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TOTALLY SYMMETRIC n-QUASIGROUPS WHICH SATISFY THE THREE SYMMETRIES THEOREM

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Abstract. Totally symmetric n-quasigroups in which every product of three symmetries is a symmetry are studied in this paper. The structure of such n-quasigroup is given in the terms of an abelian group.

Let (Q, A) be a totally symmetric n-quasigroup (shortly TS-n-quasigroup), $n \geqslant 2$, i. e. an n-quasigroup in which $A(x_1^n) = x_{n+1}$ implies $A(x_{\varphi(1)}^{\varphi(n)}) = x_{\varphi(n+1)}$, for any permutation φ on $\{1, 2, \ldots, n+1\}$ and all $x_1, x_2, \ldots, x_{n+1} \in Q$, [1]. A permutation $R_{\mathbf{a}}^i : Q \to Q$, $R_{\mathbf{a}}^i(x) = A(a_1^{i-1}, x, a_i^{n-1})$ is usually called the i-th translation in (Q, A) generated by the $(n-1)$-tuple $\mathbf{a} = (a_1, \ldots, a_{n-1})$, $i = 1, 2, \ldots, n$. Obviously, for a TS-n-quasigroup, all i-th translations generated by the same $\mathbf{a} \in Q^{n-1}$, $i = 1, 2, \ldots, n$, coincide and we will denote them by $s_{\mathbf{a}}$. Further, it is easy to show that $s_{\mathbf{a}}^2$ is the identity, i. e. $s_{\mathbf{a}}^2 = 1$. Therefore, $s_{\mathbf{a}}$ will be called the symmetry in (Q, A) generated by $\mathbf{a} \in Q^{n-1}$.

Let $\mathscr{S}$ be the set of all symmetries in a TS-n-quasigroup (Q, A). We denote by $S(Q)$ the group generated by $\mathscr{S}$, and by $S^0(Q)$ the subgroup of $S(Q)$ generated by the products of the even numbers of symmetries.

In this paper we investigate TS-n-quasigroups in which the three symmetries theorem holds, i. e. TS-n-quasigroups satisfying the following condition:

(A) *Every product of three symmetries is a symmetry.*

This axiom gives a new, geometrical justification for the introduced term »symmetry«.

In the sequel, (Q, A) is a TS-n-quasigroup satisfying (A), in short ATS-n-quasigroup. The elements of Q will be called the points.

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We will prove that the ATS-n-quasigroups have a very simple representation in terms of an abelian group $(Q, +)$ isomorphic to $S^0(Q)$ (Theorems 4 and 7). In fact, any ATS-n-quasigroup is a C^{n+1}-system (cf [2], [3]) and vice versa (Theorem 6).

PROPOSITION 1. *Let (Q, A) be an ATS-n-quasigroup. Then $S^0(Q)$ is an abelian group. If the identity is a symmetry, then $S^0(Q) = S(Q)$. If the identity is not a symmetry, then the index of $S^0(Q)$ in $S(Q)$ is equal 2 and the coset of $S^0(Q)$ is $\mathcal{S}$.*

Proof. For any three symmetries $s_1, s_2, s_3 \in \mathcal{S}$, the product $s_1 s_2 s_3$ is a symmetry, which implies $(s_1 s_2 s_3)^2 = 1$. Therefore $s_1 s_2 s_3 = s_3 s_2 s_1$ and the commutativity of $S^0(Q)$ follows immediately. The remaining statements are obious from the fact that any element in $S(Q)$ is a symmetry or a product of two symmetries.

PROPOSITION 2. *Let $s_1, s_2, s_3, s_4 \in \mathcal{S}$ be given symmetries and $x \in Q$ a point.*

a) If $s_1(x) = s_2(x)$, then $s_1 = s_2$.

b) If $s_2 s_1(x) = s_4 s_3(x)$, then $s_2 s_1 = s_4 s_3$.

