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SPECTRA OF SOME OPERATIONS ON INFINITE GRAPHS

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Abstract. This paper is a continuation of [6] and the author's previous paper [3]. We consider some binary and n-ary operations on infinite graphs and investigate the finiteness of the spectrum of so obtained graphs (the strong and lexicographic product and p-sum of graphs).

1. Introduction

Throughout the paper G is an infinite, undirected graph without loops or multiple edges whose vertex set is $X = \{x_1, x_2, \ldots\}$.

The adjacency matrix $\mathscr{A} = (a_{ij})$ of G is an infinite $N \times N$ matrix, where $a_{ij} = a^{i+j-2}$ if x_i and x_j are adjacent and $a_{ij} = 0$ if they are not adjacent (a is a fixed constant, $0 < a < 1$).

The infinite matrix $\mathscr{A}$ can be regarded as the matrix of a bounded linear operator A in a separable Hilbert space H with an orthonormal basis $\{e_j\}$. This operator is always nuclear (see [2]).

$\sigma(G)$ denotes the spectrum of G which is defined to be the spectrum $\sigma(A)$ of the operator A. It consists of zero and a sequence $\lambda_1, \lambda_2, \ldots$ of non-zero eigenvalues, where each of them is of finite multiplicity.

The vertex set X of G can be partitioned in a unique way into a finite or infinite number of disjoint subsets $X_1, X_2, \ldots$ so that any two vertices from the same subset are not adjacent, and any two subsets are completely connected or completely non-connected in G. The subsets $X_1, X_2, \ldots$ are equivalence classes under the equivalence relation which is defined in the following way: vertices x and y are equivalent if and only if they have the same neighbours. Subsets $X_1, X_2, \ldots$ are called characteristic subsets of G. The graph G is of finite type if it has finite number of characteristic subsets. Otherwise it is of infinite type (see [4]).

A subgraph g of G obtained by choosing an arbitraty vertex from each of characteristic subsets is said to be a canonical image of G. If G is of finite type k, we often denote it by $G = g(X_1, \ldots, X_k)$.

We quote some known facts about spectra of graphs of finite type, which will be used in this paper.

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THEOREM 1.1. *An infinite graph is of finite type iff it has finite spectrum.*

THEOREM 1.2. *Each induced subgraph G_0 of an infinite graph G of finite type is a graph of finite type, too.*

These theorems were proved by A. Torgašev in [4] and [5].

2. Main results

2.1. *Strong product of two infinite graphs*

Definition 2.1. The strong product $G_1 * G_2$ of two infinite graphs $G_1 = (X_1, U_1)$ and $G_2 = (X_2, U_2)$ is a graph $G = (X, U)$, where $X = X_1 \times X_2$ and the edge set U is defined as follows: Vertices (x_1, x_2) and (y_1, y_2) are adjacent in G iff either $(x_1, y_1) \in U_1$, $(x_2, y_2) \in U_2$ or $x_1 = y_1$, $(x_2, y_2) \in U_2$ or $(x_1, y_1) \in U_1, x_2 = y_2$.

THEOREM 2.1. *The strong product $G_1 * G_2$ of infinite graphs G_1 and G_2 without isolated vertices is always a graph of infinite type.*

Proof. We are proving that $G_1 * G_2$ does not have two equivalent distinct vertices. It is sufficient to prove that any two non-adjacent vertices (x_1, x_2) and (y_1, y_2) of $G_1 * G_2$ do not have the same neighbours. Since $(x_1, x_2) \neq (y_1, y_2)$, they have at least one coordinate distinct; let $x_2 \neq y_2$. We distinguish the following two cases:

1° Let $(x_2, y_2) \notin U_2$. Because x_1 is not isolated in G_1 there is $z_1 \in X_1$ such that $(x_1, z_1) \in U_1$. Then (z_1, x_2) is adjacent to (x_1, x_2) but not adjacent to (y_1, y_2).

