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APPROXIMATION THEOREMS FOR FIELDS AND COMMUTATIVE RINGS

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Abstract. We give another proof of the approximation theorems for incomparable valuations. The proofs are shorter than the proofs in [5]. They can also be applied to valuations in commutative rings which is not the case for the proofs in [5].

1. *Introduction.* Let (v, Γ) and (w, Δ) be two valuations on a commutative ring R and $w = \varphi \circ v$ where φ is an order homomorphism of the group Γ onto the group Δ. Then we say that w dominates v and denote $w \geqslant v$. Valuations v and v' are called dependent if there exists a valuation w with $w \geqslant v$ and $w \geqslant v'$ and $w(R) \neq \{w(1), w(0)\}$; and they are called independent otherwise. Note that $w \geqslant v$ implies that $v^{-1}(\infty) = w^{-1}(\infty)$. It is easy to show that $w \geqslant v$ if and only if $A_v \subseteq \subseteq A_w$ and $v^{-1}(\infty) \subseteq P_w \subseteq P_v$, where A_v and A_w are valuation rings and P_v and P_w are positive ideals of v and w ([4], Proposition 4). Let (R, P) be a Prüfer valuation pair and let R_1 be an overring of R, i. e. let R_1 be a ring with $R \subseteq R_1 \subseteq T(R)$ where $T(R)$ is the total quotient ring of R. Then there exists a prime ideal P_1 of R such that $P_1 \subseteq P$ and (R_1, P_1) is a Prüfer valuation pair ([1], Theorem 2.5). Therefore, if v and w are Prüfer valuations of the total quotient ring $T(R)$, then $w \geqslant v$ if and only if $A_w \supseteq A_v$, where A_v and A_w are valuation rings of v and w.

Let v_i, v_j be two incomparable valuations on a commutative ring R, let A_i, A_j be corresponding valuation rings, let P_i, P_j be corresponding positive ideals and let Γ_i, Γ_j be corresponding value groups. Let $v_i^{-1}(\infty) = v_j^{-1}(\infty)$ and let P be the maximal prime ideal of A_i and A_j such that $P \subseteq P_i$ and $P \subseteq P_j$. Certainly, $P \supseteq v_i^{-1}(\infty) = = v_j^{-1}(\infty)$ and $P = v_i^{-1}(\infty) = v_j^{-1}(\infty)$ if and only if the valuations v_i and v_j are independent, i. e. the valuation $v_i \wedge v_j$ is trivial. Since the valuations v_i and v_j are incomparable it follows that $P \neq P_i$ and $P \neq P_j$. Let Δ_{ij}, Δ_{ji} be the isolated subgroups of the groups Γ_i, Γ_j respectively corresponding to P. $\Delta_{ij} = \Gamma_i$, $\Delta_{ji} = \Gamma_j$ if and only if the valuations v_i and v_j are independent. If $v_i^{-1}(\infty) \neq v_j^{-1}(\infty)$, then the valuations v_i and v_j are independent and let again $\Delta_{ij} = \Gamma_i$, $\Delta_{ji} = \Gamma_j$. Let $\Theta_{ij} : \Gamma_i \to \Gamma_i/\Delta_{ij}$, $\Theta_{ji} : \Gamma_j \to \Gamma_j/\Delta_{ji}$ be the natural

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homomorphisms. The groups Γ_i/Δ_{ij} and Γ_j/Δ_{ji} are ordered isomorphic with the value group of $v_i \wedge v_j$, and consequently they can be identified.

