GLASNIK MATEMATIČKI
Vol. 18 (38) (1983), 39-46.

CHARACTERIZATIONS OF THE 0-DIMENSIONAL RINGS

M. Arapović, Sarajevo

Abstract. Throughout this paper rings are understood to be commutative with unity, and subrings are understood to have same identity as their overrings. In this paper the following results are given: *i*) a new proof of the theorem that a commutative ring R is a π-regular ring if and only if R is a 0-dimensional ring; *ii*) different characterizations of the 0-dimensional rings; *iii*) if R is a 0-dimensional ring, then $T(R[X])$ is also a 0-dimensional ring, where $R[X]$ is a polynomial ring and $T(R[X])$ is the total quotient ring of $R[X]$; *iv*) if R is a 0-dimensional ring and if A is a finitely generated ideal of R, then $A^n = (e)$ for some positive integer n and for some idempotent element e of R and *v*) a new proof of the fact that the ring R, whose total quotient ring $T(R)$ is 0-dimensional, is an additively regular ring.

A commutative ring with unity will be denoted by R. We will also use the symbol $T(R)$ when we want to emphasize the fact that $T(R)$ is a total quotient ring, i. e. a ring in which every regular element is invertible.

We next recall the following definitions.

Definition 1. A prime ideal P of R is called regular if P contains a regular element of R.

Definition 2. A commutative ring R is called semiprime if R contains no nonzero nilpotent elements.

Definition 3. A commutative ring R is called regular if for $a \in R$ there exists $a' \in R$ such that $a^2 a' = a$.

Definition 4. A commutative ring R is called π-regular if for $a \in R$ there exist $a' \in R$ and a positive integer n such that $a^n = (a^n)^2 a'$.

Definition 5. Let A be an ideal of a commutative ring R. We call A a radical ideal if it is equal to its radical, i. e. if A can be expressed as the intersection of the prime ideals of R that contain A.

Definition 6. Let R be a commutative ring with the total quotient ring $T(R)$. We say that R is an additively regular ring if R satisfies the condition: If $a \in T(R)$, then there exists $b \in R$ such that $a + b$ is a regular element of $T(R)$.

Mathematics subject classifications (1980): Primary 13 A 99; Secondary 13 B 25, 16 A 30, 13 C 15.

Key words and phrases: 0-dimensional rings, π-regular rings, regular rings, polynomial rings, additively regular rings, idempotent ideals, radical ideals.

It is well known that a minimal prime ideal of R consists of zero-divisors of R. ([3], Theorem 84). For completeness, we give a short proof of this fact here.

LEMMA 1. *If P is a regular prime ideal of a commutative ring R, then there exists a prime ideal P_1 of R properly contained in P. Therefore, every minimal prime ideal of R is non-regular.*

Proof. Let us form the ring R_P. Take a prime ideal $\overline{P}_1$ of R_P, maximal with respect to disjointness from the multiplicative system of the regular elements of R_P. If P_1 is a prime ideal of R corresponding to $\overline{P}_1$ in the natural way, then P_1 is a prime ideal of R properly contained in P.

Let R be a regular ring. It is easy to conclude that R has no non-zero nilpotent elements. Namely, let a be a nilpotent element of R. Then there exists $a' \in R$ such that $a = a^2 a' = a^3 a'^2 = a^4 a'^3 = \ldots = 0$. Hence, if $\{P_\lambda\}$ is the set of the prime ideals of R, $\cap P_\lambda = (0)$ and it is possible to present R as a subdirect product of the domains $\{R/P_\lambda\}$.

The following theorem characterizes the regular rings. The implications $a) \Rightarrow d)$ and $a) \Rightarrow e)$, can be found in the literature in one form or another. They are however included here for the following reasons: first it is nicer to have all these facts collected in one theorem and second the proofs given here are somewhat different, for example, the proof $a) \Rightarrow e)$ is different than the one presented in ([3] p. 64).

