GLASNIK MATEMATIČKI
Vol. 18 (38) (1983), 149-152.

ON THE SEPARATION AXIOM R_0

Jingcheng Tong, Detroit, Michigan, USA

Abstract. In this paper a characteristic property of the separation axiom R_0 is established, which shows the symmetry of R_0 in another sense; a new separation axiom R_T is introduced, it is strictly weaker than R_0 and stronger than all the axioms given by D. N. Misra and K. K. Dube [7].

1. *Introduction.* The separation axiom R_0 was introduced by N. A. Shanin [10]: A topological space (X, τ) is R_0 iff for each $U \in \tau$, $x \in U$ implies $\overline{\{x\}} \subset U$. Many authors have studied the separation axiom R_0 (cf. [1 – 9]). Several characterizations were given in [4], [5], [8]. A few separation axioms weaker then R_0 were given in [7]. In this paper, we give another characterization of R_0, and a new separation axiom R_T, which is weaker then R_0 and stronger than all the axioms in [7].

2. *Characterizations of* R_0. Let X be a topological space, $x \in X$. Let $\{\hat{x}\}$ denote the intersection of all open sets containing x, $\overline{\{x\}}$ be the closure of $\{x\}$. The following Lemma 1 is Theorem 2.2 (*b*) in [5], Lemma 2 is a special case of Theorem 2.2 (c).

LEMMA 1. *A topological space* X *is* R_0 *iff* $\overline{\{x\}} \subset \{\hat{x}\}$ *for all* $x \in X$.

LEMMA 2. *A topological space* X *is* R_0 *iff* $\overline{\{x\}} = \{\hat{x}\}$ *for all* $x \in X$.

Since $\overline{\{x\}}$ is the intersection of all closed sets containing x, Lemma 2 suggests a natural definition of R_0.

Definition 1. A topological space X is R_0 iff for all $x \in X$, the intersection of all open sets containing x coincides with the intersection of all closed sets containing x.

Lemma 1 and the following Theorem 1 show the symmetry of R_0 in another sense.

THEOREM 1. *In a topological space* X, *if for each* $x \in X$, *we have* $\overline{\{x\}} \supset \{\hat{x}\}$, *then* X *is* R_0.

Mathematics subject classifications (1980): Primary 54D10.
Key words and phrases: Separation axioms R_0, R_{ys}, R_y, R_D, R_{UD}.

Proof. If $y \in \overline{\{x\}}$, then $x \in \{\hat{y}\} \subset \overline{\{y\}}$, hence X is symmetric, X is R_0.

3. *The separation axiom* R_T. There are four separation axioms weaker than R_0 in [7]: R_{ys}, R_y, R_D, R_{UD}. It has been proved that $R_{ys} \Rightarrow R_y \Rightarrow R_{UD}$, $R_D \Rightarrow R_{UD}$. We write the definitions of R_{ys} and R_D in the following.

Definition 2. A topological space X is R_{ys} iff for $x, y \in X$, $\overline{\{x\}} \neq \overline{\{y\}}$ implies $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset, \{x\}$ or $\{y\}$.

Definition 3. A topological space X is R_D iff for $x \in X$, $\overline{\{x\}} \cap \cap \{\hat{x}\} = \{x\}$ implies that $\{x\}' = \overline{\{x\}} \setminus \{x\}$ is closed.

From Lemma 1,2 and Theorem 1 we know that if X is not R_0, then there are some x, $\{\hat{x}\} \setminus \overline{\{x\}} \neq \emptyset$, and there are some x, $\overline{\{x\}} \setminus \setminus \{\hat{x}\} \neq \emptyset$. This suggests a new separation axiom. In the following definition, a set that contains at most one point is called to be degenerate.

Definition 4. A topological space is R_T iff for each $x \in X$, both $\{\hat{x}\} \setminus \overline{\{x\}}$ and $\overline{\{x\}} \setminus \{\hat{x}\}$ are degenerate.

Obviously $R_0 \Rightarrow R_T$.

Example 1. $R_T \not\Rightarrow R_0$. Let $X = \{a, b, c, d\}$ with topology $\{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$. Then X is R_T but not R_0 since $\overline{\{a\}} \setminus \setminus \{\hat{a}\} = \{b\}$.

Example 2. R_D; $R_{ys} \not\Rightarrow R_T$. Let $X = \{a, b, c\}$ with topology $\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then X is R_D and R_{ys}, but X is not R_T since $\overline{\{a\}} = X$ and $\{\hat{a}\} = \{a\}$.

Example 3. $T_0 \not\Rightarrow R_T$. Let $X = \{a, b, c\}$ with topology $\{\emptyset, \{a\}, \{a, b\}, X\}$. Then X is T_0 but not R_T since $\overline{\{a\}} = X$ and $\{\hat{a}\} = \{a\}$.

THEOREM 2. $R_T \Rightarrow R_D$.

