HYPERELLIPTIC MODULAR CURVES $X_0(n)$ AND ISOGENIES
OF ELLIPTIC CURVES OVER QUADRATIC FIELDS

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Abstract. Let $n$ be an integer such that the modular curve $X_0(n)$ is hyperelliptic of genus $\geq 2$ and such that the Jacobian of $X_0(n)$ has rank 0 over $\mathbb{Q}$. We determine all points of $X_0(n)$ defined over quadratic fields, and we give a moduli interpretation of these points. As a consequence, we show that up to $\mathbb{Q}$-isomorphism, all but finitely many elliptic curves with $n$-isogenies over quadratic fields are in fact $\mathbb{Q}$-curves, and we list all exceptions. We also show that, again with finitely many exceptions up to $\mathbb{Q}$-isomorphism, every $\mathbb{Q}$-curve $E$ over a quadratic field $K$ admitting an $n$-isogeny is $d$-isogenous, for some $d | n$, to the twist of its Galois conjugate by some quadratic extension $L$ of $K$; we determine $d$ and $L$ explicitly.

1. Introduction

The study of possible torsion groups of elliptic curves over number fields has seen a lot of progress in the last few decades. See, just to mention some, [19] for results over $\mathbb{Q}$, [11, 18] for results over quadratic fields, [9, 22] for results over cubic fields, and [10] for results over quartic fields. We also mention the uniform boundness conjecture, proved by Merel [21], which states that the order of the torsion group is bounded from above by an integer depending only on the degree of the number field.

Unfortunately, much less is known about the possible degrees of isogenies of elliptic curves over number fields. A complete classification of possible isogeny degrees is only known over $\mathbb{Q}$. Mazur [20] found all the isogenies of prime degree and gave a list of isogeny degrees which he believed to be complete, after which Kenku, in a series of papers [13, 14, 15, 16], proved that the list is indeed complete.

We say that an elliptic curve has an $n$-isogeny if it has an isogeny with cyclic kernel of degree $n$. As usual, the (compact) modular curve classifying elliptic curves with an $n$-isogeny will be denoted by $X_0(n)$, and the Jacobian of $X_0(n)$ will be denoted by $J_0(n)$.

In this paper, we study quadratic points on the curves $X_0(n)$ for $n$ such that $X_0(n)$ is hyperelliptic of genus at least 2 and such that the group of $\mathbb{Q}$-rational points of $J_0(n)$ is finite. These assumptions are satisfied for

$$n \in \{22, 23, 26, 28, 29, 30, 31, 33, 35, 39, 40, 41, 46, 47, 48, 50, 59, 71\}.$$  

Throughout this paper, unless stated otherwise, $n$ will denote an integer in the above set.

Remark 1. For $n = 31$ and $n = 37$, the curve $X_0(n)$ is hyperelliptic, but $J_0(n)$ has positive rank; see [23].

Quadratic points on $X_0(n)$ (except for the cusps) correspond to $\mathbb{Q}$-isomorphism classes of elliptic curves over quadratic fields with an $n$-isogeny. Hence studying
these points gives us information about the properties of elliptic curves with $n$-isogenies over quadratic fields.

Taking inverse images of $\mathbb{Q}$-points under the hyperelliptic map $X_0(n) \to \mathbb{P}^1_{\mathbb{Q}}$ gives an infinite set of points of $X_0(n)$ over quadratic fields. Apart from these, there are only finitely many quadratic points; we will call these quadratic points exceptional.

Recall that a $\mathbb{Q}$-curve is an elliptic curve that is isogenous to all of its Galois conjugates. If $E$ is a $\mathbb{Q}$-curve over a quadratic field $K$ and $\sigma$ is the non-trivial automorphism of $K$, then $E$ is a $\mathbb{Q}$-curve if and only if $E$ is $\mathbb{Q}$-isogenous to $E^\sigma$. We prove that all elliptic curves over quadratic fields obtained from points of $\mathbb{P}^1(\mathbb{Q})$ as above are in fact $\mathbb{Q}$-curves. The fact that there are only finitely many exceptional quadratic points means that up to $\mathbb{Q}$-isomorphism, there are only finitely many elliptic curves over quadratic fields with an $n$-isogeny that are not $\mathbb{Q}$-curves.

