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ON LINEAR FUNCTIONAL EQUATIONS IN THE DETERMINATE CASE

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Abstract. Let I be the real interval and ξ a fixed point of I. Suppose that f, $g: I \to R$ are continuous, $0 < (f(x) - \xi)/(x - \xi) < 1$ for $x \in I \setminus \{\xi\}$ and $|g(\xi)| < 1$. Under these assumptions it is shown that if for every continuous function $h: I \to R$ there exists a continuous solution of equation (1), then f is strictly increasing in a neighbourhood of ξ. It is the inversion of a result of Kordylewski and Kuczma.

Further, the existence of continuous solutions of equations (1) is studied from the Baire category point of view.

1. This paper is devoted to the study of a set of linear functional equations

$$\varphi \circ f = g\varphi + h \tag{1}$$

having continuous solutions.

Suppose that I is a real interval and ξ is an element of I. Denote by C the set of all real continuous functions defined on I and endow it, if I is compact, with the uniform metric. Define

$$C' = \{g \in C : g(x) \neq 0, x \in I, x \neq \xi\},$$

$$C_* = \{g \in C' : |g(\xi)| < 1\}$$

and

$$F = \{f \in C : 0 < (f(x) - \xi) / (x - \xi) < 1, x \in I, x \neq \xi\}.$$

A theory of continuous solutions of equation (1), with $(f, g, h) \in F \times C' \times C$, has been developed in [1], [5] and [6], Chapter II. Some results have been obtained under additional assumption of strict monotonicity of f in a neighbourhood of ξ. In particular the following result is an immediate consequence of Theorems 2.10 and 3.2 of [6].

THEOREM 1. *Suppose that $f \in F$ and $g \in C_*$. If f is strictly increasing in a neighbourhood of ξ, then for every $h \in C$ there exists a solution $\varphi \in C$ of equation* (1).

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In the present paper we shall show (see Theorem 3) that if we omit the assumption of strict monotonicity of f in a neighbourhood of ξ, then the existence of continuous solutions of equation (1) is a very rare property in the set $F \times C_* \times C$. The word »rare« is here meant in the sense of the Baire category.

The similar problem, concerning the case $|g(\xi)| = 1$ from the Baire category point of view, has been treated in [2] and [3]. The strict monotonicity of f in a neighbourhood of ξ is also essential for the existence of continuous solutions of the homogeneous linear equation

$$\varphi \circ f = g\varphi$$

(see [4]).

2. Suppose that $f: I \to I$, $g: I \to R$. Denote by f^n, $n \in N_0$, the n-th iterate of f and put

$$G_n(x) = \prod_{k=0}^{n-1} g[f^k(x)], \quad x \in I, \ n \in N_0.$$

Remark 1. If $f \in F$, then ξ is the unique fixed point of f.

We start with the following useful

LEMMA 1. *Suppose that I is compact. If $f \in F$, then $\bigcap_{n=1}^{\infty} f^n(I) = \{\xi\}$ (in particular $\{f^n\}_{n\in N}$ converges uniformly to ξ). If, moreover, $g \in C$ and $|g(\xi)| < 1$, then $\{G_n\}_{n\in N}$ converges uniformly to zero.*

Proof. In view of the continuity of f every set $f^n(I)$ is a compact interval, and so is $\bigcap_{n=1}^{\infty} f^n(I)$. Let $\bigcap_{n=1}^{\infty} f^n(I) = [a, b]$. Since $b \in \bigcap_{n=1}^{\infty} f^n(I) \subset f(f^k(I))$ for every $k \in N$, $b = f(x_k)$, where $x_k \in f^k(I)$. Choose a subsequence of $\{x_k\}_{k\in N}$ converging to $x_0 \in I$. Then $x_0 \in \bigcap_{n=1}^{\infty} f^n(I) = [a, b]$ and $b = f(x_0)$. Since $\xi \in \bigcap_{n=1}^{\infty} f^n(I)$, we have $\xi \leqslant b$ and $b = f(x_0) \leqslant x_0 \leqslant b$, so $x_0 = b$ and $f(b) = b$. From Remark 1, $b = \xi$. Analogously, $a = \xi$, which ends the proof of the first part of our assertion.