Proof. a) For any point $y \in Q$, there is a symmetry $s \in \mathcal{S}$, for which $s(x) = y$, because (Q, A) is an n-quasigroup. Then

$$s_1(y) = s_1 s(x) = s_1 s s_2 s_1(x) = s_2 s s_1 s_1(x) = s_2 s(x) = s_2(y),$$

so that $s_1 = s_2$

b) From the assumption $s_2 s_1(x) = s_4 s_3(x)$, we have $s s_2 s_1(x) = s s_4 s_3(x)$, for any symmetry $s \in \mathcal{S}$. Since $s s_2 s_1$ and $s s_4 s_3$ are both symmetries, the statement *a)* implies $s s_2 s_1 = s s_4 s_3$. Therefore $s_2 s_1 = s_4 s_3$.

Definition 1. For any two $(a, b), (c, d) \in Q^2$ we define $(a, b) \approx (c, d)$ iff there is a symmetry $s \in \mathcal{S}$, such that $s(a) = d$ and $s(b) = c$.

Without difficulties, one could prove the following proposition:

PROPOSITION 3. *For any four points $a, b, c, d \in Q$ hold:*

a) $(a, a) \approx (b, b)$,

b) $(a, a) \approx (b, c)$ iff $b = c$,

c) $(a, b) \approx (c, d)$ implies $(b, a) \approx (d, c)$.

PROPOSITION 4. *$\approx$ is an equivalence relation.*

Proof. For any two points $a, b \in Q$, there is a symmetry $s \in \mathcal{S}$, such that $s(a) = b$, which means that the relation $\approx$ is reflexive. Obviously, $\approx$ is a symmetric relation. To prove transitivity, let us suppose that $(a, b) \approx (c, d)$ and $(c, d) \approx (e, f)$, for $a, b, c, d, e, f \in Q$. Then there are symmetries $s_1, s_2, s_3 \in \mathcal{S}$ such that

$$s_1(a) = d,\ s_1(b) = c,\ s_2(c) = f = s_3(a),\ s_2(d) = e,$$

which gives

$$s_3(b) = s_3 s_1(c) = s_3 s_1 s_2(f) = s_2 s_1 s_3(f) = s_2 s_1(a) = s_2(d) = e$$

and hence $(a, b) \approx (e, f)$, i. e. $\approx$ is transitive.

The equivalence classes, i. e. the elements of $Q^2/\approx$ will be denoted by ab, where $(a, b) \in ab$.

The following proposition is an easy consequence of Proposition 2.

PROPOSITION 5. *For any three points $a, b, c \in Q$, there is exactly one point $d \in Q$, such that $ab = cd$.*

COROLLARY. *Let $o \in Q$ be a fixed point. For any two points $a, b \in Q$, the equality $oa = ob$ is equivalent to $a = b$.*

PROPOSITION 6. *For any six points $a, b, c, p, q, r \in Q$, $ab = pq$ and $bc = qr$ imply $ac = pr$.*

Proof. We have $(a, b) \approx (p, q)$ and $(b, c) \approx (q, r)$ and there are symmetries $s_1, s_2, s_3 \in \mathcal{S}$ such that

$$s_1(a) = q,\ s_1(b) = p,\ s_2(b) = r = s_3(a),\ s_2(c) = q.$$

Hence

$$s_3(c) = s_3 s_2(q) = s_3 s_2 s_1(a) = s_1 s_2 s_3(a) = s_1 s_2(r) = s_1(b) = p,$$

so that $(a, c) \approx (p, r)$, i. e. $ac = pr$.

Definition 2. For any two $ab, bc \in Q^2/\approx$ we define their sum by

$$ab + bc = ac.$$

From Propositions 5 and 6, it follows that the addition is well defined on $Q^2/\approx$.

THEOREM 1. *$(Q^2/\approx, +)$ is an abelian group.*

Proof. Associativity follows immediately. Further, for any two points $a, b \in Q$, aa is the zero and ba is the inverse of ab. The group $(Q^2/\approx, +)$ is abelian, since for all $ab, bc \in Q^2/\approx$ and any symmetry $s \in \mathcal{S}$ satisfying $s(c) = b$, we have

$$ab + bc = ac = bd = bc + cd = bc + ab,$$

where $d = s(a)$.