Let $E/K$ be a $\mathbb{Q}$-curve without complex multiplication with an $n$-isogeny as obtained in our construction and such that $E$. Then $E$ is not in general isogenous over $K$ to its Galois conjugate $E^\sigma$, but it is by construction $K$-isogenous to a quadratic twist of $E^\sigma$. It follows that $E$ becomes isogenous to $E^\sigma$ after base extension to a quadratic extension $L$ of $K$. We explicitly construct such an extension $L$. Moreover, we prove that $E$ has even rank over $L$.

Finally, we show that in the cases $n = 28$ and $n = 40$, for all but a few explicitly listed elliptic curves over quadratic fields $E/K$ with an $n$-isogeny, the quadratic field $K$ is real.

Similar results were proved in [5] for elliptic curves over quadratic fields with prescribed torsion groups $\mathbb{Z}/n\mathbb{Z}$, where $n \in \{13, 16, 18\}$. This uses the arithmetic properties of the relevant modular curves $X_1(n)$. In [5] it was also proved that for $n = 13$ and $n = 18$, the endomorphism ring of the restriction of scalars $\text{Res}_{K/\mathbb{Q}} E$ of such an elliptic curve $E/K$ contains an order in a quadratic field, which implies that $\text{Res}_{K/\mathbb{Q}} E(\mathbb{Q}) \simeq E(K)$ has even rank. One says that $E$ has false complex multiplication; see [5] for a precise definition. In all of the cases that we consider, our construction implies that $E$ has false complex multiplication (in the sense of [5]) over $L$.

**Remark 2.** The difficulty that $E$ is not isogenous to $E^\sigma$, but only to a twist of $E^\sigma$, does not happen for the elliptic curves with prescribed torsion studied in [5]. This is explained by the fact that $X_0(n)$ is only a coarse moduli space, while $X_1(n)$ is a fine moduli space.

The computer calculations were done using Magma [4]. In particular, we made use of plane models of $X_0(n)$ given by polynomials in $\mathbb{Q}[x,y]$, and the $q$-expansions of the modular functions $x$ and $y$ for every $n$. These data were obtained from M. Harrison’s Small Modular Curve database, which is included in Magma.

### 2. Quadratic Points on $X_0(n)$

In this section we describe the set of all quadratic points on $X_0(n)$. We will denote $X_0(n)$ by $X$ and the Jacobian of $X_0(n)$ by $J$. We fix a hyperelliptic equation of the form

$$X: y^2 = f(x).$$

Using Magma’s functionality for 2-descent [26], one proves in all the cases that we consider that $J$ has rank 0 over $\mathbb{Q}$. One could alternatively prove that $J$ has rank 0 using $L$-functions [25, Section 3.10]. Thus in this section $J(\mathbb{Q})$ is finite.
We first observe that finding the quadratic points on $X$ amounts to finding the rational points on the symmetric square $X^{(2)}$ of $X$. Let $\iota$ be the hyperelliptic involution of $X$ and fix a cusp $C \in X(\mathbb{Q})$. Then the map

$$\phi : X^{(2)} \to J, \quad \{P,Q\} \mapsto [P + Q - C - \iota(C)],$$

is an isomorphism away from the fibre above 0. Thus

$$X^{(2)}(\mathbb{Q}) = \phi^{-1}(0) \cup \phi^{-1}(J(\mathbb{Q}) \setminus \{0\}).$$

Note that $\phi^{-1}(J(\mathbb{Q}) \setminus \{0\})$ is easy to compute and is finite, since $J(\mathbb{Q})$ is finite by assumption. In Section 7, we explicitly list all the quadratic points on $X$ that are not in the set $\phi^{-1}(0)$.

The set $\phi^{-1}(0)$ is the set of pairs of points which are fixed by $\iota$, in other words

$$\phi^{-1}(0) = \{P, \iota(P)\}.$$ 

Let $P = (x, \sqrt{f(x)}) \in X(K)$, where $K$ is some quadratic field, be as above, and let $\sigma$ be the generator of $\text{Gal}(K/\mathbb{Q})$. It follows easily that $\iota(P) = \iota(x, \sqrt{f(x)}) = (x, -\sqrt{f(x)})$. We now conclude that $\iota$ acts on $P$ in the same way as $\sigma$. Let $E$ be the elliptic curve which is, up to twist (and together with a subgroup), parameterized by $P \in X(K)$. If $\iota = w_d$ (which will usually be the case), it follows that $E$ is $d$-isogenous to $E^\sigma$, up to twist. Thus $E$ is a $\mathbb{Q}$-curve.