A pair $(\alpha_i, \alpha_j) \in \Gamma_i \times \Gamma_j$ is called compatible if, by the preceding identification, $\Theta_{ij}(\alpha_i) = \Theta_{ji}(\alpha_j)$. Let $v_1, \ldots, v_s$ $(s \geqslant 2)$ be pairwise incomparable valuations of R. $(\alpha_1, \alpha_2, \ldots, \alpha_s) \in \Gamma_1 \times \Gamma_2 \times \times \ldots \times \Gamma_s$ is called compatible if and only if every pair (α_i, α_j) $(i \neq j)$ is compatible. If $\alpha_i = v_i(x)$, $\alpha_j = v_j(x)$ $(x \in R)$, then the pair (α_i, α_j) is compatible, since $\overline{v_i(x)} = w(x) = \overline{v_j(x)}$, where $w = v_i \wedge v_j$, $\overline{v_i(x)} = \Theta_{ij}(v_i(x))$, $\overline{v_j(x)} = \Theta_{ji}(v_j(x))$.

If the valuations $v_1, v_2, \ldots, v_s$ are pairwise independent, then every $(\alpha_1, \alpha_2, \ldots, \alpha_s) \in \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_s$ is compatible.

2. THEOREM 1. (Approximation theorem in the neighbourhood of zero). *Let $v_1, v_2, \ldots, v_n$ be noncomparable valuations of the field K, $V_1, V_2, \ldots, V_n$ valuation rings, $M_1, M_2, \ldots, M_n$ maximal ideals and $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ value groups of these valuations respectively and let $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n$ be compatible. Then there exists $x \in K$ such that $v_i(x) = \alpha_i$ $(i = 1, 2, \ldots, n)$.*

Proof. We first show that there exists $a_1 \in K$ such that $v_1(a_1) = = 0$, $v_i(a_1) < 0$ $(i = 2, 3, \ldots, n)$. We will prove this by induction on n. Let $n = 2$. Take $x_1 \in V_1 \setminus V_2$. If $x_1 \in V_1 \setminus M_1$, then $v_1(x_1) = = 0$, $v_2(x_1) < 0$. If $x_1 \in M_1$, then for $1 + x_1$ we have $v_1(1 + x_1) = = 0$, $v_2(1 + x_1) < 0$ and so we may take $a_1 = 1 + x_1$. Let $n > 2$. Suppose that there exist $a_1', a_1'' \in K$ such that $v_1(a_1') = 0$, $v_i(a_1') < 0$ $(i = 2, 3, \ldots, n-1)$; $v_1(a_1'') = 0$, $v_i(a_1'') < 0$ $(i = 3, 4, \ldots, n)$ and prove that there exists $a_1 \in K$ such that $v_1(a_1) = 0$, $v_i(a_1) < 0$ $(i = 2, 3, \ldots, n)$. It is easy to conclude that for some positive integer m $v_i(a_1'^m) \neq v_i(a_1'')$ $(i = 2, 3, \ldots, n)$. If $v_1(a_1'^m + a_1'') = 0$, then for $a_1 = a_1'^m + a_1''$ we have $v_1(a_1) = 0$, $v_i(a_1) < 0$ $(i = 2, 3, \ldots, n)$. If $v_1(a_1'^m + a_1'') > 0$, then for $a_1 = 1 + (a_1'^m + a_1'')$ we have $v_1(a_1) = = 0$, $v_i(a_1) < 0$ $(i = 2, 3, \ldots, n)$. Therefore, there exists $a_1 \in K$ such that $v_1(a_1) = 0$, $v_i(a_1) < 0$ $(i = 2, 3, \ldots, n)$. For $\frac{1}{a_1}$ we have $v_1\left(\frac{1}{a_1}\right) = = 0$, $v_i\left(\frac{1}{a_1}\right) > 0$ $(i = 2, 3, \ldots, n)$.

Let $P_2, \ldots, P_n$ be prime ideals of $V_2, \ldots, V_n$ respectively such that $P_i \nsubseteq M_1$ $(i = 2, 3, \ldots, n)$. Valuation rings V_1, $(V_i)_{P_i}$ are incomparable $(i = 2, 3, \ldots, n)$, therefore there exists $a_1 \in K$ such that $v_1(a_1) = = 0$, $(v_i)_{P_i}(a_1) > 0$ $(i = 2, 3, \ldots, n)$; and since P_i is the maximal ideal of $(V_i)_{P_j}$ $(i = 2, 3, \ldots, n)$ it follows that $a_1 \in (V_1 \setminus M_1) \cap \cap (\bigcap_{i=2}^{n} P_i)$.

