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A CHARACTERIZATION OF REAL AND COMPLEX HILBERT SPACE

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Abstract. Let X be a real or complex Banach space and $L(X)$ the algebra of all bounded linear operators on X. Suppose there exists a $*$-algebra $B(X) \subset L(X)$ which contains the identity operator I and all linear operators with finite dimensional range. The main result: If each operator $U \in B(X)$ with the property $U^*U = UU^* = I$ has norm one then X is a Hilbert space.

This work is a continuation of our earlier research (see [6] and [7]). Throughout this paper we denote by $L(X)$ the algebra of all bounded linear operators on a real or complex Banach space X. We write $B(X)$ for each subalgebra of $L(X)$ which contains the identity operator I and all linear operators with finite dimensional range. By involution we mean a linear (in the complex case a conjugate linear) mapping $A \mapsto A^*$ on $L(X)$ or $B(X)$ such that $(AB)^* = B^* A^*$ and $A^{**} = A$. Let a $B(X)$ equipped with an involution be given. An operator $U \in B(X)$ will be called unitary if $U^*U = UU^* = I$, and $A \in B(X)$ will be called normal if $A^* A = AA^*$.

Let X be such a real or complex Banach space that there exists an involution $A \mapsto A^*$ on $L(X)$ satisfying the condition $AA^* \neq 0$ for each nonzero $A \in L(X)$. According to the classical result of S. Kakutani and G. W. Mackey (see [2] and [3]) there exists an inner product on X such that the corresponding norm is equivalent to the given norm on X and that A^* is the adjoint of A relative to the inner product. Recently J. Bognár obtained a simple and elementary proof of this result (see [1]). It seems natural to ask under what conditions concerning an involution on $L(X)$ does there exist an inner product on X such that the norm induced by this inner product coincides with the given norm. By our knowledge the first result in this direction is due to N. Prijatelj [4], who proved the following theorem.

THEOREM 1 (N. Prijatelj [4]). *Let X be a complex Banach space. If $L(X)$ is a B^*-algebra then there exists an inner product on X such that the norm induced by this inner product is equal to the given norm.*

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For an improvement of N. Prijatelj's result see [6]. The main purpose of this paper is to prove the result below which also characterizes Hilbert space among all Banach spaces.

THEOREM 2. *Let X be a real or complex Banach space. Suppose that there exists an involution $A \mapsto A^*$ on some $B(X)$. If each unitary operator has norm one then there exists an inner product on X such that the corresponding norm is equal to the given norm. For each $A \in B(X)$ the relation $(Ax, y) = (x, A^* y)$ is fulfilled for all pairs $x, y \in X$.*

For the proof of Theorem 2 we need the result below.

THEOREM 3. *Let X be a real or complex Banach space. Suppose there exists an involution $A \mapsto A^*$ on some $B(X)$. If $P^*P \neq 0$ is fulfilled for each one-dimensional projector P then an inner product can be introduced into X with norm equivalent to the given norm and such that A^* is the adjoint of A relative to the inner product.*

Proof. See the proof of Theorem 4 in [7].

The proof of the lemma below is simple and will therefore be omitted.

LEMMA 4. *Let X be a real or complex Hilbert space. For arbitrary pairs $x, y \in X$, $\|x\| = \|y\| = 1$ there exists a unitary operator U, contained in each $B(X)$, such that $Ux = y$.*

Proof of Theorem 2. First we shall prove that for an arbitrary projector $P \in B(X)$ the relation $P^*P = 0$ implies $P = 0$. Suppose on the contrary that there exists $P \neq 0$ such that $P^*P = 0$. Then a routine calculation shows that for any real number t the operator U_t defined by the relation $U_t = I + (\exp t - 1) P + (\exp(-t) - 1) P^* - \frac{1}{2}(\exp t + \exp(-t) - 2) PP^*$ is unitary. Let $e \in X$ be such that $Pe = e$, $\|e\| = 1$. Then $P^*P = 0$ implies $P^*e = 0$. Using this we obtain $U_t e = (\exp t) e$. Therefore $\|U_t\| \geqslant \|U_t e\| = \exp t$. This inequalityi s in contradiction with the requirement of the theorem that $\|U_t\| = 1$. So the implication $P^*P = 0 \Rightarrow P = 0$ is proved. Hence by Theorem 3 there exists an inner product on X such that the corresponding norm is equivalent to the given norm and that the involution induced by the inner product coincides with the given involution. Denote the given norm by $\|\cdot\|$ and the norm induced by the inner product by $\|\cdot\|_0$. We have to prove that there exists such an inner product on X that the corresponding norm is equal to the given norm. It sufficies to prove that there exists a constant k such that

$$\|x\| = k\|x\|_0 \tag{1}$$

is fulfilled for each $x \in X$. Let $e \in X$, $\|e\|_0 = 1$ be a fixed and $x \neq 0$ an arbitrary vector. Since X equipped with the norm $\|.\|_0$ is a Hilbert space, there exists by Lemma 4 a unitary operator U such that

$$\|x\|_0^{-1} x = Ue \tag{2}$$

We may assume that $U \in B(X)$ and that U is unitary also with respect to the given involution since both norms are equivalent and both involutions coincide. From the requirement of the theorem that each unitary operator has norm one it follows that $\|Ux\| = \|x\|$ for each $x \in X$. Combining this with (2) we obtain $\|x\|_0^{-1} \|x\| = \|Ue\| = \|e\|$. Hence $\|x\| = \|e\| \, \|x\|_0$. Thus the relation (1) is proved and the proof of the theorem is complete.

COROLLARY 5. *Let X be a real or complex Banach space. Suppose there exists an involution $A \mapsto A^*$ on some $B(X)$ such that the relation $\|A^*A\| = \|A\|^2$ is fulfilled for each normal operator $A \in B(X)$. In this case there exists an inner product on X such that the corresponding norm is equal to the given norm. For each $A \in B(X)$ the relation $(Ax, y) = (x, A^* y)$ is fulfilled for all pairs $x, y \in X$.*

As we mentioned above, the complex version of Corollary 5 was first proved by N. Prijatelj [4] by different methods and stronger assumptions. He required that the relation $\|A^*A\| = \|A\|^2$ is fulfilled for each $A \in L(X)$ and used methods from Banach $*$-algebra theory. For another improvement of N. Prijatelj's result see [6].

COROLLARY 6. *Let X be a real or complex Banach space. Suppose that there exists an involution on $L(X)$ such that each unitary operator has norm one. Then $L(X)$ is a B^*-algebra.*

It is well known that a complex Banach $*$-algebra with the identity element is a B^*-algebra if each unitary element has norm one (see [5, Theorem 10.1]). Therefore the result above is not surprising for the complex case, but for the real case it seems to be new even in this special case.

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KARAKTERIZACIJA REALNEGA IN KOMPLEKSNEGA HILBERTOVEGA PROSTORA

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Povzetek

Naj bo $L(X)$ algebra omejenih linearnih operatorjev realnega ali kompleksnega Banachovega prostora X in $B(X) \subset L(X)$ takšna $*$-podalgebra, ki vsebuje identični operator I in vse izrojene linearne operatorje. Če ima vsak operator $U \in B(X)$ z lastnostjo $U^*U = UU^* = I$ normo ena, potem je prostor X Hilbertov.