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ON BANACH ALGEBRAS WITHOUT ZERO DIVISORS

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Abstract. In this article we generalize Edwards' variant of Gel'fand-Mazur theorem for complex Banach algebras to any nonassociative Banach algebras. From this generalization we also obtain that if in a complex nonassociative Banach algebra there is

$$\lambda \|x\| \cdot \|y\| \leqq \|xy\| \leqq \mu \|x\| \cdot \|y\|$$

for some fixed positive λ, μ and any elements x, y of the algebra, this algebra is one-dimensional.

Throughout the paper let H be a (real or complex) normed space which is also a (not neccessarily associative) algebra with continuous multiplication. Such an H we call a *normed algebra* or, in the case of complete normed space, a *Banach algebra*. It is well known that if $\|\cdot\|$ is any norm, equivalent to the original norm, there is a positive constant μ such that

$$\|xy\| \leqq \mu \|x\| \cdot \|y\| \quad (x, y \in H). \tag{1}$$

The following theorem was proved by Edwards:

THEOREM 1. *Let H be a complex associative Banach algebra with the norm satisfying* (1) *with $\mu = 1$, and with a unit whose norm is* 1. *If*

$$\|x^{-1}\| \leqq \|x\|^{-1} \tag{2}$$

for any invertible element $x \in H$, then H is isometrically isomorphic to the complex field.

Let L be a regular representation: $L_x y = xy$. Since $L_{x^{-1}} = L_x^{-1}$, we can write the inequality (2) in the form

$$\|x\| \cdot \|L_x^{-1}\| \leqq 1. \tag{3}$$

We intend to generalize Theorem 1 to nonassociative case. Our proof will follow closely the original proof of Edwards.

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THEOREM 2. *Let H be a Banach algebra with norm $\|\cdot\|$, let $G = \{x \in H \mid \exists L_x^{-1}\}$ be nonempty and suppose that for some $\delta > 0$*

$$\|x\| \cdot \|L_x^{-1}\| \leqq \delta \quad (x \in G). \tag{4}$$

Then $G = H - \{0\}$.

Proof. If $\dim H = 0$, we have $G = \emptyset$. If $\dim H = 1$, then the proof is trivial. So suppose: $\dim H > 1$.

Define $A_\varrho = \{z \in H \mid \|z\| \geqq \varrho\}$ for any $\varrho > 0$. A_ϱ is a connected set. If x and y are noncolinear elements in A_ϱ, they are joined by the path

$$\tau \to f(\tau) = [(1-\tau)\|x\| + \tau\|y\|] \cdot \|(1-\tau)x + \tau y\|^{-1} \cdot [(1-\tau)x + \tau y]$$

in A_ϱ. But if $y = ax$ for some number a, we take z in A_ϱ, which is not colinear with x, and compose the path $f(\tau)$ from x to z with another one from z to y.

Observe that $G \cap A_\varrho$ is nonempty for every ϱ, since $x \in G$ implies $ax \in G$ for any number $a \neq 0$.

Since we have (1) it follows that $\|L_x\| \leqq \mu\|x\|$ $(x \in H)$. If $x \in G$, there is an open ball in $B(H)$ (the operator algebra on H) of radius ε and with center at L_x, in which all the elements are invertible. So, if $z \in H$, $\|z\| < \varepsilon/\mu$, then $\|L_x - L_{x+z}\| = \|L_z\| \leqq \mu\|z\| < \varepsilon$, which means that L_{x+z} is invertible and $x + z \in G$. Therefore G is open in H and so $G \cap A_\varrho$ is open in the relative topology of A_ϱ.