2° If $(x_2, y_2) \in U_2$ then $x_1 \neq y_1$ and $(x_1, y_1) \notin U_1$ (since (x_1, x_2) and (y_1, y_2) are non-adjacent). Now, by applying 1° to $(x_1, y_1) \notin U_1$ the desired result is obtained.

Because $G_1 * G_2$ does not have two distinct equivalent vertices, it is a graph of infinite type.

2.2 *Lexicographic product of two infinite graphs*

Definition 2.2. The lexicographic product $G_1 [G_2]$ of two infinite graphs $G_1 = (X_1, U_1)$ and $G_2 = (U_2, X_2)$ is a graph $G = (X, U)$, where $X = X_1 \times X_2$ and the edge set U is defined in the following way: Vertices (x_1, x_2) and (y_1, y_2) are adjacent in G iff $x_1 = y_1, (x_2, y_2) \in U_2$ or $(x_1, y_1) \in U_1$, $(x_2, y_2) \in U_2$.

Definition 2.3. An infinite graph $G = (X, U)$ is said to be a graph complete in parts iff vertex set X can be partitioned into a finite number of disjoint subsets $M_1, \ldots, M_n$ so that

1° Each set M_i $(i = 1, \dots, n)$ is completely connected in G;

2° Each two sets M_i and M_j $(i \neq j)$ are either non-connected or completely connected in G.

THEOREM 2.2. *The lexicographic product $G_1 [G_2]$ of two infinite non-trivial graphs G_1 and G_2 is of finite type iff G_2 is of finite type and G_1 is complete in parts.*

Proof. Necessity. Let $G_1 [G_2]$ be of finite type k, i. e. $G_1 [G_2] = g(N_1, \dots, N_k)$, and let $X_1 = \{x_1, y_1, z_1, v_1, \dots\}$, $X_2 = \{x_2, y_2, z_2, v_2, \dots\}$. If, contrary, G_2 is of infinite type, then $G_1 [G_2]$ contains an induced subgraph G_0, whose vertex set is $\{(x_1, x_2), (x_1, y_2), (x_1, z_2), \dots\}$, which is of infinite type. Indeed, graphs G_2 and G_0 are isomorphic so they have the same type. Then, by Theorem 1.2 the graph $G_1 [G_2]$ is of infinite type, which is impossible.

Let x_2 and y_2 be adjacent in G_2. Consider vertices

$$Z_1 = \{(x_1, x_2), (y_1, x_2), (z_1, x_2), \dots\}$$

$$Z_2 = \{(x_1, y_2), (y_1, y_2), (z_1, y_2), \dots\}$$

of $G_1 [G_2]$. Vertices from Z_1 lie in p $(1 \leqslant p \leqslant k)$ sets

$$N_{i_1}, N_{i_2}, \dots, N_{i_p}$$

$(i_1 < i_2 < \dots < i_p)$. Denote by

$$N'_{i_1}, N'_{i_2}, \dots, N'_{i_p} \tag{1}$$

the projections of

$$N_{i_1} \cap Z_1, N_{i_2} \cap Z_1, \dots, N_{i_p} \cap Z_1$$

onto X_1. Then obviously these sets are disjoint and their union is X_1.

We prove that the sets (1) are completely connected in G_1. Let x_1 and y_1 be two vertices from N'_{i_1}. Then $((x_1, x_2), (x_1, y_2)) \in U$, so that $((y_1, x_2), (x_1, y_2)) \in U$, since vertices from N_{i_1} have the same neighbours. Whence it follows that $(x_1, y_1) \in U_1$. Since x_1 and y_1 are arbitrary vertices from N'_{i_1}, this set must be completely connected in G_1.