THEOREM 2. *Let R be a commutative ring with unity. The following statements are equivalent:*

a) R is a regular ring;

b) R is a total quotient ring and for $a \in R$ there exists an idempotent $e \in R$ such that $a = ae$ and $a + (1 - e)$ is a regular element of R;

c) R is a total quotient ring and for $a \in R$ there exists $b \in R$ such that $ab = 0$ and $a + b$ is a regular element of R;

d) R is a total quotient ring and every $a \in R$ has the form $a = re$, where r is a regular and e is an idempotent element of R;

e) R is a semiprime 0-dimensional ring.

Proof. Let R be a semiprime ring and let $\{P_\lambda\}$ be the set of non-regular prime ideals of R. Clearly $\cap P_\lambda = (0)$, R/P_λ is a domain for every λ, and R is a subdirect product of the domains $\{R/P_\lambda\}$. An element $r \in R$ is regular if and only if its component in R/P_λ is not equal to zero for every λ. An element e of R is idempotent if and only if its component in R/P_λ is unity or zero for every λ.

a) ⇒ *b*). Let a be a regular element of R. Then there exists an $a' \in R$ such that $a = a^2 a'$, i. e. $a(aa' - 1) = 0$, furthermore, since a is regular $aa' = 1$, i. e. a is invertible; therefore R is a total quotient ring. Suppose $a \in R$, then there exists $a' \in R$ such that $a = a^2 a'$. It is easy to see that $e = aa'$ is an idempotent element of R whose component is unity in the places where the component of a is different from zero, and zero in the places where the component of a is zero. $1 - e$ is an idempotent element of R that has unity in those places and only in those places where the component of a is zero. Hence $a + 1 - e$ is a regular element of R.

b) ⇒ *c*). Obvious.

c) ⇒ *d*). Let $n \in R$ be a nilpotent element of R. Then $n + r$ is a regular element of R for $r \in R$ if and only if r is a regular element of R. Therefore, if *c*) is valid, then R has no nonzero nilpotent elements. Let $a \in R$. Then there exists $b \in R$ such that $ab = 0$ and $a + b$ is a regular element of R. $a = (a + b)\dfrac{a}{a+b}$ and it is easy to conclude that $\dfrac{a}{a+b}$ is an idempotent element of R that has the unity in those places where the component of a is different from zero, and in the other places has the component zero.

d) ⇒ *e*). Let P_1, P_2 be prime ideals of R, $P_1 \subseteq P_2$. Let $a \in P_2$. $a = re$, where r is a regular and e is an idempotent element of R. Since $r \notin P_2$, we have $e \in P_2$; furthermore $e \in P_2$ implies $1 - e \notin P_2$. Since $e(1 - e) = 0$, it follows that $e \in P_1$. Therefore, $a \in P_1$. i. e. $P_1 = P_2$.

e) ⇒ *b*). Let $a \in R$. Let S be the subring of the direct product $\prod_\lambda R/P_\lambda$ that is generated by R and the idempotent element $e \in \prod_\lambda R/P_\lambda$ that has component the unity in those places where the component of a is different from zero, and in all other places has the component zero. Therefore, $S = R[e]$. Since e is a root of the polynomial $x^2 - x \in R[x]$, S is an integral overring of R. Hence, S is a 0-dimensional ring and therefore S is a total quotient ring. $1 - e + a$ is a regular element of S, and so $\dfrac{1}{1 - e + a} \in S$, therefore, $\dfrac{1}{1 - e + a}$ has the form $\dfrac{1}{1 - e + a} = r_1 + r_2 e$; $r_1, r_2 \in R$. It is easy to see that $e = \dfrac{1}{1 - e + a}\, a = (r_1 + r_2 e)\, a = (r_1 + r_2)\, a$ and therefore $e \in R$.

b) ⇒ *a*). Let $a \in R$. Then there exists an idempotent $e \in R$ such that $a = ae$ and $a + 1 - e$ is a regular element of R. The equation $a = a^2 a'$ holds for $a' = \dfrac{1}{a + 1 - e} \in R$.