Now choose $\vartheta \in (0, 1)$ such that $|g(\xi)| < \vartheta < 1$. Since $\lim_{n\to\infty} f^n(x) = \xi$ uniformly in I and g is uniformly continuous in I, there exists $n_0 \in N$ such that

$$|g[f^n(x)]| < \vartheta \text{ for } n \geqslant n_0 \text{ and } x \in I.$$

Then

$$|G_n(x)| = |G_{n_0}(x)| \prod_{k=n_0}^{n-1} |g[f^k(x)]| \leqslant \sup_I |G_{n_0}| \vartheta^{n-n_0}$$

for $n \geqslant n_0$ and $x \in I$, which shows that $\lim_{n\to\infty} G_n(x) = 0$ uniformly in I.

Suppose that $f: I \to I$ and introduce the following equivalence relation in I: two points $x, y \in I$ are said to be equivalent, $x \sim y$, iff there exist $n, m \in N_0$ such that $f^n(x) = f^m(y)$ (cf. [6]). The equivalence classes under »$\sim$« are called orbits. Denote by $C(x)$ the orbit of $x \in I$.

Remark 2. Let $f \in F$ and $x, y \in I$. If $x \sim y$, then

$$P_{x,y} = \{(p, q) \in N_0^2 : f^p(x) = f^q(y)\}$$

is a non-empty set. In the set $P_{x,y}$ the following relation can be defined:

$$(p_1 q_1) \leqslant (p_2, q_2) \text{ iff } p_1 \leqslant p_2 \text{ and } q_1 \leqslant q_2.$$

According to Lemma 0.2 of [6] there exists an integer k such that $p - q = k$ for every $(p, q) \in P_{x,y}$. So it is obvious that the relation »$\leqslant$« is a linear order and there exists the smallest element in the space $(P_{x,y}, \leqslant)$.

LEMMA 2. *Suppose that X is an arbitrary non-void set, $x_0 \in X$, $f: X \to X$, $g, h: X \to R$, $g(x) \neq 0$ for every $x \in X$. A solution φ of equation* (1) *on $C(x_0)$ is completely determined by the value of φ at x_0. Moreover, if $x \in C(x_0)$, then*

$$\varphi(x) = \frac{G_p(x_0)}{G_q(x)} \left(\varphi(x_0) + \sum_{k=0}^{p-1} \frac{h[f^k(x_0)]}{G_{k+1}(x_0)}\right) - \sum_{k=0}^{q-1} \frac{h[f^k(x)]}{G_{k+1}(x)},$$

where numbers $p, q \in N_0$ are choosen in such a manner that $f^p(x_0) = f^q(x)$.

Proof. Induction yields

$$\varphi(y) = \frac{\varphi[f^n(y)]}{G_n(y)} - \sum_{k=0}^{n-1} \frac{h[f^k(y)]}{G_{k+1}(y)}, \quad y \in I \setminus \{\xi\}, \ n \in N_0.$$

Applying this formula twice we get

$$\varphi(x) = \frac{\varphi[f^q(x)]}{G_q(x)} - \sum_{k=0}^{q-1} \frac{h[f^k(x)]}{G_{k+1}(x)}$$

$$= \frac{\varphi[f^p(x_0)]}{G_q(x)} - \sum_{k=0}^{q-1} \frac{h[f^k(x)]}{G_{k+1}(x)}$$

$$= \frac{G_p(x_0)}{G_q(x)} \left(\varphi(x_0) + \sum_{k=0}^{p-1} \frac{h[f^k(x_0)]}{G_{k+1}(x_0)}\right) - \sum_{k=0}^{q-1} \frac{h[f^k(x)]}{G_{k+1}(x)}.$$

The following lemmas will be useful in the next considerations. The proof of first of them is similar to that of well known Stolz theorem, so we omit it.

