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COMPLETION OF ADDITIVE SET FUNCTIONS WITH VALUES IN A UNIFORM SEMIGROUP

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Abstract. We establish two theorems on a Peano-Jordan-type completion of additive set functions with values in an Abelian uniform semigroup with neutral element which improve earlier results by the author and D. Butković.

Let G be an Abelian semigroup with neutral element 0 equipped with a (Hausdorff) uniformity $\mathbf{U}$ under which the addition $+$ is uniformly continuous as a mapping from $G \times G$ into G ([3], p. 75, and [6], p. 2). G is further termed an *Abelian uniform semigroup with* 0. The letters W and V always denote arbitrary elements of $\mathbf{U}$ which are called entourages. Our terminology and notation concerning uniform spaces follows, in principle, [2], Chapter 8.

Throughout the paper $\mathbf{M}$ denotes a ring of subsets of a set X. Following [5], p. 20, we associate with every set function $\mu : \mathbf{M} \to G$ the family $\mathbf{M}_\mu$ of all $E \subset X$ for which to every V there exist $M, N \in \mathbf{M}$ such that

$$(*)\ M \subset E \subset N \text{ and } (\mu(S), 0) \in V \text{ provided } N \setminus M \supset S \in \mathbf{M}.$$

PROPOSITION 1 (cf. [5], Proposition 1 and Remark 3). *Let $\mu : \mathbf{M} \to G$ be additive and $\mu(\emptyset) = 0$. Then*

(*a*) *$\mathbf{M}_\mu$ is a ring of sets containing $\mathbf{M}$.*

(*b*) *Given $E \in \mathbf{M}_\mu$, to every W there exist $M, N \in \mathbf{M}$ such that $M \subset E \subset N$ and $(\mu(S'), \mu(S'')) \in W$ provided $M \subset S', S'' \subset N$ and $S', S'' \in \mathbf{M}$.*

Proof. (*a*) If $M_i \subset E_i \subset N_i$ for $i = 1, 2$, then

$$M_1 \cup M_2 \subset E_1 \cup E_2 \subset N_1 \cup N_2, \quad M_1 \setminus N_2 \subset E_1 \setminus E_2 \subset N_1 \setminus M_2,$$

$$(N_1 \cup N_2) \setminus (M_1 \cup M_2), \quad (N_1 \setminus M_2) \setminus (M_1 \setminus N_2) \subset$$

$$\subset (N_1 \setminus M_1) \cup (N_2 \setminus M_2).$$

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Hence the additivity of μ and the uniform continuity of $+$ imply that $\mathbf{M}_\mu$ is a ring. That $\mathbf{M} \subset \mathbf{M}_\mu$ follows from the assumption that $\mu(\emptyset) = 0$.

(*b*) Fix W and take V so that $(x_1, y_1), (x_2, y_2) \in V$ implies $(x_1 + x_2, y_1 + y_2) \in W$. Let E, V and M, N satisfy $(*)$. Then for S', $S'' \in \mathbf{M}$ with $M \subset S'$, $S'' \subset N$ we have $S' \setminus S''$, $S'' \setminus S' \subset N \setminus M$ whence $(\mu(S' \setminus S''), 0)$, $(\mu(S'' \setminus S'), 0) \in V$. Since, clearly, $(\mu(S' \cap S''), \mu(S' \cap S'')) \in V$, it follows from the additivity of μ that $(\mu(S'), \mu(S'')) \in 2W$.

If G is a group, the condition given in (*b*) characterizes $\mathbf{M}_\mu$ (cf. [5], Remark 3). In general, this is not so as shown by the following simple

Example. Let $X = \{a, b, c\}$ be a three-element set. Put $\mathbf{M} = \{\emptyset, X, \{a\}, \{b, c\}\}$ and $\mu(\emptyset) = 0$, $\mu(X) = \mu(\{a\}) = \infty$ and $\mu(\{b, c\}) = 1$. Then $\mu : \mathbf{M} \to [0, \infty]$ is additive. Moreover, when $[0, \infty]$ is equipped with the discrete uniformity, the condition given in (*b*) holds for $E = \{a, b\}$ while $\mathbf{M}_\mu = \mathbf{M}$. Note that in our situation this condition does not determine a ring of sets.

A set function $\mu : \mathbf{M} \to G$ is called $\mathbf{K}$-*tight* ($\mathbf{K}$ being a subfamily of $\mathbf{M}$) if to every $M \in \mathbf{M}$ and V there exists $K \in \mathbf{K}$ such that

$K \subset M$ and $(\mu(S), 0) \in V$ provided $M \setminus K \supset S \in \mathbf{M}$ (cf. [4], Definition 2). In case μ is additive, this condition is seen to imply that to every $M \in \mathbf{M}$ and W there exists $K \in \mathbf{K}$ such that

$K \subset M$ and $(\mu(M), \mu(M')) \in W$ if $K \subset M' \subset M$, $M' \in \mathbf{M}$ (cf. [5], Definition 2). If, moreover, G is a group, then the converse also holds.