Let $(\alpha_1, \ldots, \alpha_n) \in \Gamma_1 \times \ldots \times \Gamma_n$, $\alpha_1 = 0$, $\alpha_2 > 0$, $\alpha_3 > 0, \ldots \ldots, \alpha_n > 0$, be compatible. Take $x_i \in V_i$ such that $v_i(x_i) = \alpha_i$ and let P_i be the minimal prime ideal of V_i that contains x_i, $(i = 2, 3, \dots, n)$. Then $P_i \nsubseteq M_1$ $(i = 2, 3, \dots, n)$. Take $a_1 \in (V_1 \setminus M_1) \cap (\bigcap_{i=2}^{n} P_i)$. For some positive integer m we have $v_1(a_1^m) = 0$, $v_i(a_1^m) > \alpha_i$ $(i = 2, 3, \dots, n)$.

Let $(\alpha_1, \dots, \alpha_n) \in \Gamma_1 \times \dots \times \Gamma_n$ be compatible. Take $x_i \in K$ such that $v_i(x_i) = \alpha_i$ $(i = 1, 2, \dots, n)$. Take $a_i \in K$ $(i = 1, 2, \dots, n)$ such that $v_i(a_i) = 0$, $v_j(a_i) > \alpha_j - v_j(x_i)$ $(i, j = 1, 2, \dots, n;\ i \neq j)$. For $x = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$ we have $v_i(x) = \alpha_i$ $(i = 1, 2, \dots, n)$.

THEOREM 2. (General approximation theorem). *Let $v_1, v_2, \dots, v_n$ be pairwise incomparable valuations of the field K, let $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \Gamma_1 \times \dots \times \Gamma_n$ be compatible and let $b_1, b_2, \dots, b_n \in K$. Then there exists $x \in K$ such that $v_i(x - b_i) = \alpha_i$ $(i = 1, 2, \dots, n)$, if and only if*

$$v_i(b_i - b_j) < \alpha_i \Rightarrow \alpha_i - v_i(b_i - b_j) \in \Delta_{ij}. \tag{1}$$

Proof. Suppose that (1) is satisfied. Let $V_1, V_2, \dots, V_n$ be valuation rings of $v_1, v_2, \dots, v_n$ respectively, and let $D = \bigcap_{i=1}^{n} V_i$. From Theorem 1 it follows that $V_i = D_{M_i}$, where M_i is the center of v_i on D $(i = 1, 2, \dots, n)$ and if M is a maximal ideal of D, then $M = M_i$ for some i. Suppose first that $b_i \in D$ $(i = 1, 2, \dots, n)$. We will first prove that there exists $b \in K$ such that $v_i(b - b_i) \geqslant \alpha_i$ $(i = 1, 2, \dots, n)$. Let $Q_i = \{b \in D \mid v_i(b) \geqslant \alpha_i\}$, $(i = 1, 2, \dots, n)$ and let $i, j \in \{1, 2, \dots, n\}$, $i \neq j$. We will show that $b_i - b_j \in (Q_i + Q_j) V_k$ $(k = 1, 2, \dots, n)$. Since $v_i(b_i - b_k) < \alpha_i \Rightarrow \alpha_i - v_i(b_i - b_k) \in \Delta_{ik}$ it follows easily that $b_i - b_k \in Q_i V_k \subseteq (Q_i + Q_j) V_k$, and since $v_j(b_j - b_k) < \alpha_j \Rightarrow \alpha_j - v_j(b_j - b_k) \in \Delta_{jk}$ it follows that $b_j - b_k \in Q_j V_k \subseteq (Q_i + Q_j) V_k$, $(k = 1, 2, \dots, n)$. Therefore $b_i - b_j = (b_i - b_k) + (b_k - b_j) \in (Q_i + Q_j) V_k$, $(k = 1, 2, \dots, n)$. Therefore, $b_i - b_j \in Q_i + Q_j$ $(i, j = 1, 2, \dots, n)$ and by Chinese remainder theorem there exists $b \in D$ such that $b_i - b \in Q_i$ $(i = 1, 2, \dots, n)$. Clearly, $v_i(b - b_i) \geqslant \alpha_i$ $(i = 1, 2, \dots, n)$.