LEMMA 3. *Let A be an arbitrary non-void set, $\alpha_n, \beta_n : A \to R$, $n \in N$. Suppose that $\beta_n(x) \leqslant \beta_{n+1}(x)$ for $x \in A$, $n \in N$, and $\lim\limits_{n\to\infty} \beta_n = \infty$ uniformly in A. If $\lim\limits_{n\to\infty} (\alpha_{n+1} - \alpha_n)/(\beta_{n+1} - \beta_n) = a \in \overline{R}$ uniformly A, in then $\lim\limits_{n\to\infty} \alpha_n/\beta_n = a$ uniformly in A.*

LEMMA 4. *Suppose that $\{b_n\}_{n\in N}$ and $\{u_n\}_{n\in N}$ are real sequences such that $u_n \to \infty$ and $b_n/u_n \to 0$. Then there exists a sequence $\{t_n\}_{n\in N}$ such that $t_n \to 0$ and for every convergent (to a real number) sequence $\{a_n\}_{n\in N}$ the sequence $\{a_n b_n - t_n u_n\}_{n\in N}$ is divergent (it has not finite limit).*

Proof. Choose a number $k_0 \in N$ such that $u_k > 0$ for every $k \geqslant k_0$. For $k < k_0$ we define t_k arbitrarily, If $k \geqslant k_0$ and $\liminf |b_n| > 0$, then we put $t_k = |b_k/u_k|^{1/2}$. If $k \geqslant k_0$ and $\liminf |b_n| = 0$, then we put $t_k = |1/u_k|^{1/2}$. The sequence $\{t_k\}_{k\in N}$ has the required property.

The next result is the inversion of Theorem 1.

THEOREM 2. *Suppose that $f \in F$ and $g \in C_*$. If for every $h \in C$ there exists a solution $\varphi \in C$ of equation* (1), *then f is strictly increasing in a neighbourhood of ξ.*

Proof. Without loss of generality we may assume that I is a compact interval and ξ is its left endpoint. Further, we may assume that $|g(x)| < 1$ for $x \in I$. The proof is divided into three parts.

1°. Let $x_0 \in I$, $\{x_n\}_{n\in N} \subset C(x_0)$ and denote by (p_n, q_n) the smallest element of the set P_{x_n,x_0} (see Remark 2). Then the set $\{p_n\}_{n\in N}$ is finite.

Assume, for the indirect proof, that the set $\{p_n\}_{n\in N}$ is infinite. Since

$$f^{p_n}(x_n) = f^{q_n}(x_0) \text{ for } n \in N, \tag{2}$$

we get

$$x_n \neq \xi \text{ for } n \in N_0 \tag{3}$$

(otherwise $x_n = x_0 = \xi$ and $p_n = 0$ for $n \in N$). On account of Lemma 1, (2) and the infiniteness of the set $\{p_n\}_{n\in N}$ we have

$$\xi = \liminf f^{p_n}(x_n) = \liminf f^{q_n}(x_0).$$

Obviously $x_0 \neq \xi$, thus the set $\{q_n\}_{n\in N}$ is infinite, too.

Let $\{x_{k_n}\}_{n\in N}$ be a convergent subsequence of $\{x_n\}_{n\in N}$ such that

$$p_{k_n} \nearrow \infty, \, q_{k_n} \nearrow \infty. \tag{4}$$

Define

$$x'_n = x_{k_n}, \; p'_n = p_{k_n}, \; q'_n = q_{k_n} \text{ for } n \in N \text{ and } \overline{x} = \lim x'_n.$$

Obviously $\overline{x} \in I$. Now we must distinguish two cases:

Case A: $\bar{x} \notin C(x_0)$. At first we shall show that

$$f^i(x'_n) = f^j(x'_m),\ i \leqslant p'_n,\ j \leqslant p'_m \Rightarrow n = m \text{ and } i = j. \tag{5}$$

We have

$$f^{q'_n}(x_0) = f^{p'_n}(x'_n) = f^{p'_n - i}(f^j(x'_m)) = f^{p'_n - i + j}(x'_m),$$

whence

$$q'_n \geqslant q'_m \text{ and } p'_n - i + j \geqslant p'_m.$$

Similarly we get

$$q'_m \geqslant q'_n \text{ and } p'_m - j + i \geqslant p'_n.$$

Consequently,

$$q'_n = q'_m \text{ and } p'_n - i = p'_m - j,$$

and thus $n = m$ and $i = j$.