Now let $\{x_n\} \subset G \cap A_\varrho$ be a sequence converging to x. Clearly, $x \in A_\varrho$. We will show that $x \in G$. Since $\|L_{x_n}^{-1}\| \leqq \delta/\|x_n\| \leqq \delta/\varrho$, we have:

$$\|L_{x_m}^{-1} - L_{x_n}^{-1}\| = \|L_{x_m}^{-1}(L_{x_n} - L_{x_m})L_{x_n}^{-1}\| \leqq \|L_{x_m}^{-1}\| \cdot \|L_{x_n}^{-1}\| \cdot$$
$$\cdot \|L_{x_n} - L_{x_m}\| \leqq (\mu\delta^2/\varrho^2)\|x_n - x_m\|,$$

which implies that $\{L_{x_n}^{-1}\}$ is a Cauchy sequence in $B(H)$ and so it converges to a $U \in B(H)$. We have

$$\|L_x U - I\| \leqq \|L_x U - L_x L_{x_n}^{-1}\| + \|L_x L_{x_n}^{-1} - L_{x_n} L_{x_n}^{-1}\| \leqq$$
$$\leqq \|L_x\| \cdot \|U - L_{x_n}^{-1}\| + \mu\|x - x_n\| \cdot \delta/\varrho,$$

which implies that $L_x U = I$. Similarly, $UL_x = I$. Consequently, $U = L_x^{-1}$, so $x \in G \cap A_\varrho$. This shows that $G \cap A_\varrho$ is closed.

As A_ϱ is connected, it follows that $G \cap A_\varrho = A_\varrho$, and since $H - \{0\} = \bigcup_{\varrho > 0} A_\varrho$, the proof is complete.

COROLLARY 3. *Let H be the algebra from Theorem 2. In the complex case H is topologically isomorphic to the complex field.*

Proof. This is a direct consequence of the well known theorem, that a complex Banach algebra in which L_x is invertible for any nonzero $x \in H$ is one-dimensional.

COROLLARY 4. *Let H be a complex normed algebra with unit and with norm $\|\cdot\|$. Suppose that there exists a positive number λ such that*

$$\lambda \|x\| \cdot \|y\| \leqq \|xy\| \quad (x, y \in H). \tag{5}$$

Then H is topologically isomorphic to the complex field.

Proof. Let $\hat{H}$ be the completion of H. Denote by $\|.\|$ also the norm, extended from H to $\hat{H}$. Then by the properties of completion (5) remains true for any $x, y \in \hat{H}$.

Because of the existence of unit in $\hat{H}$ the set G from Theorem 2 is not empty. Let $x \in G$. Then for any $y \in \hat{H} - \{0\}$ we have $\|y\| = \|x \cdot L_x^{-1} y\| \leq \lambda \|x\| \cdot \|L_x^{-1} y\|$, or $\|x\| \cdot \|L_x^{-1} y\|/\|y\| \leqq 1/\lambda$, which implies that $\|x\| \cdot \|L_x^{-1}\| \leqq 1/\lambda$.

Now the conditions of Theorem 2 are satisfied for $\delta = 1/\lambda$ and so Corollary 4 follows from Corollary 3.

Conjecture. If the number field is real then the class of algebras satisfying the assumptions of Corollary 3 coincides with the class of algebras satisfying the assumption of Corollary 4; these algebras have dimension 1, 2, 4, or 8.

We hope to prove this conjecture by showing that these algebras cannot be infinite dimensional and then applying Bott-Milnor theorem about finite dimensional algebras with division.

REFERENCES:

[1] *I. Kaplansky,* Algebraic and Analytic Aspects of Operator Algebras. Providence, Amer. Math. Society 1970.

[2] *R. Larsen,* Banach Algebras, An Introduction. Marcel Dekker, Inc., New York 1973.

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O BANACHOVIH ALGEBRAH BREZ DELJITELJEV NIČA

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Vsebina

V članku posplošimo Edwardsovo varianto izreka Gelfand-Mazur na neasociativne Banachove algebre. Kot posledico pa dokažemo še, da če v neasociativni kompleksni Banachovi algebri velja

$$\lambda \|x\| \cdot \|y\| \leqq \|xy\| \leqq \mu \|x\| \cdot \|y\|$$

za neka pozitivna λ, μ ter poljubna elementa x, y algebre, je algebra enodimenzionalna.