Now let $b_i \in K$ $(i = 1, 2, \dots, n)$. Take $b_i', d \in D$ such that $b_i = \frac{b_i'}{d}$ $(i = 1, 2, \dots, n)$, and let $b' \in D$ be such that $v_i(b' - b_i') \geqslant \alpha_i + v_i(d)$. Then for $b = \frac{b'}{d}$ we have $v_i(b - b_i) \geqslant \alpha_i$ $(i = 1, 2, \dots, n)$.

Take $\beta_i \in \bigcap_{j \neq i} \Delta_{ij}$, $\beta_i > 0$ $(i = 1, 2, \dots, n)$ and $\alpha_i' = \alpha_i + \beta_i$ $(i = 1, 2, \dots, n)$. Then there exists $b \in K$ such that $v_i(b - b_i) \geqslant \alpha_i'$ $(i = 1, 2, \dots, n)$. Now, by the approximation theorem in the neighbourhood of zero, there exists $a \in K$ such that $v_i(a) = \alpha_i$ $(i = 1, 2, \dots, n)$. For $x = a + b$ we have $v_i(x - b_i) = \alpha_i$ $(i = 1, 2, \dots, n)$.

Conversely, suppose that there exists $x \in K$ such that $v_i(x - b_i) = \alpha_i$ $(i = 1, 2, \dots, n)$. It is easy to check that then (1) holds ([5], Theorem 3, page 136).

Remark. It is easy to conclude that

$$(v_i(b_i - b_j) < a_i \Rightarrow a_i - v_i(b_i - b_j) \in \Delta_{ij}) \Leftrightarrow (v_i(b_i - b_j) < a_i,$$

$$v_j(b_j - b_i) < a_j \Rightarrow a_i - v_i(b_i - b_j) \in \Delta_{ij}).$$

THEOREM 3. *Let* $v_1, v_2, \ldots, v_n$ *be pairwise incomparable valuations of the field* K, *let* $(a_1, \ldots, a_n) \in \Gamma_1 \times \ldots \times \Gamma_n$ *be compatible and let* $b_1, \ldots, b_n \in K$ *be such that* $v_i(b_i) < a_i \Rightarrow a_i - v_i(b_i) \in \bigcap_{j \neq i} \Delta_{ij}$. *Then there exists* $b \in K$ *such that* $v_i(b - b_i) = a_i$ $(i = 1, 2, \ldots, n)$.

Proof. By the preceding theorem and by the preceding remark it is sufficient to show that

$v_i(b_i - b_j) < a_i$, $v_j(b_j - b_i) < a_j \Rightarrow a_i - v_i(b_i - b_j) \in \Delta_{ij}$, $(i, j = 1, 2, \ldots, n; i \neq j)$. Suppose that $v_i(b_i - b_j) < a_i$, $v_j(b_j - b_i) < a_j$.

1) If $v_i(b_i) \leqslant v_i(b_j)$, then $v_i(b_i) < a_i$ and consequently $a_i - v_i(b_i) \in \Delta_{ij}$, therefore especially $a_i - v_i(b_i - b_j) \in \Delta_{ij}$.