Now put

$$b_n = \frac{G_{q'_n}(x_0)}{G_{p'_n}(x'_n)}, \quad u_n = \sum_{i=0}^{p'_n - 1} \frac{\varepsilon^{i+1}}{G_{i+1}(x'_n)}, \quad n \in N, \tag{6}$$

where $\varepsilon = \operatorname{sgn} g\,|_{I\setminus\{\xi\}}$. We shall verify that these sequences fulfil the assumption of Lemma 4. In view of Lemma 1 there exists an $i_0 \in \in N$ such that

$$|G_i(x'_n)| < 1 \text{ for } i > i_0 \text{ and } n \in N.$$

For $n \in N$, such that $p'_n > i_0$, we get

$$u_n = \sum_{i=0}^{p'_n - 1} \frac{\varepsilon^{i+1}}{G_{i+1}(x'_n)} = \sum_{i=0}^{p'_n - 1} |G_{i+1}(x'_n)|^{-1}$$

$$> \sum_{i=0}^{i_0 - 1} |G_{i+1}(x'_n)|^{-1} + (p'_n - i_0),$$

which implies $u_n \to \infty$. Furthermore, observe that Lemma 1 gives

$$\lim_{n\to\infty} \frac{\varepsilon^{n+1}/G_{n+1}}{\varepsilon^{n+1}/G_{n+1} - \varepsilon^n/G_n} = \lim_{n\to\infty} (1 - |g \circ f^n|)^{-1}$$
$$= (1 - |g(\xi)|)^{-1}$$

uniformly in I. Hence and by Lemmas 1 and 3 we get for $x \in I$

$$\lim_{n\to\infty} \left(\sum_{i=0}^{n-1} \frac{\varepsilon^{i+1}}{G_{i+1}(x)} \right) \Big/ \frac{\varepsilon^n}{G_n(x)} = (1 - |g(\xi)|)^{-1}.$$

Consequently,

$$\lim_{n\to\infty} b_n/u_n = \lim_{n\to\infty} G_{q_n'}(x_0)\left(G_{p_n'}(x_n')\sum_{i=0}^{p_n'-1}\varepsilon^{i+1}/G_{i+1}(x_n')\right)^{-1} =$$
$$= 0\cdot(1-|g(\xi)|) = 0.$$

Choose a sequence $\{t_n\}_{n\in N}$ according to Lemma 4. In the set

$$D = \{f^i(x_0)\}_{i\in N} \cup \{f^i(\overline{x})\}_{i\in N} \cup \{\xi\} \cup \bigcup_{n=1}^{\infty}\bigcup_{i=0}^{p_n'-1}\{f^i(x_n')\}$$

we define a function $\overline{h}$ as follows:

$$\overline{h}[f^i(x_0)] = \overline{h}(\xi) = \overline{h}[f^i(\overline{x})] = 0, \quad i\in N \tag{7}$$

and

$$\overline{h}[f^i(x_n')] = \varepsilon^{i+1}t_n, \quad n\in N,\ i\in\{0,\dots,p_n'-1\}.$$

The correctness of this definition may be deduced from (3), (5), the assumption $\overline{x}\notin C(x_0)$ and the definition of the sequence $\{(p_n, q_n)\}_{n\in N}$. The points ξ and $f^i(\overline{x})$, $i\in N$, are the unique accumulation points of the set D. Then the convergence of $\{t_n\}_{n\in N}$ to zero implies the continuity of $\overline{h}$ in D. The set D is closed, thus in virtue of Tietze's theorem there exists a function $h\in C$, which is an extension of $\overline{h}$.

Let $\varphi\in C$ be a solution of equation (1). By Lemma 2, (6) and (7) we have for $n\in N$

$$\varphi(x_n') = \frac{G_{q_n'}(x_0)}{G_{p_n'}(x_n')}\left(\varphi(x_0) + \sum_{k=0}^{q_n'-1}\frac{h[f^k(x_0)]}{G_{k+1}(x_0)}\right) - \sum_{k=0}^{p_n'-1}\frac{h[f^k(x_n')]}{G_{k+1}(x_n')}$$
$$= b_n\varphi(x_0) - \sum_{k=0}^{p_n'-1}\frac{\varepsilon^{k+1}t_n}{G_{k+1}(x_n')} = b_n\varphi(x_0) - t_n u_n,$$

which, in view of Lemma 4, contradicts continuity of φ at $\overline{x}$.