This construction is similar to the one with hyperelliptic modular curves $X_1(n)$ [5, Section 4], but there is a fundamental difference. Namely, the curves $X_1(n)$ are fine moduli spaces, while $X_0(n)$ are coarse moduli spaces. The consequence of this is that the $\mathbb{Q}$-curves obtained from $X_1(n)$ in [5] are $K$-isogenous to their Galois conjugates, while our curves obtained from $X_0(n)$ will only be $K$-isogenous to their Galois conjugates up to twist, meaning that $E$ will be isogenous to a twist of $E^\sigma$. An alternative way of saying this is to say that $E$ and $E^\sigma$ are $\mathbb{Q}$-isogenous. We refer to Elkies [7, Section 3] for a detailed exposition of the relevant properties of the curves $X_0(n)$.

From what we have seen, determining $X^{(2)}(\mathbb{Q})$ now amounts to determining the group $J(\mathbb{Q})$. Note that for all the $X$ we consider, $X(\mathbb{Q})$ consists entirely of cusps. The cuspidal subgroup of $J$, denoted by $C_X(\mathbb{Q})$, is the subgroup of $J(\mathbb{Q})$ generated by the classes of the divisors $P_1 - P_2$, where $P_1, P_2$ are $\mathbb{Q}$-rational cusps of $X$.

**Proposition 1.** Let $n$ be an integer such that $X_0(n)$ is hyperelliptic and that its Jacobian $J_0(n)$ has rank 0 over $\mathbb{Q}$. Then $J_0(n)(\mathbb{Q})$ is equal to its cuspidal subgroup.

**Proof.** One can obtain an upper bound on the size of $J(\mathbb{Q})$ by using the fact that, for a prime of good reduction $p$, the prime-to-$p$ part of $J(\mathbb{Q})$ injects into $J(\mathbb{F}_p)$.

The function $\text{TorsionBound}$ in Magma does exactly this. For the Jacobians of all $X_0(n)$, except for $n \in \{30, 33, 39, 48\}$, this immediately shows that $C_X(\mathbb{Q}) = J(\mathbb{Q})$.

For $n \in \{30, 33, 39, 48\}$, the bound obtained in this way is unfortunately larger than the order of the cuspidal subgroup. For $n = 30$ and $n = 48$, we only obtain that the index $(J(\mathbb{Q}) : C_X(\mathbb{Q}))$ is a divisor of 4, while in the cases $n = 33$ and $n = 39$ the index is 1 or 2.

We deal with the cases $n = 33$ and $n = 39$ by studying the group structures of the reductions, not just their orders.

Let $n = 33$. One computes the group structure of $C_X(\mathbb{Q})$ to be $(\mathbb{Z}/10\mathbb{Z})^2$. We compute

$$J(\mathbb{F}_3) \simeq \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/20\mathbb{Z}.$$
and

\[ J(F_7) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/10\mathbb{Z})^2. \]

Since \( J(\mathbb{Q}) \) injects into both of these groups, it follows that \( J(\mathbb{Q}) = C_X(\mathbb{Q}) \).

Similarly, in the case \( n = 39 \), one computes

\[
\begin{align*}
C_X(\mathbb{Q}) &\simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/28\mathbb{Z}, \\
J(F_5) &\simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/28\mathbb{Z}, \\
J(F_7) &\simeq (\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z}/28\mathbb{Z},
\end{align*}
\]

from which it follows that \( J(\mathbb{Q}) = C_X(\mathbb{Q}) \).