Proof. If X is R_T, and denote $\langle x \rangle = \overline{\{x\}} \cap \{\hat{x}\}$, then $\overline{\{x\}} = \langle x \rangle \cup \cup D$, $\{\hat{x}\} = \langle x \rangle \cup E$, where D, E are degenerate sets and $D \not\subset \{\hat{x}\}$, $E \not\subset \overline{\{x\}}$. If $\langle x \rangle = \{x\}$, then $\overline{\{x\}} = \{x\} \cup D$, $\{\hat{x}\} = \{x\} \cup E$. We prove that $\overline{\{x\}}' = \overline{\{x\}} \setminus \{x\} = D$ is a closed set. Let U be an open set containing $\{\hat{x}\}$. Then $X \setminus U$ is a closed set, and $(X \setminus U) \cap \overline{\{x\}} = D$ or $\emptyset$. If $(X \setminus U) \cap \overline{\{x\}} = D$, then D is the intersection of two closed sets hence is also closed. If $(X \setminus U) \cap \overline{\{x\}} = \emptyset$, then $\overline{\{x\}} \subset U$, $D \subset U$. Since $D \not\subset \{\hat{x}\}$, there is an open set V such that $x \in V$ and $D \not\subset V$. Then $\overline{\{x\}} \cap (X \setminus V) = D$ is a closed set. Therefore $\{x\}'$ is closed whenever $\langle x \rangle = \{x\}$, X is R_D.

THEOREM 3. $R_T \Rightarrow R_{ys}$.

Proof. Let X be R_T and $x, y \in X$. If $\overline{\{x\}} \neq \overline{\{y\}}$ and there is an $a \in X$ such that $a \neq x$, $a \neq y$ but $a \in \overline{\{x\}} \cap \overline{\{y\}}$, then $a \in \overline{\{x\}}$, $a \in \overline{\{y\}}$ hence $x \in \{\hat{a}\}$, $y \in \{\hat{a}\}$. Since $\{\hat{a}\} = \langle a \rangle \cup E$, where E is a degenerate set and $E \not\subset \overline{\{a\}}$, thereare following four possible cases for $x \in \{\hat{a}\}$, $y \in \{\hat{a}\}$:

(i) $x \in \langle a \rangle$ and $y \in \langle a \rangle$. We have $x \in \overline{\{a\}}$, $y \in \overline{\{a\}}$, but $a \in \overline{\{x\}}$, $a \in \overline{\{y\}}$, hence $\overline{\{x\}} = \overline{\{y\}} = \overline{\{a\}}$, impossible.

(ii) $\{x\} = E$ and $y \in \langle a \rangle$. We have $x \notin \overline{\{a\}}$ and $y \in \overline{\{a\}}$. Since $a \in \overline{\{y\}}$, we have $\overline{\{y\}} = \overline{\{a\}}$. There are two cases to discuss about the relationship between y and $\overline{\{x\}}$. (1) $y \in \overline{\{x\}}$. Then $\overline{\{a\}} = \overline{\{y\}}$. Since $x \notin \overline{\{a\}}$, $x \in X \setminus \overline{\{a\}}$, where $X \setminus \overline{\{a\}}$ is open, $\{\hat{x}\} \in X \setminus \overline{\{a\}}$, $\overline{\{x\}} \setminus \{\hat{x}\} \supset \overline{\{a\}} \supset \{y, a\}$, hence $\overline{\{x\}} \setminus \{\hat{x}\}$ is not a degenerate set, a contradiction to the fact that X is R_T. (2) $y \notin \overline{\{x\}}$. Since $y \in \overline{\{a\}}$ and $a \in \overline{\{x\}}$, we have $y \in \overline{\{x\}}$, a contradiction.

(iii) $x \in \langle a \rangle$ and $\{y\} = E$. Similar to Case (ii).

(iv) $\{x\} = \{y\} = E$. We have $\overline{\{x\}} = \overline{\{y\}}$, impossible.

Therefore if $\overline{\{x\}} \neq \overline{\{y\}}$, we have $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$, $\{x\}$ or $\{y\}$, X is R_{ys}.

Acknowledgement. The author thanks the referee for his valuable help.

REFERENCES:

[1] *K. Császár*, Untersuchungen über Trennungsaxiome, Publ. Math. **14** (1967), 353—364.

[2] ————, New results on separation axioms, Proceedings of the International Symposium on Topology and Its Applications, Beograd 1969, 118—120.

[3] ————, On F. Riesz' separation axiom, Topics in Topology edited by *Á.* Császár, Colloquia Matematica Societatis Janos Bolyai 8, North Holland 1972, 173—180.

[4] *A. S. Davis*, Indexed systems of neighborhoods for general topological spaces, Amer. Math. Monthly **68** (1961), 886—893.

[5] *K. K. Dube*, A note on R_0-topological spaces, Mat. Vesnik **11** (26) (1974), 203——208.

[6] *M. W. Lodato*, On topologically induced generalized proximity relations, Proc. Amer. Math. Soc. **15** (1964), 417—422.

[7] *D. N. Misra* and *K. K. Dube*, Some axioms weaker than the R_0-axiom. Glasnik Mat. Ser. III **8** (1973), 145—147.

[8] *K. Morita*, On the simple extension of a space with respect to a uniformity I, Proc. Japan Acad. **27** (1951), 65—72.

[9] *S. A. Naimpally*, On R_0-topological spaces, Ann. Univ. Sci. Budapest Eötvös, Sect. Math. **10** (1967), 53—64.

[10] *S. A. Shanin,* On separation in topological space, Doklady Akad. Nauk. USSR **38** (1943), 110—113.

(Received November 6, 1981)
(Revised April 16, 1982)

Department of Mathematics
Wayne State University
Detroit, Michigan 48202
and
Institute of Mathematics
Academia Sinica
Peking, China

O AKSIOMU SEPARACIJE R_0

Jingcheng Tong, Detroit, Michigan, SAD

Sadržaj

U članku je nađeno karakteristično svojstvo aksioma separacije R_0. Nadalje je uveden novi aksiom separacije R_T koji je slabijio d R_0 i jači od svih aksioma uvedenih u referenciji [7].