Case B: $\overline{x}\in C(x_0)$. Let $k, l\in N_0$ be choosen in such a manner that $f^k(\overline{x}) = f^l(x_0)$. Our assumptions imply that $\{x_n'\}_{n\in N}\subset C(\overline{x})$. Thus let (i_n, j_n), $n\in N$, be the smallest element of the set $P_{x_n',\overline{x}}$. Then

$$f^{i_n+k}(x_n') = f^{j_n+k}(\overline{x}) = f^{j_n+l}(x_0), \quad n\in N,$$

whence

$$i_n + k \geqslant p_n' \text{ and } j_n + l \geqslant q_n' \text{ for } n\in N.$$

Hence, according to (4), we obtain $\lim\limits_{n\to\infty} i_n = \lim\limits_{n\to\infty} j_n = \infty$. Let $\{l_n\}_{n\in N}$ be such a subsequence of the sequence of positive integers that $i_{l_n}\nearrow\infty$ and $j_{l_n}\nearrow\infty$. Set $x_n'' = x_{l_n}'$, $i_n' = i_{l_n}$, $j_n' = j_{l_n}$ for $n\in N$. In the sequel the proof is similar to that of case A, where x_0, $\{x_n'\}_{n\in N}$, $\{p_n'\}_{n\in N}$, $\{q_n'\}_{n\in N}$ must be replaced by $\overline{x}$, $\{x_n''\}_{n\in N}$, $\{i_n'\}_{n\in N}$, $(j_n')_{n\in N}$, respectively.

2°. Now we wish to show that

$$\exists n \in N \, \forall x, y \in I \, (\bar{f}^n(x) = \bar{f}^n(y) \Rightarrow \bar{f}^{n-1}(x) = \bar{f}^{n-1}(y)).$$

Suppose, on the contrary, that there exist sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ such that $x_n, y_n \in I$ and

$$f^n(x_n) = f^n(y_n) \text{ and } f^{n-1}(x_n) \neq f^{n-1}(y_n), \quad n \in N. \tag{8}$$

Obviously

$$x_n, y_n \neq \xi \text{ for } n \in N. \tag{9}$$

Again we distinguish two cases:

Case A: For a certain subsequence $\{y_{k_n}\}_{n\in N}$ of $\{y_n\}_{n\in N}$ and an $x_0 \in I$ we have $y_{k_n} \in C(x_0)$, $n \in N$.

Then set $x'_n = x_{k_n}$, $y'_n = y_{k_n}$, $n \in N$. It follows, by (8), that k_n is the smallest positive integer such that

$$f^{k_n}(x'_n) = f^{k_n}(y'_n), \, n \in N. \tag{10}$$

Choosing, if necessary, suitable subsequences, we may assume that $\{x'_n\}_{n\in N}$ and $\{y'_n\}_{n\in N}$ converge, i. e.

$$x'_n \to \bar{x} \text{ and } y'_n \to \bar{y}. \tag{11}$$

Since $x'_n \in C(y'_n)$, it follows that $x'_n \in C(x_0)$ for every $n \in N$. Let (p_n, q_n) be the smallest element of the set $P_{x'_n, x_0}$ and (i_n, j_n) the same for the set $P_{y'_n, x_0}$, $n \in N$. In particular we have

$$f^{p_n}(x'_n) = f^{q_n}(x_0), \quad n \in N, \tag{12}$$

and

$$f^{i_n}(y'_n) = f^{j_n}(x_0), \quad n \in N. \tag{13}$$

Hence we obtain

$$f^{p_n+j_n}(x'_n) = f^{q_n+j_n}(x_0) = f^{q_n+i_n}(y'_n), \quad n \in N,$$

which implies

$$p_n + j_n \geqslant k_n, \; q_n + i_n \geqslant k_n \text{ for } n \in N$$

and $\lim\limits_{n\to\infty} j_n = \lim\limits_{n\to\infty} q_n = \infty \; (k_n \to \infty)$. In virtue of part 1° the sets $\{p_n\}_{n\in N}$ and $\{i_n\}_{n\in N}$ are finite, whence, by (11), (12) and (13), we get $\bar{x} = \bar{y} = \xi$. Thus ξ is the unique accumulation point of the sets $E_x = \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{i=0}^{k_n} \{f^i(x'_n)\}$ and $E_y = \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{i=0}^{k_n} \{f^i(y'_n)\}$. Consequently, since $\xi \notin E_x$, none of the elements of E_x has an infinite number of representations of the form $f^i(x'_n)$, where $n \in N$ and $i \in \{0, \ldots, k_n\}$. The set E_y has the analogous property.