Let \( n = 30 \). We compute

\[
\begin{align*}
C_X(\mathbb{Q}) &\simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}, \\
J(F_7) &\simeq (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/48\mathbb{Z}, \\
J(F_{23}) &\simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z} \oplus (\mathbb{Z}/24\mathbb{Z})^2.
\end{align*}
\]

Therefore \( J(\mathbb{Q}) \) is isomorphic to a subgroup of \((\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}\). Thus to prove that \( C_X(\mathbb{Q}) = J(\mathbb{Q}) \), it suffices to show that \( J(\mathbb{Q})[2] \simeq (\mathbb{Z}/2\mathbb{Z})^3 \). Now the splitting field of \( J[2] \) (the same as that of the discriminant of \( X \)) is \( \mathbb{Q}(\sqrt{-3}, \sqrt{5}) \). By computing \( J[2] \) over this field and taking Galois invariants, we obtain the desired claim.

Finally, let \( n = 48 \). This case is harder, and we will use the following method, proposed to us by Samir Siksek. We have

\[
\begin{align*}
C_X(\mathbb{Q}) &\simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}, \\
J(F_5) &\simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/8\mathbb{Z})^2.
\end{align*}
\]

By computing Galois invariants in \( J[2] \) as above, we find that the 2-torsion of \( J(\mathbb{Q}) \) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^3\), and hence is contained in \( C_X(\mathbb{Q}) \). Suppose that \( [J(\mathbb{Q}) : C_X(\mathbb{Q})] \neq 1 \); then there exists a \( P \in C_X(\mathbb{Q}) \) and \( Q \in J(\mathbb{Q})C_X(\mathbb{Q}) \) such that \( 2Q = P \). This implies that \( P + 2C_X(\mathbb{Q}) \) is in the kernel of the map

\[ C_X(\mathbb{Q})/2C_X(\mathbb{Q}) \to J(\mathbb{Q})/2J(\mathbb{Q}), \]

and hence also in the kernel of

\[ C_X(\mathbb{Q})/2C_X(\mathbb{Q}) \to J(F_5)/2J(F_5). \]

An explicit calculation shows that the map (3) is injective. Thus it follows that \( P \) is necessarily in \( 2C_X(\mathbb{Q}) \), say \( P = 2R \) for some \( R \in C_X(\mathbb{Q}) \). It follows that \( P = 2R \) and \( 2(Q - R) = 0 \) in \( J(\mathbb{Q}) \). Since \( C_X(\mathbb{Q}) \) contains the complete 2-torsion of \( J(\mathbb{Q}) \), it follows that \( Q \in C_X(\mathbb{Q}) \), which is a contradiction. Hence \( J(\mathbb{Q}) = C_X(\mathbb{Q}) \). \hfill \Box

3. Moduli interpretation of the normalizer of \( \Gamma_0(n) \)

Let \( B(\Gamma_0(n)) = \text{Norm}_{\text{GL}_2(\mathbb{Q})}^+(\Gamma_0(n)) \) denote the normalizer of \( \Gamma_0(n) \) in the group \( \text{GL}_2(\mathbb{Q})^+ \) of \( 2 \times 2 \)-matrices over \( \mathbb{Q} \) with positive determinant. In this section, we describe a canonical action of \( B(\Gamma_0(n)) \) on \( X_0(n) \), and we give a “moduli interpretation” of this action.
3.1. **The action of** $B(\Gamma_0(n))$. For any positive integer $n$, let $C_n$ be the category defined as follows. The objects of $C_n$ are the triples $(E \to S, C, \omega)$ consisting of an elliptic curve $E$ over some $\mathbb{Q}$-scheme $S$, a cyclic subgroup scheme $C$ of order $n$ of $E$ and a nowhere-vanishing global relative differential $\omega$ on $E$. A morphism $(E' \to S', C', \omega') \to (E \to S, C, \omega)$ in $C_n$ is a Cartesian diagram of schemes

$$
\begin{array}{ccc}
E' & \to & E \\
\downarrow & & \downarrow \\
S' & \to & S
\end{array}
$$

that is compatible with the group structure and satisfies $\phi^* C = C'$ and $\phi^* \omega = \omega'$. We will usually omit $S$ from the notation.

We let $B(\Gamma_0(n))$ act by equivalences of categories on $C_n$ as follows. Let $\gamma \in B(\Gamma_0(n))$. First we consider the case where $\gamma$ is a scalar matrix $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a \in \mathbb{Q}^\times$. For such $\gamma$, we define $\gamma(E, C, \omega) = (E, C, a^{-1}\omega)$.