The following result is a common generalization of Theorem 1 and the essential part of Theorem 2 in [5].

THEOREM 1. *If* $(G, \mathbf{U})$ *is complete, then every additive set function* $\mu : \mathbf{M} \to G$ *with* $\mu(\emptyset) = 0$ *has a unique extension to an* $\mathbf{M}$-*tight additive set function* $\nu : \mathbf{M}_\mu \to G$. *Moreover,* $(\mathbf{M}_\mu)_\nu = \mathbf{M}_\mu$.

Proof. According to Proposition 1 (*a*), $\mathbf{M}_\mu$ is a ring and $\mathbf{M} \subset \mathbf{M}_\mu$. In view of part (*b*) of the same proposition, for every $E \in \mathbf{M}_\mu$ the net $\{\mu(M) : E \supset M \in \mathbf{M}\}$, where the index set is directed by $\subset$, satisfies the Cauchy condition. Put

$$\nu(E) = \lim \{\mu(M) : E \supset M \in \mathbf{M}\}.$$

Clearly, ν extends μ.

To prove that ν is additive (cf. [5], Lemma 2 (iii)), take $E_1, E_2 \in \mathbf{M}_\mu$ with $E_1 \cap E_2 = \emptyset$. Then $E = E_1 \cup E_2$ is in $\mathbf{M}_\mu$, too. Fix V and take a closed W such that $(x_1, y_1), (x_2, y_2) \in W$ implies $(x_1 + x_2, y_1 + y_2) \in V$. Let M, $N \in \mathbf{M}$, E and W be as in Proposition 1 (*b*). Also, let M_i, $N_i \in \mathbf{M}$, E_i and W be chosen according to Proposition 1 (*b*) for $i = 1, 2$. Put

$$\widetilde{M}_1 = M_1 \cup ((M \setminus M_2) \cap N_1) \text{ and } \widetilde{M}_2 = (N_2 \cap N) \setminus \widetilde{M}_1.$$

We have $M_i \subset \widetilde{M}_i \subset N_i$ for $i = 1, 2$ and $M \subset \widetilde{M}_1 \cup \widetilde{M}_2 \subset N$ and $\widetilde{M}_1 \cap \widetilde{M}_2 = \emptyset$. Hence $(\nu(E_i), \mu(\widetilde{M}_i)) \in W$ and $(\nu(E), \mu(\widetilde{M}_1 \cup \cup \widetilde{M}_2)) \in W$. It follows that $(\nu(E_1) + \nu(E_2), \mu(\widetilde{M}_1 \cup \widetilde{M}_2)) \in V$, and so $(\nu(E), \nu(E_1) + \nu(E_2)) \in 2V$.

To prove that ν is $\mathbf{M}$-tight, given $E \in \mathbf{M}_\mu$ and W, choose $M, N \in \in \mathbf{M}$ so that $(*)$ holds. Let $F \in \mathbf{M}_\mu$ and $F \subset E \setminus M$. Then there is $T \in \mathbf{M}$ with $T \subset F$ and $(\nu(F), \mu(T)) \in W$. Since $T \subset N \setminus M$, we have $(\mu(T), 0) \in W$. It follows that $(\nu(F), 0) \in 2W$.

The uniqueness assertion is clear while the equality $(\mathbf{M}_\mu)_\nu = \mathbf{M}_\mu$ is a consequence of the following

PROPOSITION 2. *Let $\mu : \mathbf{M} \to G$ be additive and $\mu(\emptyset) = 0$. If $\nu : \mathbf{M}_\mu \to G$ is an extension of μ, then $(\mathbf{M}_\mu)_\nu = \mathbf{M}_\mu$.*

Proof (cf. [5], the last part of the proof of Theorem 1). Suppose $F \in (\mathbf{M}_\mu)_\nu$, fix W and choose V such that $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$ implies $(x_1 + x_2 + x_3,\ y_1 + y_2 + y_3) \in W$. There are $E_1, E_2 \in \mathbf{M}_\mu$ with $E_1 \subset F \subset E_2$ and $(\mu(S), 0) \in V$ whenever $E_2 \setminus E_1 \supset S \in \mathbf{M}$. Take $M_i, N_i \in \mathbf{M}$ such that $M_i \subset E_i \subset N_i$ and $(\mu(S), 0) \in V$ whenever $N_i \setminus M_i \supset S \in \mathbf{M}$, $i = 1, 2$. We have $M_1 \subset F \subset N_2$, and

$$N_2 \setminus M_1 \subset (N_1 \setminus M_1) \cup (E_2 \setminus E_1) \cup (N_2 \setminus M_2).$$

Hence, by the additivity of μ and our choice of V, $(\mu(S), 0) \in W$ whenever $N_2 \setminus M_1 \supset S \in \mathbf{M}$. Thus $F \in \mathbf{M}_\mu$.