We next prove that if two sets N'_{i_1} and N'_{i_2} are connected, then they are completely connected. Let $x_1 \in N'_{i_1}$ and $z_1 \in N'_{i_2}$ be adjacent in G_1, i. e. $(x_1, z_1) \in U_1$. Let y_1 and v_1 be any vertices from N'_{i_1} and N'_{i_2}, respectively. Since $(x_1, z_1) \in U_1$ we have $((x_1, y_2), (z_1, x_2)) \in U$. But then $((v_1, x_2), (x_1, y_2)) \in U$ (since vertices from N_{i_2} have the same neighbours). Hence, $((x_1, x_2), (v_1, y_2)) \in U$ so that $((y_1, x_2), (v_1, y_2)) \in U$ (since vertices from N_{i_1} have the same neighbours). Therefrom it follows that $(y_1, v_1) \in U_1$. In a similar way, it can be proved that the vertex z_1 is adjacent to each vertex from N'_{i_1} and the vertex x_1 to each vertex from N'_{i_2}. Thus, N'_{i_1} and N'_{i_2} are completely connected in G_1, and we conclude that G_1 is a graph complete in parts.

Sufficiency. Let G_2 be a graph of finite type k, i. e. $G_2 = g(N_1, \ldots, N_k)$ and let G_1 be a graph complete in parts.

Then the vertex set $X = X_1 \times X_2$ of $G_1[G_2]$ can be partitioned into $n \cdot k$ mutually disjoint subsets $M_i \times N_j$ $(i = 1, \ldots, n; j = 1, \ldots, k)$. Any two vertices of set $M_i \times N_j$ are not adjacent, since their second coordinates are non-adjacent in G_2.

Let (x_0, y_0) and (x, y) be arbitrary vertices of $M_i \times N_j$, and let (u, v) be arbitrary vertex of X adjacent to (x_0, y_0). Then $u = x_0$ (or $(u, x_0) \in U_1$) and $(v, y_0) \in U_2$. If $u \in M_i$ then either $u = x$ or $(u, x) \in U_1$, since M_i is completely connected in G_1. If $u \in M_l$ $(l \neq i)$ then $(u, x) \in U_1$, since M_i and M_l are completely connected in G_1. Therefore, either $u = x$ or $(u, x) \in U_1$. On the other side, since $y_0, y \in N_j$, they have the same neighbours in G_2, so that $(v, y) \in U_2$. Thus (u, v) is adjacent to (x, y). So, we have proved that all vertices from $M_i \times N_j$ have the same neighbours in $G_1[G_2]$. Hence, all the vertices of $M_i \times N_j$ are equivalent. Since there are exactly $n \cdot k$ such sets, the number of equivalence classes must be less (or equal) to $n \cdot k$. Thus $G_1[G_2]$ is a graph of finite type.

Example 2.1. Let $G_1 = K_\infty$ and $G_2 = K_{N_1, N_2}$. Then $G_1[G_2]$ is a complete bipartite graph $K_{X_1 \times N_1, X_1 \times N_2}$. If its vertex set is $X = \{x_1, x_2, \ldots\}$, then its spectrum is

$$\sigma(G_1[G_2]) = \left\{0, \pm \frac{1}{a^2} \sqrt{A_1 A_2}\right\},$$

where $A_1 = \sum\limits_{x_i \in X_1 \times N_1} a^{2i}$, $A_2 = \sum\limits_{x_i \in X_1 \times N_2} a^{2i}$.

2.3 *p-sum of infinite graphs*

Definition 2.4. The p-sum of infinite graphs $G_1 = (X_1, U_1), \ldots, G_n = (X_n, U_n)$ is a graph $G = (X, U)$, where $X = X_1 \times \ldots \times X_n$ and U is defined in the following way: Vertices $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in X$ are adjacent in G iff exactly p of n pairs (x_i, y_i) $(i = 1, \ldots, n)$ are adjacent in the corresponding graphs G_i, and $x_i = y_i$ for remaining pairs.