Next we may assume that $\gamma$ is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$. We put $\delta = \det \gamma = ad - bc$; this is a positive integer. After étale localization on $S$, we can choose an isomorphism $\phi: (\mathbb{Z}/\mathbb{Z})^2 \xrightarrow{\sim} E[n\delta]$.

We may assume that the subgroup $\langle \phi(0, \delta) \rangle$ generated by $\phi(0, \delta)$ equals $C$. We put $C_\gamma = n\langle \phi(a, b), \phi(c, d) \rangle \subset E$.

This is a subgroup of order $\delta$ whose structure (elementary divisors) is given by the Smith normal form of $\gamma$. Furthermore, we put $E' = E/C_\gamma$

and $C' = (\langle \phi(c, d) \rangle + C_\gamma)/C_\gamma \subset E'$;

one easily sees that $C'$ is cyclic of order $n$. There exists a unique differential $\omega'$ on $E/C_\gamma$ whose pull-back to $E$ equals $\omega$. We define $\gamma(E, C, \omega) = (E', C', \omega')$.

Because $X_0(n)$ is the coarse moduli space of pairs $(E, C)$ as above and the above construction commutes with scaling $\omega$, every $\gamma \in B(\Gamma_0(n))$ induces an automorphism $\iota_\gamma: X_0(n) \xrightarrow{\sim} X_0(n)$.

This automorphism only depends on the image of $\gamma$ in $B(\Gamma_0(n))/\Gamma_0(n)$. Thus we get an action of the latter group by automorphisms of the curve $X_0(n)$ over $\mathbb{Q}$.

3.2. **The complex perspective.** We now study an analogous construction over the complex numbers, where we can express elliptic curves using lattices.

Let $L_\mathbb{Q}$ denote the set of all group homomorphisms $\psi: \mathbb{Q}^2 \to \mathbb{C}$ such that the group $L_\psi = \psi(\mathbb{Z}^2) = \mathbb{Z}\psi(1,0) + \mathbb{Z}\psi(0,1)$ is a “positively-oriented” lattice in $\mathbb{C}$, in the sense that $\psi(1,0)$ and $\psi(0,1)$ are $\mathbb{R}$-linearly independent and $\psi(1,0)/\psi(0,1)$ has positive imaginary part.
We define a free left action of \( \text{GL}_2(\mathbb{Q})^+ \) on \( \mathcal{L}_Q \) as follows: given \( \gamma \in \text{GL}_2(\mathbb{Q})^+ \) and \( \psi : \mathbb{Q}^2 \rightarrow \mathbb{C} \) in \( \mathcal{L}_Q \), we define \( \gamma \psi \in \mathcal{L}_Q \) by

\[
(\gamma \psi)(v) = \frac{1}{\det \gamma} \psi(v \gamma),
\]

where \( v \gamma \) denotes the usual right action of \( \text{GL}_2(\mathbb{Q})^+ \) on \( \mathbb{Q}^2 \) ("row vectors"). More concretely, the bases \((\omega_1, \omega_2)\) of \( L_\psi \) and \((\omega'_1, \omega'_2)\) of \( L_{\gamma \psi} \) defined by

\[
\omega_1 = \psi(1, 0), \quad \omega_2 = \psi(0, 1),
\]

\[
\omega'_1 = (\gamma \psi)(1, 0), \quad \omega'_2 = (\gamma \psi)(0, 1),
\]

satisfy the relation

\[
\omega'_1 = \frac{1}{\det \gamma} (a \omega_1 + b \omega_2), \quad \omega'_2 = \frac{1}{\det \gamma} (c \omega_1 + d \omega_2).
\]

**Lemma 2.**

1. Let \( \psi \) and \( \psi' \) be in \( \mathcal{L}_Q \). Then \( L_\psi = L_{\psi'} \) if and only if the orbits \( \text{SL}_2(\mathbb{Z})\psi \) and \( \text{SL}_2(\mathbb{Z})\psi' \) are equal.

2. Let \( \psi \) be in \( \mathcal{L}_Q \), and let \( \gamma \) and \( \gamma' \) be in \( \text{GL}_2(\mathbb{Q})^+ \). Then \( L_{\gamma \psi} = L_{\gamma' \psi} \) if and only if the cosets \( \text{SL}_2(\mathbb{Z})\gamma \) and \( \text{SL}_2(\mathbb{Z})\gamma' \) are equal.