Examples of regular rings. The direct product of fields is a regular ring. The subdirect product of fields is a regular ring if and only if it is a total quotient ring that has the property that for each of its elements a there exists an idempotent element which has the unity in those components where the component of a is different from zero and has the zero element in the remaining components. A Noetherian semiprime ring that is a total quotient ring (a ring is of this type if and only if it is a semiprime total quotient ring having only finitely many prime ideals) is a regular ring; namely, every such ring is a direct product of finitely many fields. The ring of funcions, defined on an interval $[a, b]$ taking values in the field of the complex numbers, is a regular ring. It is shown in [4] that the complete quotient ring of a semiprime commutative ring R is a regular ring. Suppose the semiprime ring S is an integral overring of the regular ring R. Since R is a 0-dimensional ring, S is a 0-dimensional ring also. Therefore, S is a regular ring.

Let R be a regular ring. Let $a_1, a_2 \in R$ and let e_1 and e_2 be the idempotent elements of R such that $a_1 = a_1 e_1, a_2 = a_2 e_2$ and $a_1 + (1 - e_1)$, $a_2 + (1 - e_2)$ are regular elements of R. It is easy to verify that $e = e_1 + (1 - e_1) e_2 = e_2 + (1 - e_2) e_1$ is an idempotent element and that $(a_1, a_2) = (e)$. So we have derived the known result that every finitely generated ideal of R is principal and generated by an idempotent element of R, i. e. if $a_1, a_2, \ldots, a_n \in R$ then there exists an idempotent element e of R such that $(a_1, a_2, \ldots, a_n) = (e)$. ([4], § 3.5. Lemma (von Neumann)). It is easy to prove that $(a_1, a_2, \ldots, a_n, 1 - e) = (1)$ and $(a_1, a_2, \ldots, a_n)(1 - e) = (0)$ If we realize R as a subdierct product of fields, then e is the idempotent element of that subdirect product that has the unity in those components where at least one of the elements $a_1, a_2, \ldots, a_n$ has non-zero component, and has zero in the remaining components.

PROPOSITION 3. *Let R be a regular ring and let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a set of indeterminates over R. Then the total quotient ring $T(R[X])$ of $R[X]$ is a regular ring. (We write $R[X]$ instead of $R[\{X_\lambda\}_{\lambda \in \Lambda}]$.)*

Proof. If $f \in R[X]$, let A_f denote the ideal of R generated by the coefficients of f. It is known that an element f of $R[X]$ is a zero divisor if and only if there is a non zero element r of R such that $rA_f = (0)$. ([2], Proposition 24.7). Since R is a regular ring, $A_f = (e)$ for some idempotent element e of R. $f(1 - e) = 0$ and $f + (1 - e)$ is a regular element of $R[X]$. It follows by Theorem 2 that the total quotient ring $T(R[X])$ of $R[X]$ is a regular ring.

It is easy to verify that a homomorphic image of the regular ring R is a regular ring.

PROPOSITION 4. *Let $\varphi(R)$ be a homomorphic image of the regular ring R. Then $\varphi(R)$ is a regular ring.*

Proof. Let $\varphi(a) \in \varphi(R)$, where $\varphi(a)$ is the image of $a \in R$. Since R is a regular ring, there exists $a' \in R$ such that $a^2 a' = a$. Then $(\varphi(a))^2 \varphi(a') = \varphi(a)$, therefore, $\varphi(R)$ is also a regular ring.

We can characterize the regular rings in terms of radical ideals.

THEOREM 5. *Let R be a commutative ring. The following statements are equivalent:*

a) R is a regular ring;

b) Every ideal of R is a radical ideal;

c) Every ideal of R is an idempotent ideal.