If $p = 1$ one obtains the sum $G_1 + G_2 + \ldots + G_n$ of graphs, and if $p = n$ one obtains the Descartes product $G_1 \times G_2 \times \ldots \times G_n$.

THEOREM 2.3. *If $G_1, G_2, \ldots, G_n$ are infinite graphs without isolated vertices, then their p-sum $(1 \leqslant p < n)$ is always a graph of infinite type.*

Proof. In the vertex set X consider the subset

$$Y = \{(x_1, x_2, \ldots, x_n) \mid x_n \in X_n\}.$$

Let $(x_1, x_2, \ldots, x_n)$ and $(x_1, x_2, \ldots, y_n)$ be any two vertices from Y and let z_i be a vertex in X_i adjacent to x_i $(i = 1, \ldots, p)$. Then the vertices

$$(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n)$$
$$(z_1, \ldots, z_p, x_{p+1}, \ldots, x_n)$$

are adjacent and the vertices

$$(x_1, \ldots, x_p, x_{p+1}, \ldots, y_n)$$
$$(z_1, \ldots, z_p, x_{p+1}, \ldots, x_n)$$

are non-adjacent. We conclude that no two vertices in Y are equivalent, since they do not have the same neighbours in G. Hence, the vertices of Y belong to distinct equivalence classes of G. Since Y is an infinite set, there is an infinite number of equivalence classes, or G is of infinite type.

COROLLARY. *The sum $G_1 + G_2 + \ldots + G_n$ of infinite graphs $G_1, G_2, \ldots, G_n$ without isolated vertices is always a graph of infinite type.*

THEOREM 2.4. *The Descartes product $G_1 \times G_2 \times \ldots \times G_n$ of infinite graphs $G_1, G_2, \ldots, G_n$ without isolated vertices is of finite type iff the graphs $G_1, G_2, \ldots, G_n$ are of finite type. Furthermore, if G_i is of finite type k_i $(i = 1, \ldots, n)$, then $G_1 \times G_2 \times \ldots \times G_n$ is of finite type $k_1 \cdot k_2 \cdot \ldots \cdot k_n$.*

Proof. Sufficiency. Let G_i be of finite type k_i, i. e. $G_i = g_i(X_1^i, X_2^i, \ldots$ $\ldots X_{k_i}^i)$ $(i = 1, \ldots, n)$.

The vertex set X of $G_1 \times G_2 \times \ldots \times G_n$ can be partitioned into $k_1 \cdot k_2 \cdot \ldots \cdot k_n$ mutually disjoint subsets

$$Y_{i_1 \ldots i_n} = X_{i_1}^1 \times X_{i_2}^2 \times \ldots \times X_{i_n}^n \; (i_1 = 1, \ldots, k_1; \ldots; i_n = 1, \ldots, k_n). \quad (2)$$

First we prove that each set (2) contains only equivalent vertices.

Let $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in Y_{i_1 \ldots i_n}$. Then the vertices $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are not adjacent, since no pair of vertices (x_i, y_i) is adjacent in G_i $(i = 1, \ldots, n)$. If a vertex $(z_1, z_2, \ldots, z_n)$ is adjacent to $(x_1, x_2, \ldots, x_n)$, then it is adjacent to $(y_1, y_2, \ldots, y_n)$ too (because the vertices x_1 and y_1, x_2 and $y_2, \ldots, x_n$ and y_n have the same neighbours in $G_1, G_2, \ldots, G_n$, respectively). Thus, $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are not adjacent and have the same neighbours in $G_1 \times G_2 \times \ldots \times G_n$. We conclude that $Y_{i_1 \ldots i_n}$ contains equivalent vertices only.

Next we prove that if $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equivalent in $G_1 \times G_2 \times \ldots \times G_n$ then x_1 and y_1, x_2 and $y_2, \ldots, x_n$ and y_n are equivalent in $G_1, G_2, \ldots, G_n$, respectively.