Therefore, we may define by induction a sequence $\{m_n\}_{n\in N}$ of positive integers as follows: Put $m_1 = 1$ and, m_n being defined, define m_{n+1} as the smallest positive integer $m > m_n$ such that

$$f^j(x'_{m_n}) \notin \bigcup_{n=m}^{\infty} \bigcup_{i=0}^{k_n} \{f^i(x'_n)\} \cup \{f^i(y'_n)\} \text{ for } j \in \{0, \ldots, k_{m_n}\}.$$

Set $x''_n = x'_{m_n}, y''_n = y'_{m_n}, k'_n = k_{m_n}$ for $n \in N$. It follows from the construction of the sequence $\{m_n\}_{n\in N}$ that

$$f^i(x''_n) = f^j(x''_l),\ i \leqslant k'_n, j \leqslant k'_l \Rightarrow n = l \text{ and } i = j, \tag{14}$$

and the same holds for the sequence $\{y''_n\}$. We shall show that

$$f^i(y''_n) = f^j(x''_l),\ i \leqslant k'_n, j \leqslant k'_l \Rightarrow n = l \text{ and } i = j. \tag{15}$$

Assume, for instance, that $k'_n - i \leqslant k'_l - j$. Then by (10) we get

$$f^{k'_n}(x''_n) = f^{k'_n}(y''_n) = f^{k'_n - i}(f^i(y''_n)) = f^{k'_n - i + j}(x''_l).$$

Since $k'_n - i + j \leqslant k'_l$, (14) implies that $n = l$ and $k'_n = k'_n - i + j$, i. e. $i = j$.

Now put

$$b_n = \frac{G_{k'_n}(y''_n)}{G_{k'_n}(x''_n)}, \quad u_n = \sum_{i=0}^{k'_n - 1} \frac{\varepsilon^{i+1}}{G_{i+1}(x''_n)}, \ n = N. \tag{16}$$

Analogously as in the part 1° one can verify that these sequences fulfil assumptions of Lemma 4.

Choose a sequence $\{t_n\}_{n\in N}$ according to this lemma. In the set

$$D = \bigcup_{n=1}^{\infty} \bigcup_{i=0}^{k'_n - 1} \{f^i(x''_n)\} \cup \bigcup_{n=1}^{\infty} \bigcup_{i=0}^{k'_n - 1} \{f^i(y''_n)\} \cup \{\xi\}$$

we define a function $\bar{h}$:

$$\bar{h}[f^i(x''_n)] = \varepsilon^{i+1} t_n,\ \bar{h}[f^i(y''_n)] = \bar{h}(\xi) = 0, \tag{17}$$
$$n \in N, i \in \{0, \ldots, k'_n - 1\}.$$

The correctness of the definition follows from (9), (14) and (15). The set D is closed and ξ is the unique accumulation point of it. Thus $\bar{h}$ is continuous ($t_n \to 0$) and in virtue of Tietze's theorem it can be extended to a function $h \in C$.

Let $\varphi \in C$ be a solution of equation (1). In view of Lemma 2 and by (10), (16) and (17) we have

$$\varphi(x_n'') = \frac{G_{k_n'}(y_n'')}{G_{k_n'}(x_n'')} \left(\varphi(y_n'') + \sum_{i=0}^{k_n'-1} \frac{h[f^i(y_n'')]}{G_{i+1}(y_n'')}\right) - \sum_{i=0}^{k_n'-1} \frac{h[f^i(x_n'')]}{G_{i+1}(x_n'')} =$$

$$= b_n \varphi(y_n'') - \sum_{i=0}^{k_n'-1} \frac{\varepsilon^{i+1} t_n}{G_{i+1}(x_n'')} = b_n \varphi(y_n'') - t_n u_n, \; n \in N.$$

Thus on account of (11) and the continuity of φ, sequences $\{\varphi(x_n'')\}_{n \in N}$ and $\{\varphi(y_n'')\}_{n \in N}$ converge to $\varphi(\xi)$, contrary to the Lemma 4.

Case B: Every orbit contains a finite number of points of the set $\{y_n\}_{n \in N}$.