Proof. a) $\Rightarrow$ *b).* Let A be an ideal of R. Since R/A is a regular ring, A is a radical ideal of R.

b) $\Rightarrow$ *c).* Let A be an ideal of R. It is easy to verify: If $A \neq A^2$ then A^2 is not a radical ideal of R. Therefore $A = A^2$.

c) $\Rightarrow$ *a).* Let $a \in R$. Since $(a^2) = (a)$, there exists $a' \in R$ such that $a = a^2 a'$. Therefore R is a regular ring.

It is easy to show that a π-regular ring is a total quotient ring and it is known that a commutative ring R is a π-regular if and only if R is a 0-dimensional ring. The following theorem characterizes the π-regular rings.

THEOREM 6. *The following statements are equivalent:*

a) R is a π-regular ring;

b) R is a total quotient ring and for every $r \in R$ there exists an idempotent $e_r \in R$ such that $r + (1 - e_r)$ is a regular element of R and $r(1 - e_r)$ is a nilpotent element of R;

c) R is a total quotient ring and for every $a \in R$ there exists $b \in R$ such that $a + b$ is a regular element and ab is a nilpotent element of R;

d) R is a total quotient ring and for every $a \in R$ there exists a positive integer n such that $a^n = re$, where r is a regular and e is an idempotent element of R.

e) R is a 0-dimensional commutative ring.

Proof. a) $\Rightarrow$ *b).* Let $r \in R$, r a regular element. Then there exists $r' \in R$ and a positive integer n such that $r^n = (r^n)^2 r'$, i. e. $r^n(r^n r' - 1) = 0$. It follows, since r is a regular element of R, that $r^n r' - 1 = 0$, i. e. $r(r^{n-1} r') = 1$. Therefore, π-regular rings are total quotient rings. Let $r \in R$. Then there exist $r' \in R$ and a positive integer n such that $(r^n)^2 r' = r^n$. It is easy to verify that $r^n r'$ is an idempotent element of R, and, therefore, $1 - r^n r'$ is also an idempotent element of R. Furthermore, $[r(1 - r^n r')]^n = r^n(1 - r^n r')^n = r^n(1 - r^n r') = 0$, hence, $r(1 - r^n r')$ is a nilpotent element of R. We will show that $r + (1 - r^n r')$ is a regular element of R. It is sufficient to show that $r + (1 - r^n r') \notin P$, for every prime ideal P of R. Let P be a prime ideal of R. Since $r(1 - r^n r')$ is a nilpotent element of R, $r(1 - r^n r') \in P$. Hence, either $r \in P$ or $1 - r^n r' \in P$. If both $r \in P$ and $1 - r^n r' \in P$, then $1 \in P$. Therefore, $r + (1 - r^n r') \notin P$, i. e. $r + (1 - r^n r')$ is a regular element of R.

b) $\Rightarrow$ *c*). Obvious.

c) $\Rightarrow$ *d*). Let $a \in R$. Then there exist $b \in R$ and a positive integer n such that $(ab)^n = 0$ and $a + b$ is a regular element of R. $a^n + b^n$ is also a regular element of R. (Namely, if $a^n + b^n \in P$, P a prime ideal of R, then $a^n b^n = 0$ implies both $a^n \in P$ and $b^n \in P$, hence, both $a \in P$ and $b \in P$, therefore, $a + b \in P$, but this is not the case.) $a^n = (a^n + b^n)\dfrac{a^n}{a^n + b^n}$ and $\dfrac{a^n}{a^n + b^n}$ is an idempotent element of R, as $a^n(a^{2n} + b^{2n}) = a^{2n}(a^n + b^n)$ implies $\dfrac{a^n}{a^n + b^n} = \dfrac{a^{2n}}{a^{2n} + b^{2n}} = \left(\dfrac{a^n}{a^n + b^n}\right)^2$.