Let z_i be adjacent to x_i in G_i. Then x_i and $y_i\,(i = 1, \ldots, n)$ are not adjacent in G_i, respectively. Indeed, if, for instance, vertices x_1 and y_1 are adjacent in G_1, then $(y_1, z_2, \ldots, z_n)$ is adjacent to $(x_1, x_2, \ldots, x_n)$ and not adjacent to $(y_1, y_2, \ldots, y_n)$, which is impossible. Since $(z_1, z_2, \ldots \ldots, z_n)$ is adjacent to $(x_1, x_2, \ldots, x_n)$ in $G_1 \times G_2 \times \ldots \times G_n$ it is adjacent to $(y_1, \ldots, y_n)$ in $G_1 \times G_2 \times \ldots \times G_n$. Hence, it follows that the vertices z_1 and y_1, z_2 and $y_2, \ldots, z_n$ and y_n are adjacent in $G_1, G_2, \ldots, G_n$, respectively. So we have proved that x_i and y_i $(i = 1, \ldots, n)$ are not adjacent and they have the same neighbours in G_i; thus x_i and y_i are equivalent in G_i.

Finally, we conclude that any two vertices from distinct sets (2) are not equivalent, whence the sets (2) must be characteristic sets of the graph $G_1 \times G_2 \ldots \times G_n$. This means that $G_1 \times G_2 \times \ldots \times G_n$ is of finite type k, and $k = k_1 \cdot k_2 \cdot \ldots \cdot k_n$.

Necessity. Let at least one of the graphs $G_1, G_2, \ldots, G_n$ be of infinite type. Similarly to the previous proof, one can prove that the Descartes product of the characteristic subsets of $G_1, G_2, \ldots, G_n$ forms the characteristic subsets of the graph $G_1 \times G_2 \times \ldots \times G_n$. Since this set is infinite, we conclude that $G_1 \times G_2 \times \ldots \times G_n$ is a graph of infinite type.

Hence, if $G_1 \times G_2 \times \ldots \times G_n$ is a graph of finite type, then the graphs $G_1, G_2, \ldots, G_n$ must be of finite type, too.

Remark. Theorem 2.4, for the Descartes product of two infinite graphs, was proved by A. Torgašev [6].

Example 2.2. Let $G_1 = K_{M_1,M_2}$ and $G_2 = K_{N_1,N_2}$. Then $G_1 \times G_2$ is a disconnected graph with connected components $K_{M_1 \times N_1, M_2 \times N_2}$ and $K_{M_1 \times N_2, M_2 \times N_1}$. If its vertex set is $X = \{x_1, x_2, \ldots\}$, then its spectrum is

$$\sigma(G_1 \times G_2) = \sigma(K_{M_1 \times N_1, M_2 \times N_2}) \cup \sigma(K_{M_1 \times N_2, M_2 \times N_1}) =$$

$$= \left\{0, \pm \frac{1}{a^2} \sqrt{A_1 A_2}, \quad \pm \frac{1}{a^2} \sqrt{A_3 A_4}\right\},$$

where $A_1 = \sum_{x_i \in M_1 \times N_1} a^{2i}$, $A_2 = \sum_{x_i \in M_2 \times N_2} a^{2i}$, $A_3 = \sum_{x_i \in M_1 \times N_2} a^{2i}$,

$A_4 = \sum_{x_i \in M_2 \times N_1} a^{2i}$ (see [2] and [4]).

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SPEKTRI NEKIH OPERACIJA SA BESKONAČNIM GRAFOVIMA

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Sadržaj

U članku se posmatraju neke binarne i n-arne operacije sa beskonačnim grafovima i ispituje konačnost spektra tako dobijenih grafova (jaki i leksikografski proizvod i p-suma grafova). Nekim teoremama se utvrđuje da je spektar grafova tako dobijenih uvek beskonačan a druge teoreme daju potrebne i dovoljne uslove za konačnost spektra tih grafova.