Applying the corollary of Proposition 5 and Theorem 1 we get the following theorem.

THEOREM 2. *For any fixed point* $o \in Q$, *let* $f_o : Q \to Q^2/_{\approx}$ *be the one-to-one mapping given by the equality*

$$f_o(x) = ox.$$

Suppose that the addition $\oplus$ *on* Q, *with respect to the point* o, *is defined by the formula*

$$a \oplus b = f_o^{-1}(f_o(a) + f_o(b)).$$

Then $(Q, \oplus)$ *is an abelian group.*

COROLLARY. *Abelian groups* $(Q, \oplus)$ *do not depend on the choice of the point* $o \in Q$.

THEOREM 3. *For any* n *points* $x_1, \ldots, x_n \in Q$, *the equality*

$$x_1 \oplus x_2 \oplus \ldots \oplus x_n = \underset{o}{s_{n-1}}(A(x_1^n))$$

is valid.

Proof. Let us define the points $y_0, y_1, \ldots, y_{n-1} \in Q$ such that $y_0 = x_1$ and $y_i y_{i+1} = o x_{i+2}$, for all $i = 0, 1, \ldots, n-2$. It follows that

$$x_1 \oplus x_2 \oplus \ldots \oplus x_n = f_o^{-1}(f_o(x_1) + f_o(x_2) + \ldots + f_o(x_n)) =$$
$$= f_o^{-1}(ox_1 + ox_2 + \ldots + ox_n) = f_o^{-1}(oy_0 + y_0 y_1 +$$
$$+ \ldots + y_{n-2} y_{n-1}) = f_o^{-1}(oy_{n-1}) = y_{n-1}.$$

On the other hand, for all $a_1, \ldots, a_{n-3} \in Q$ and any $i = 0, 1, \ldots, n-2$,

$$s[y_i, x_{i+2}, a_1^{n-3}] = s[y_{i+1}, o, a_1^{n-3}]$$

holds, where $s[x_1^{n-1}]$ denotes $s_{\mathbf{x}}$, for $\mathbf{x} = (x_1, \ldots, x_{n-1}) \in Q^{n-1}$. Hence we obtain

$$A(x_1^n) = s[x_1^{n-1}](x_n) = s[y_0, x_2, x_3^{n-1}](x_n) =$$
$$= s[y_1, o, x_3^{n-1}](x_n) = s[y_1, x_3, x_4^n](o) = \ldots =$$
$$= s[y_2, o, x_4^n](o) = s[y_2, x_4, x_5^n, o](o) =$$
$$= s\left[y_{n-2}, x_n, \overset{n-3}{o}\right](o) = s\left[y_{n-1}, \overset{n-2}{o}\right](o) = s\left[\overset{n-1}{o}\right](y_{n-1}).$$

Therefore,

$$s\left[\overset{n-1}{o}\right](A(x_1^n)) = y_{n-1} = x_1 \oplus x_2 \oplus \ldots \oplus x_n.$$

COROLLARY. *For any two points $x, y \in Q$*

$$x \oplus y = s\begin{bmatrix} n-1 \\ o \end{bmatrix}\left(A\left(x, y, \overset{n-2}{o}\right)\right)$$

holds.

THEOREM 4. *For any n points $x_1, \ldots, x_n \in Q$, the equality*

$$A(x_1^n) = \ominus x_1 \ominus x_2 \ominus \ldots \ominus x_n \oplus s\begin{bmatrix} n-1 \\ o \end{bmatrix}(o)$$

is valid.