d) $\Rightarrow$ *e*). Let P_1, P_2 be prime ideals of R, $P_1 \subseteq P_2$ and let $a \in P_2$. $a^n = re$, r a regular element of R and e an idempotent element of R. Since $r \notin P_2$, $e \in P_2$. Since $e \in P_2$, $1 - e \notin P_2$. $e(1 - e) = 0$ implies $e \in P_1$. Hence, $a \in P_1$, i. e. $P_1 = P_2$.

e) $\Rightarrow$ *c*). Let $a \in R$ and let N be the nilradical of R. Since R is a 0-dimensional ring, R/N is also a 0-dimensional ring, hence, R/N is a regular ring. It follows that there exists $b \in R$ such that, for some positive integer n, $(ab)^n = 0$ and $a + b$ is a regular element of R.

c) $\Rightarrow$ *a*). Let $a \in R$. Then there exists $b \in R$ such that, for some positive integer n, $(ab)^n = 0$ and $a + b$ is a regular element of R. From $a^n(a^n + b^n) = (a^n)^2$ we have $a^n = (a^n)^2 \dfrac{1}{a^n + b^n}$.

Examples of the π-regular rings. Every regular ring is π-regular. Artinian rings are π-regular. Let R be a regular ring and let M be a nonzero R-module. If we define multiplication in the direct sum $R \oplus M$ of the abelian groups R and M by: $(r, m)(r', m') = (rr', rm' + r'm)$, then $R \oplus M$ is a ring which is a π-regular ring that is not a regular ring. Let the ring S be an integral overring of the π-regular ring R. Since R is a 0-dimensional ring, S is a 0-dimensional ring also. Therefore, S is a π-regular ring.

PROPOSITION 7. *Let $A = (a_1, a_2, \ldots, a_k)$ be a finitely generated ideal of a 0-dimensional ring R. Then there exists a positive integer n and an idempotent element e of R such that $A^n = (e)$. In addition, $A + (1 - e) = (1)$, $A^n + (1 - e) = (1)$, $A^n(1 - e) = (0)$.*

Proof. Since R is a 0-dimensional ring, there exist positive integers $n_1, n_2, \ldots, n_k$ and idempotent elements $e_1, e_2, \ldots, e_k$ of R such that

$a_1^{n_1} = a_1^{n_1} e_1$, $a_2^{n_2} = a_2^{n_2} e_2, \ldots, a_k^{n_k} = a_k^{n_k} e_k$ and $a_1 + (1 - e_1), \ldots, a_k + (1 - e_k)$ are regular elements of R. Construct the idempotent element e of R such that $(e_1, e_2, \ldots, e_k) = (e)$. (For example, $(e_1, e_2) = (e)$, where e is the idempotent element $e = e_1 + (1 - e_1) e_2 = e_2 + (1 - e_2) e_1 = e_1 + e_2 - e_1 e_2$ of R.) Then $A^n = (e)$, where $n = \sum_{i=1}^{k} n_i$. $(a_i^{n_i} = a_i^{n_i} e_i \in (e) \Rightarrow A^n \subseteq (e)$; $\quad e_i = e_i \dfrac{a_i + (1 - e_i)}{a_i + (1 - e_i)} = \dfrac{e_i}{a_i + (1 - e_i)} a_i \in A$ $(i = 1, 2, \ldots, k) \Rightarrow (e) \subseteq A \Rightarrow (e) = (e^n) \subseteq A^n$.) e is an idempotent element of R with the property: if P is a prime ideal of R, then $e \in P$ if and only if $A \subseteq P$. Therefore, $A + (1 - e) = (1)$, $A^n + (1 - e) = (1)$, $A^n (1 - e) = (0)$.

PROPOSITION 8. *Let R be a 0-dimensional ring and let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a set of indeterminates over R. Then the total quotient ring $T(R[X])$ of $R[X]$ is a 0-dimensional ring (We write $R[X]$ instead* of $R[\{X_\lambda\}_{\lambda \in \Lambda}]$.)