2) If $v_j(b_j) \leqslant v_j(b_i)$, then we similarly conclude that $a_j - v_j(b_j - b_i) \in \Delta_{ji}$, i. e. $\overline{a}_j - \overline{v_j(b_j - b_i)} = \overline{0}$, so that $\overline{a}_i - \overline{v_i(b_i - b_j)} = \overline{0}$, and therefore $a_i - v_i(b_i - b_j) \in \Delta_{ij}$.

3) If $v_i(b_i) > v_i(b_j)$ and $v_j(b_j) > v_j(b_i)$, then $\overline{v_i(b_i)} \geqslant \overline{v_i(b_j)} = \overline{v_j(b_j)} \geqslant \overline{v_j(b_i)}$, i. e. $\overline{v_i(b_i)} = \overline{v_i(b_j)}$, and therefore $a_i - v_i(b_i - b_j) \in \Delta_{ij}$.

3. Let R be a Prüfer ring, i. e. a ring in which each finitely generated regular ideal is invertible, let $\{M_\lambda\}$ be the set of maximal ideals and let $\{P_\lambda\}$ be the set of prime ideals of R. It is well known that R is a Prüfer ring if and only if $(R_{[M_\lambda]}, [M_\lambda] R_{[M_\lambda]})$ is a valuation pair for every $M_\lambda \in \{M_\lambda\}$. Also, $(R_{[P_\lambda]}, [P_\lambda] R_{[P_\lambda]})$ is a valuation pair for every $P_\lambda \in \{P_\lambda\}$. Conversely, if V is a valuation overring of R, then $V = R_{[P_\lambda]}$ for some $P_\lambda \in \{P_\lambda\}$ ([3], Chapter X).

Let $\{V_\lambda\}$ be the set of valuation overrings of R and let $\{v_\lambda\}$ be the corresponding valuations. It is easy to see that Theorems 1, 2, and 3 can be applied to valuations $\{v_\lambda\}$.

THEOREM 4. *Let* R *be a Prüfer ring,* $V_1, V_2, \ldots, V_n$ *pairwise incomparable valuation overrings of* R, *let* $v_1, v_2, \ldots, v_n$ *be the corresponding valuations,* $\Gamma_1, \ldots, \Gamma_n$ *the corresponding value groups, and let* $(a_1, \ldots, a_n) \in \Gamma_1 \times \ldots \times \Gamma_n$ *be compatible. Then there exists* $x \in T(R)$ *such that* $v_i(x) = a_i$ $(i = 1, 2, \ldots, n)$, *where* $T(R)$ *is the total quotient ring of* R.

Proof. First let $(a_1, a_2, \ldots, a_n) \in \Gamma_1 \times \ldots \times \Gamma_n$ be compatible and such that $a_1 = 0$, $a_i > 0$ $(i = 2, 3, \ldots, n)$. Then there exists $x \in R$ such that $v_1(x) = 0$, $v_i(x) > a_i$ $(i = 2, 3, \ldots, n)$. Namely, take $x_i \in T(R)$ such that $v_i(x_i) = a_i$ $(i = 2, 3, \ldots, n)$ and let P_i be the minimal prime ideal of V_i that contains x_i $(i = 2, 3, \ldots, n)$. Take $x \in (V_1 \setminus M_1) \cap (\bigcap_{i=2}^{n} P_i) \cap R$, where M_1 is the positive ideal of V_1.

Then for some positive integer m we have $v_1(x^m) = 0$, $v_i(x) > a_i$ $(i = 2, 3, \ldots, n)$.

Let $(a_1, a_2, \ldots, a_n) \in \Gamma_1 \times \ldots \times \Gamma_n$ be compatible. Take x_i, $a_i \in T(R)$ such that $v_i(x_i) = a_i$ $(i = 1, 2, \ldots, n)$, $v_i(a_i) = 0$, $v_j(a_i) > a_j - v_j(x_i)$ $(i, j = 1, 2, \ldots, n;, i \neq j)$. For $x = x_1 a_1 + x_2 a_2 + \ldots + \ldots + x_n a_n$ we have $v_i(x) = a_i$ $(i = 1, 2, \ldots, n)$.