Remark. In case the uniformity $\mathbf{U}$ has a countable base and $\mathbf{M}$ is a σ-ring, Theorem 1 holds without the completeness assumption on $\mathbf{U}$. Indeed, then $E \in \mathbf{M}_\mu$ if and only if there exist $M, N \in \mathbf{M}$ such that

$M \subset E \subset N$ and $\mu(S) = 0$ provided $N \setminus M \supset S \in \mathbf{M}$ (cf. [5], Proposition 2). Hence it is enough to apply Theorem 1 to G equipped with the discrete uniformity. In the latter case the resulting set function ν is called the Lebesgue (or null) completion of μ (cf. [1], p. 34).

In view of what we have just said, the following result generalizes [1], Theorem 1.1 and Proposition 1.2.

THEOREM 2. *Let $\{G_i : i \in I\}$ be a family of Abelian uniform semigroups with 0 and let p_i be the projection of $\prod_{i \in I} G_i$ onto G_i. If $\mu : \mathbf{M} \to \prod_{i \in I} G_i$ is additive and $\mu(\emptyset) = 0$, then $\mathbf{M}_\mu = \bigcap_{i \in I} \mathbf{M}_{p_i \circ \mu}$. If the G_i s are, moreover, complete, then $p_i \circ \nu(E) = \nu_i(E)$ for all $E \in \in \mathbf{M}_\mu$ and $i \in I$, where ν and ν_i are the unique $\mathbf{M}$-tight additive extensions of μ and $p_i \circ \mu$ on $\mathbf{M}_\mu$ and $\mathbf{M}_{p_i \circ \mu}$, respectively, given by Theorem 1.*

Proof. As p_i is uniformly continuous, we have $\mathbf{M}_\mu \subset \mathbf{M}_{p_i \circ \mu}$, and so $\mathbf{M}_\mu \subset \bigcap_{i \in I} \mathbf{M}_{p_i \circ \mu}$. To prove the other inclusion, fix an entourage V in $\prod_{i \in I} G_i$ and choose entourages V_{i_k} in G_{i_k} for $k = 1, \ldots, n$ so that $(\{x_i\}, (y_i\}) \in V$ whenever $(x_{i_k}, y_{i_k}) \in V_{i_k}$ for every k. Given $E \in \bigcap_{i \in I} \mathbf{M}_{p_i \circ \mu}$, take $M_{i_k}, N_{i_k} \in \mathbf{M}$ with $M_{i_k} \subset E \subset N_{i_k}$ and $(p_{i_k} \circ \circ \mu (S), 0) \in V_{i_k}$ whenever $N_{i_k} \setminus M_{i_k} \supset S \in \mathbf{M}$. It follows that for V and $M = \bigcup_{k=1}^{n} M_{i_k}$ and $N = \bigcap_{k=1}^{n} N_{i_k}$ (*) holds.

Clearly, $p_i \circ \nu$ is an additive extension of $p_i \circ \mu$. Since p_i is uniformly continuous, $p_i \circ \nu$ is, moreover, $\mathbf{M}$-tight. It follows that $p_i \circ \nu$ and ν_i coincide on $\mathbf{M}_\mu$.

REFERENCES:

[1] *D. Butković*, On integration with respect to measures with values in arbitrary topological vector spaces, Glasnik Mat., Ser. III, **15 (35)** (1980), 33—40.

[2] *R. Engelking*, Outline of general topology, North-Holland and Polish Scientific Publishers, Amsterdam—Warsaw, 1968.

[3] *C. T. Ionescu Tulcea*, Funcţii de mulţime şi integralele lor, Stud. Cerc. Mat. **5** (1954), 73—142.

[4] *D. Landers* and *L. Rogge*, The Hahn-Vitali-Saks and the uniform boundedness theorem in topological groups, Manuscripta Math. **4** (1971), 351—359.

[5] *Z. Lipecki*, Extensions of additive set functions with values in a topological group, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **22** (1974), 19—27.

[6] *M. Sion*, A theory of semigroup valued measures, Lecture Notes in Math. 355, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

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UPOTPUNJENJE ADITIVNIH SKUPOVNIH FUNKCIJA S VRIJEDNOSTIMA U UNIFORMNIM POLUGRUPAMA

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Sadržaj

Dokazuju se dva teorema o upotpunjenju Peano-Jordanovog tipa aditivnih skupovnih funkcija s vrijednostima u Abelovim uniformnim polugrupama s neutralnim elementom. Teoremi poboljšavaju ranije rezultate koje su postigli autor i D. Butković.