Let $\{y_{k_n}\}_{n \in N}$ be a subsequence of $\{y_n\}_{n \in N}$ having at most one point in common with every orbit and put $w_n = x_{k_n}$, $z_n = y_{k_n}$, $n \in N$. In view of (8) k_n is the smallest positive integer such that

$$f^{k_n}(w_n) = f^{k_n}(z_n), \; n \in N. \tag{18}$$

Moreover, since $z_i \neq z_j$ for $i \neq j$ and $C(w_i) = C(z_i)$ for every $i \in N$,

$$C(z_i) \cap C(z_j) = \emptyset, \; C(w_i) \cap C(w_j) = \emptyset \quad \text{for} \quad i \neq j. \tag{19}$$

Choosing, if necessary, suitable subsequences, we may assume that $\{w_n\}_{n \in N}$ and $\{z_n\}_{n \in N}$ converge, i. e. $w_n \to \overline{w}$, $z_n \to \overline{z}$. Observe, that by (19) there exists $n_0 \in N$ such that

$$\overline{w} \notin C(w_n), \; \overline{z} \notin C(z_n), \; n \geqslant n_0. \tag{20}$$

It follows from (19) and the assumption $f \in F$ that

$$f^i(w_n) = f^j(w_m) \Rightarrow n = m \quad \text{and} \quad i = j. \tag{21}$$

Let us put

$$b_n = \frac{G_{k_n}(z_n)}{G_{k_n}(w_n)}, \; u_n = \sum_{i=0}^{k_n-1} \frac{\varepsilon^{i+1}}{G_{i+1}(w_n)}, \; n \in N. \tag{22}$$

As previously one can verify that these sequences fulfil the assumptions of Lemma 4. Thus let $\{t_n\}_{n \in N}$ be a sequence existing on account of this lemma. In the set

$$D = \bigcup_{n=n_0}^{\infty} \bigcup_{i=0}^{k_n-1} \{f^i(w_n)\} \cup \bigcup_{n=n_0}^{\infty} \bigcup_{i=0}^{k_n-1} \{f^i(z_n)\} \cup$$

$$\cup \{f^i(\overline{w})\}_{i \in N} \cup \{f^i(\overline{z})\}_{i \in N} \cup \{\xi\}$$

define a function $\bar{h}$ as follows:

$$\bar{h}\,[f^i(w_n)] = \varepsilon^{i+1}\,t_n,\ \bar{h}\,[f^i(z_n)] = 0,\ \ n \geq n_0\ \ i \in \{0, \ldots, k_n - 1\}, \tag{23}$$

$$\bar{h}\,[f^i(\bar{w})] = \bar{h}\,[f^i(\bar{z})] = \bar{h}(\xi) = 0,\ i \in N_0.$$

By (9), (18), (20) and (21) this definition is correct. The set D is closed and the points ξ, $f^n(\bar{w})$ and $f_n(\bar{z})$ for $n \in N_0$ are its unique accumulation points. Since $t_n \to 0$, $\bar{h}$ is continuous.

Thus let $h \in C$ be an extension of $\bar{h}$ and $\varphi \in C$ be a solution of equation (1). In virtue of Lemma 2, (22) and (23) we have

$$\varphi(w_n) = \frac{G_{k_n}(z_n)}{G_{k_n}(w_n)}\left(\varphi(z_n) + \sum_{i=0}^{k_n-1} \frac{h[f^i(z_n)]}{G_{i+1}(z_n)}\right) - \sum_{i=0}^{k_n-1} \frac{h\,[f^i(w_n)]}{G_{i+1}(w_n)} =$$

$$= b_n\,\varphi(z_n) - \sum_{i=0}^{k_n-1} \frac{\varepsilon^{i+1}\,t_n}{G_{i+1}(w_n)} = b_n\,\varphi(z_n) - t_n\,u_n,\ \ n \in N,$$

which, in view of Lemma 4, contradicts continuity of φ.

3°. Now, we shall show that $f\,|_{f^{\bar{n}-1}(I)}$ is strictly increasing.