Proof. If $f \in R[X]$, A_f denotes the ideal of R generated by the coefficients of f. It is known that an element f of $R[X]$ is a zero divisor if and only if there is a nonzero element r of R such that $rA_f = (0)$. ([1], Proposition 24.7). Since R is a 0-dimensional ring, there exist a positive integer n and an idempotent element e of R such that $A_f^n = (e)$. e is the idempotent element of R with the property: if P is a prime ideal of R then $e \in P$ if and only if $A_f \subseteq P$. Therefore, $f + (1 - e)$ is a regular element of $R[X]$ and $f(1 - e)$ is a nilpotent element of $R[X]$. It follows by Theorem 6 that the total quotient ring $T(R[X])$ of $R[X]$ is a 0-dimensional ring.

We can also characterize the 0-dimensional rings in the following way.

THEOREM 9. *Let R be a commutative ring and let N be the nil-radical of R. The following statements are equivalent:*

a) R is a 0-dimensional ring;

b) If A is an ideal of R, then $A + N$ is the radical of A;

c) If A is an ideal of R, then $A^2 + N = A + N$.

Proof. The proof follows easily using Theorem 5 and the fact that R is a 0-dimensional ring if and only if R/N is a regular ring.

It is known, that a ring R, whose total quotient ring $T(R)$ is π-regular, is an additively regular ring [2]. Here we shall present another proof of this fact.

THEOREM 10. *Let R be a ring whose total quotient ring $T(R)$ is π-regular. Then R is an additively regular ring.*

Proof. Let $x \in T(R)$. Then there exists $y \in T(R)$ such that xy is a nilpotent element and $x + y$ is a regular element of $T(R)$.

$y = \frac{a}{r}$; $a, r \in R$, r a regular element of $T(R)$. Then, since xa is a nilpotent element, either $x \in P$ or $a \in P$. If both $x \in P$ and $a \in P$, then both $x \in P$ and $y = \frac{a}{r} \in P$, hence, $x + y \in P$, which is impossible since $x + y$ is a regular element of $T(R)$. Therefore, $x + a \notin P$ for every prime ideal P of $T(R)$, and it follows that $x + a$ is a regular element of $T(R)$.

REFERENCES:

[1] *R. Gilmer*, Multiplicative Ideal Theory, Queen's Papers on Pure and Applied Math. No 12, Queen's Univ., Kingston, Ontario 1968.

[2] *J. Huckaba*, On valuation rings that contain zero divisors, Proc. Amer. Math. Soc. **40** (1973), 9—15.

[3] *I. Kaplansky*, Commutative rings, Allyn and Bacon, Boston, Mass., 1970.

[4] *J. Lambek*, Lectures on Rings and Modules, Waltham, Toronto, London, Blaisdell 1966.

[5] *O. Zariski* and *P. Samuel*, Commutative Algebra, Vol. I, Univ. Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1958.

(Received March 4, 1981)
(Revised March 12, 1982)

Department of Mathematics,
University of Sarajevo,
71000 *Sarajevo, Yugoslavia*

KARAKTERIZACIJA PRSTENOVA DIMENZIJE NULA

M. Arapović, Sarajevo

Sadržaj

U ovom radu se posmatraju komutativni prstenovi sa jedinicom. Dati su slijedeći rezultati: *i*) novi dokaz teoreme da je komutativni prsten π-regularan prsten ako i samo ako je prsten dimenzije nula; *ii*) različite karakterizacije prstenova dimenzije nula; *iii*) ako je R prsten dimenzije nula, onda je $T(R[X])$ također prsten dimenzije nula, gdje je $R[X]$ prsten polinoma i $T(R[X])$ totalni prsten razlomaka prstena $R[X]$; *iv*) ako je R prsten dimenzije nula i ako je A konačno generisani ideal u R, onda $A^n = (e)$ za neki prirodni broj n i za neki idempotentni element e iz R i *v*) novi dokaz činjenice da je prsten R, čiji je totalni prsten razlomaka $T(R)$ prsten dimenzije nula, aditivno regularan prsten.