Proof. Set $o' = s\begin{bmatrix} n-1 \\ o \end{bmatrix}(o)$. For any point $x \in Q$, we get

$$o' = s\begin{bmatrix} n-1 \\ o \end{bmatrix}(x \ominus x) = A\left(x, \ominus x, \overset{n-2}{o}\right),$$

i. e.

$$\ominus x = A\left(x, o', \overset{n-2}{o}\right).$$

It implies

$$o' \ominus A(x_1^n) = o' \oplus A\left(A(x_1^n), o', \overset{n-2}{o}\right) =$$

$$= s\begin{bmatrix} n-1 \\ o \end{bmatrix}\left(A\left(o', A\left(A(x_1^n), o', \overset{n-2}{o}\right), \overset{n-2}{o}\right)\right) =$$

$$= s\begin{bmatrix} n-1 \\ o \end{bmatrix}(A(x_1^n)) =$$

$$= x_1 \oplus \ldots \oplus x_n$$

which was to be proved.

THEOREM 5. *Given an abelian group $(Q, +)$. For a fixed $a \in Q$, let A_a be an n-ary operation on Q, defined by*

$$A_a(x_1^n) = -\sum_{i=1}^{n-1} x_i + a.$$

Then (Q, A_a) is an ATS-n-quasigroup.

Proof. Obviously, (Q, A_a) is a TS-n-quasigroup. Further, for $\mathbf{u} = (u_1, \ldots, u_{n-1}) \in Q^{n-1}$, set $u = \sum_{i=1}^{n} u_i$. Then

$$s_{\mathbf{u}}(x) = A_a(u_1^{n-1}, x) = a - u - x.$$

Therefore, for $\mathbf{v} = (v_1, \ldots, v_{n-1})$, $\mathbf{w} = (w_1, \ldots, w_{n-1}) \in Q^{n-1}$, $v = \sum_{i=1}^{n-1} v_i$, $w = \sum_{i=1}^{n-1} w_i$, it follows that

$$s_{\mathbf{w}}\, s_{\mathbf{u}}\, s_{\mathbf{v}}(x) = a - (u - v + w) - x,$$

i. e. $s_{\mathbf{w}}\, s_{\mathbf{u}}\, s_{\mathbf{v}}$ is a symmetry.

In [2] the notion of C^{n+1}-system was introduced as a TS-n-quasigroup (Q, A) which is bisymmetric (i. e. the expression $A(\{A(x_{i1}^{in})\}_{i=1}^{n})$ is invariant for any permutation of its elements $x_{ij} \in Q$, $i, j = 1, 2, \ldots, n$). According to the results in [2] (compare [3]) and to our Theorems 4 and 5 the following equivalence holds:

THEOREM 6. *TS-n-quasigroup (Q, A) satisfies the axiom (A) if and only if it is a C^{n+1}-system.*

THEOREM 7. *Let (Q, A) be an ATS-n-quasigroup. Then the abelian groups $S^0(Q)$ and $(Q, \oplus)$ are isomorphic.*

Proof. Any $p \in S^0(Q)$ could be written in the form $p = s_{\mathbf{b}} s_{\mathbf{a}}$, for some $\mathbf{a} = (a_1, \ldots, a_{n-1})$, $\mathbf{b} = (b_1, \ldots, b_{n-1}) \in Q^{n-1}$. Now, by setting $a = a_1 \oplus \ldots \oplus a_{n-1}$, $b = b_1 \oplus \ldots \oplus b_{n-1}$, it follows that

$$p(x) = A(b_1^{n-1}, A(a_1^{n-1}, x)) = a \ominus b \oplus x,$$

for all $x \in Q$. Let define $h : S^0(Q) \to Q$, by the equality

$$h(p) = h(s_{\mathbf{b}} s_{\mathbf{a}}) = a \ominus b,$$

for all $p \in S^0(Q)$. Obviously, h is an one-to-one mapping and $h(p_2 p_1) = h(p_1) \oplus h(p_2)$ is valid, for any two $p_1, p_2 \in S^0(Q)$, which proves the statement.

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TOTALNO SIMETRIČNE n-KVAZIGRUPE KOJE ZADOVOLJAVAJU TEOREM O TRI SIMETRIJE

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Sadržaj

U radu se istražuje struktura onih totalno simetričnih n-kvazigrupa koje imaju svojstvo da je produkt triju simetrija također simetrija. Svaka takva n-kvazigrupa može se izraziti u terminima pripadne Abelove grupe pomaka.