For, assume that $f(x) = f(y)$ for certain $x, y \in f^{\bar{n}-1}(I)$. Hence $x = f^{\bar{n}-1}(x_0)$, $y = f^{\bar{n}-1}(y_0)$, where $x_0, y_0 \in I$, and

$$f^{\bar{n}}(x_0) = f(f^{\bar{n}-1}(x_0)) = f(x) = f(y) = f(f^{\bar{n}-1}(y_0)) = f^{\bar{n}}(y_0).$$

It follows from the part 2° that $f^{\bar{n}-1}(x_0) = f^{\bar{n}-1}(y_0)$, i. e. $x = y$ and $f\,|_{f^{\bar{n}-1}(I)}$ is invertible. Consequently, since $f \in F$, $f\,|_{f^{\bar{n}-1}(I)}$ is strictly increasing, which ends the proof of our theorem.

3. This section is devoted to the qualitative description of the set of all equations (1) having continuous solutions. We start with the following

LEMMA 5. *Let I be a compact interval. Then the set F_0 of all functions $f \in F$ strictly increasing in a neighbourhood of ξ is of the first category in F.*

Proof. We have $F_0 = \bigcup_{n=1}^{\infty} F_n$, where F_n, $n \in N$, is the set of all functions $f \in F$ strictly increasing in the interval $I \cap \left(\xi - \frac{1}{n}, \xi + \frac{1}{n}\right)$. Observe, that every set $\mathrm{cl}\,F_n$ contains only increasing functions, so its interior in F is empty and F_n is a nowhere dense set.

THEOREM 3. *Let I be a compact interval. The set*

$$H = \{(f, g, h) \in F \times C_* \times C : \exists\, \varphi \in C \quad \varphi \circ f = g\varphi + h\}$$

is of the first category in $F \times C_* \times C$.

Proof. The set

$$B = \{(f, g, h, \varphi) \in F \times C_* \times C \times C : \varphi \circ f = g\varphi + h\}$$

is closed in $F \times C_* \times C \times C$. Thus H, as a projection of B on the space $F \times C_* \times C$, is analytic and has the Baire property (see [7], Chapter XIII, § 1).

Let $f \in F \setminus F_0$ and $g \in C_*$. On account of Theorem 2 the section

$$H_{f,g} = \{h \in C : \exists\, \varphi \in C \quad \varphi \circ f = g\varphi + h\}$$

of H is a proper subset of C. In virtue of Lemma 1 of [2] $H_{f,g}$ is of the first category in C. To finish our proof it is sufficient to apply Theorem 15.4 of [8] and Lemma 5.

REFERENCES:

[1] *B. Choczewski* and *M. Kuczma,* On the »indeterminate case« in the theory of a linear functional equation, Fund. Math. **58** (1966), 163—175.

[2] *W. Jarczyk,* On a set of functional equations having continuous solutions, Glasnik Mat. **17** (**37**) (1982) 59—64.

[3] ————, A category theorem for linear functional equations in the indeterminate case, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. et Phys. **29** (1981), 371—372.

[4] *J. Kalinowski* and *M. Kuczma,* Some remarks on continuous solutions of a functional equation, Uniwersytet Slaski w Katowicach— Prace Mat. **9** (1979), 64—73.

[5] *J. Kordylewski* and *M. Kuczma,* On some linear functional equaltions I, II, Ann. Polon. Math. **9** (1960), 119—136; **11** (1962), 203—207.

[6] *M. Kuczma,* Functional equations in a single variable, PWN (MM 46) Warszawa 1968.

[7] *K. Kuratowski* and *A. Mostowski,* Set theory, PWN Warszawa, North-Holland Publishing Company Amsterdam, 1976.

[8] *J. C. Oxtoby,* Measure and category, Springer-Verlag (GTM 2), New York Heidelberg Berlin 1971.

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O LINEARNIM FUNKCIONALNIM JEDNADŽBAMA

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Sadržaj

Neka je $\xi \in I$, I-realni interval a $f, g : I \to R$ neprekidne funkcije takve da je $0 < (f(x) - \xi) / (x - \xi) < 1$ za $x \in I \setminus \{\xi\}$ i $|g(\xi)| < 1$. Pod tim pretpostavkama je pokazano: ako za svaku neprekidnu funkciju $h : I \to R$ postoji neprekidno rješenje jednadžbe (1), onda je f striktno rastuća u okolini od ξ. Nadalje je studirana egzistencija neprekidnih rješenja jednadžbe (1) sa stanovišta Baireovih kategorija.