the choice of $\mathscr{V}''_{\lambda'}$, it follows that there is a $\mathscr{V}'_{\lambda'}$-homotopy $G : (I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (B_{\lambda'}, D_{\lambda'}, b_{\lambda'})$ with $G : p_{\lambda'} \widetilde{\Phi} \simeq r_{\lambda'\mu} \Phi$. Then $r_{\lambda\lambda'} G : r_{\lambda\lambda'}\, p_{\lambda'} \widetilde{\Phi} \simeq r_{\lambda\mu} \Phi$. Since $r_{\lambda\lambda'}\, p_{\lambda'} = p_\lambda\, q_{\lambda\lambda'}$, it follows $p_\lambda\, q_{\lambda\lambda'} \widetilde{\Phi} \simeq r_{\lambda\mu} \Phi$. With this in mind, (21') and (21'') imply (21).

(*vi*) We now show that $g p_{\mu *} = q_{\lambda\mu *}$.

Let $\beta \in \pi_n(E_\mu, F_\mu, e_\mu)$ be given by a map $\varphi : (I^n, \partial I^n, J^{n-1}) \to (E_\mu, F_\mu, e_\mu)$, i. e. $\beta = [\varphi]$, and let $p_{\lambda *}(\beta) = [\Phi]$, where $\Phi = p_\mu \varphi$.

We put $\widetilde{\Phi} = q_{\lambda'\mu}\,\varphi$. It is easy to see that $\widetilde{\Phi} \mid J^{n-1} = e_{\lambda'}$ and $p_{\lambda'}\widetilde{\Phi} = r_{\lambda'\mu}\Phi$, i. e. (12) and (13) hold. Therefore, $g([\Phi]) = q_{\lambda\lambda'*}([\widetilde{\Phi}])$, which means that $g\,p_{\mu*}(\beta) = q_{\lambda\mu*}(\beta)$. This proves the theorem.

If we pass to the shape groups

$$\check{\pi}_n(E, F, e) = \lim_{\leftarrow} \text{pro-}\pi_n(E, F, e)$$

$$\check{\pi}_n(B, b) = \lim_{\leftarrow} \text{pro-}\pi_n(B, b)$$

then we obtain from Theorem 5.7 the following corollary.

5.8. COROLLARY. *Let $p : E \to B$ be a shape fibration, which is a closed map of topological space E into a normal space B. If $e \in E$, $b = p(e)$ and if $F = p^{-1}(b)$ is P-embedded in E, then p induces an isomorphism of the shape groups*

$$p_* : \check{\pi}_n(E, F, e) \to \check{\pi}_n(B, b).$$

In [7], 5.2, it is shown that whenever $(\mathbf{E}, \mathbf{F}, \mathbf{e})$ is an object in pro--HCW_0^2, then the following sequence of homotopy progroups is exact.

$$\ldots \to \text{pro-}\pi_n(F, e) \to \text{pro-}\pi_n(E, e) \to \text{pro-}\pi_n(E, F, e) \to \text{pro-}\pi_{n-1}(F, e) \to.$$

Hence, Theorem 5.7 yields the following result.

5.9. THEOREM. *Let $p : E \to B$ be a shape fibration, which is a closed map of a topological space E into a normal space B. If $e \in E$, $b = p(e)$, and if $F = p^{-1}$ is P-embedded in E, then the following sequence of homotopy pro-groups is exact*

$$\ldots \to \text{pro-}\pi_n(F, e) \xrightarrow{\mathbf{i}_*} \text{pro-}\pi_n(E, e) \xrightarrow{\mathbf{p}_*} \text{pro-}\pi_n(B, b) \xrightarrow{\delta} \text{pro-}\pi_{n-1}(F, e) \to \ldots$$

Hence $\mathbf{i}_*$ and $\mathbf{p}_*$ are morphisms of pro-groups induced by the inclusion map $i : F \to E$ and by the map $p : E \to B$ respectively, and δ is the composition of the inverse of the isomorphism of pro-groups induced by $p : (E, F, e) \to (B, b, b)$ (Theorem 5.7) and of the boundary morphism pro-$\pi_n(E, F, e) \to$ pro-$\pi_{n-1}(F, e)$ induced by the boundary homomorphisms $\pi_n(E_\lambda, F_\lambda, e_\lambda) \to \pi_{n-1}(F_\lambda, e_\lambda)$.

5.10. COROLLARY. *Let $p : E \to B$ be a closed map of metric ANR spaces (not necessarily locally compact), which has the AHLP in the sense of Coram and Duvall* [3]. *If $e \in E$, $b = p(e)$, $F = p^{-1}(b)$, then the following sequence is exact*

$$\ldots \to \text{pro-}\pi_n(F, e) \xrightarrow{\mathbf{i}_*} \pi_n\,)E, e) \xrightarrow{\mathbf{p}_*} \pi_n(B, b) \xrightarrow{\delta} \text{pro-}\pi_{n-1}(F, e) \to \ldots$$

Proof. By [10], Corollary 4, p is a closed shape fibration and the assertion follows immediately from Theorem 5.9.

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EGZAKTAN NIZ FIBRACIJE OBLIKA

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Sadržaj

Koristeći definiciju fibracije oblika između proizvoljnih topoloških prostora iz [5], dokazane su slijedeće činjenice:

Neka je $p : E \to B$ zatvoreno preslikavanje topološkog prostora E u normalni prostor B koje je fibracija oblika. Tada

(*i*) Ako je B_0 zatvoren podskup od B, $E_0 = p^{-1}(B_0)$ i ako su E_0 i B_0 P-smješteni u E odnosno B, onda je i restrikcija $p \mid E_0 : E_0 \to B_0$ fibracija oblika. (Teorema 4.1).

(*ii*) Ako je $e \in E$, $b = p(e)$ i $F = p^{-1}(b)$ P-smješten u E, onda p inducira izomorfizam homotopskih pro-grupa

$$\mathbf{p}_* : \text{pro-}\pi_n(E, F, e) \to \text{pro-}\pi_n(B, b).$$

(Teorema 5.7). Kao korolar od (*ii*) dobivamo slijedeći egzaktan niz fibracije oblika

$$\ldots \to \text{pro-}\pi_n(F, e) \to \text{pro-}\pi_n(E, e) \to \text{pro-}\pi_n(B, b) \to \text{pro-}\pi_{n-1}(F, e) \to \ldots$$

(Teorema 5.9).