THEOREM 5. *Let R be a Prüfer ring, $V_1, V_2, \ldots, V_n$ pairwise incomparable valuation overrings of R, let $v_1, \ldots, v_n$ be the corresponding valuations, $\Gamma_1, \ldots, \Gamma_n$ the corresponding value groups, and let $b_1, b_2, \ldots, b_n \in T(R)$, where $T(R)$ is the total quotient ring of R. Then there exists $x \in T(R)$ such that $v_i(x - b_i) = a_i$ $(i = 1, 2, \ldots, n)$, if and only if*

$$v_i(b_i - b_j) < a_i \Rightarrow a_i - v_i(b_i - b_j) \in \Delta_{ij}. \qquad (1)$$

Proof. Let $D = \bigcap_{i=1}^{n} V_i$. Since $R \subseteq D$, D is a Prüfer ring, and $V_i = D_{[M_i]}$, where M_i is the center of v_i on D $(i = 1, 2, \ldots, n)$. Moreover, if M is a maximal ideal of D, then $M = M_i$ for some i. Suppose first that $b_i \in D$ $(i = 1, 2, \ldots, n)$. We will show that there exists $x \in T(R)$ such that $v_i(x - b_i) \geqslant a_i$ $(i = 1, 2, \ldots, n)$. Let $Q_i = \{b \in D \mid v_i(b) \geqslant a_i\}$, $(i = 1, 2, \ldots, n)$, and let $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. We will prove that $b_i - b_j \in Q_i + Q_j$. Since $Q_i + Q_j$ is a regular ideal of D it is sufficient to show that $b_i - b_j \in (Q_i + Q_j) V_k$ for every $k \in \{1, 2, \ldots, n\}$. Since (1) holds and since an ideal of a Prüfer valuation ring V is v-closed if and only if it is a regular ideal of V, it follows that $b_i - b_k \in Q_i V_k \subseteq (Q_i + Q_j) V_k$ and $b_j - b_k \in Q_j V_k \subseteq (Q_i + Q_j) V_k$ $(k = 1, 2, \ldots, n)$. Therefore, $b_i - b_j = (b_i - b_k) + (b_k - b_j) \in (Q_i + Q_j) V_k$ $(k = 1, 2, \ldots, n)$, i. e. $b_i - b_j \in Q_i + Q_j$. Since $b_i - b_j \in Q_i + Q_j$ $(i, j = 1, 2, \ldots, n)$, by Chinese remainder theorem there exists $b \in D$ such that $b - b_i \in Q_i$ $(i = 1, 2, \ldots, n)$. Clearly, $v_i(b - b_i) \geqslant a_i$ $(i = 1, 2, \ldots, n)$. The rest of the proof is the same as in Theorem 2.

THEOREM 6. *Let R be a Prüfer ring, $V_1, \ldots, V_n$ pairwise incomparable valuation overrings of R, $v_1, \ldots, v_n$ the corresponding valuations, let $(a_1, \ldots, a_n) \in \Gamma_1 \times \ldots \times \Gamma_n$ be compatible and let $b_1, \ldots, b_n \in T(R)$ be such that $v_i(b_i) < a_i \Rightarrow a_i - v_i(b_i) \in \bigcap_{j \neq i} \Delta_{ij}$. Then there exists $b \in T(R)$ such that $v_i(b - b_i) = a_i$ $(i = 1, 2, \ldots, n)$.*

Proof. Theorem 6 can be proved from Theorem 5 in the same way as Theorem 3 from Theorem 2.

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TEOREME O APROKSIMACIJI ZA POLJA I KOMUTATIVNE PRSTENE

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Sadržaj

U ovom radu dajemo drugi dokaz teorema o aproksimaciji za neuporedive valuacije. Dokazi su kraći od dokaza u [5], a mogu se primijeniti i na valuacije u komutativnim prstenima što nije slučaj za dokaze u [5]