**Proof.** The first claim is easy to verify; the second one follows from the first and the fact that \( \text{GL}_2(\mathbb{Q})^+ \) acts freely on \( \mathcal{L}_Q \). \( \square \)

Now let \( n \) be a positive integer. We write \( \mathcal{L}_n \) for the set of pairs of lattices \((L, L')\) in \( \mathbb{C} \) such that \( L' \supset L \) and \( L'/L \) is cyclic of order \( n \). Let \( \gamma_n \) be the matrix \({\begin{smallmatrix}n & 0 \\0 & 1\end{smallmatrix}}\).

There is a surjection

\[
\pi_n : \mathcal{L}_Q \rightarrow \mathcal{L}_n,
\]

\[
\psi \mapsto (L_\psi, L_{\gamma_n \psi}),
\]

We note that \( L_{\gamma_n \psi} \) is the lattice spanned by \( \psi(1, 0) \) and \( \frac{1}{n} \psi(0, 1) \).

**Lemma 3.** The map \( \pi_n \) is a quotient map for the left action of the subgroup \( \Gamma_0(n) \subset \text{GL}_2(\mathbb{Q})^+ \) on \( \mathcal{L}_Q \).

**Proof.** We have

\[
\pi_n(\gamma \psi) = (L_{\gamma \psi}, L_{\gamma_n \gamma \psi})
\]

Hence \( \pi_n(\gamma \psi) = \pi_n(\psi) \) if and only if \( L_{\gamma \psi} = L_\psi \) and \( L_{\gamma_n \gamma \psi} = L_{\gamma_n \psi} \). By Lemma 2, this condition is equivalent to \( \gamma \in \text{SL}_2(\mathbb{Z}) \) and \( \gamma_n \gamma \gamma_n^{-1} \in \text{SL}_2(\mathbb{Z}) \), which is in turn equivalent to \( \gamma \in \Gamma_0(n) \). \( \square \)

The above result yields a natural action of \( B(\Gamma_0(n)) \) on \( \mathcal{L}_n \). Viewing elements of \( \mathcal{L}_n \) as values of \( \pi_n \), we can describe this action as

\[
\gamma(L_\psi, L_{\gamma_n \psi}) = (L_{\gamma \psi}, L_{\gamma_n \gamma \psi}) \quad \text{for all} \quad \gamma \in B(\Gamma_0(n)) \text{ and } \psi \in \mathcal{L}_Q.
\]

Furthermore, there is a map from \( \mathcal{L}_Q \) to the upper half-plane \( \mathbb{H} \) sending \( \psi \) to \( \psi(1, 0)/\psi(0, 1) \). This descends to a map from \( \mathcal{L}_n \) to the non-compact analytic modular curve \( \Gamma_0(n) \backslash \mathbb{H} \). From these observations and the relation (4) we obtain a commutative diagram

\[
\begin{array}{ccc}
\mathcal{L}_Q & \rightarrow & \mathbb{H} \\
\pi_n \downarrow & & \downarrow \\
\mathcal{L}_n & \rightarrow & \Gamma_0(n) \backslash \mathbb{H}
\end{array}
\]
in which the upper horizontal map is compatible with the action of \( \text{GL}_2(\mathbb{Q})^+ \) and the lower horizontal map is compatible with the action of \( B(\Gamma_0(n)) \).

### 3.3. Compatibility

We now show that the constructions in the two preceding sections are compatible. Let \( \mathcal{C}_n^{\text{an}} \) be the set of isomorphism classes of (analytified) objects of \( \mathcal{C}_n \) over the base \( S = \text{Spec} \mathbb{C} \). The action of \( B(\Gamma_0(n)) \) on \( \mathcal{C}_n \) clearly induces an action on \( \mathcal{C}_n^{\text{an}} \). We define a map

\[
F_n : \mathcal{L}_n \rightarrow \mathcal{C}_n^{\text{an}}
\]

\[
(L, L') \mapsto (\mathbb{C}/L, L'/L, 2\pi i dz).
\]

The following result shows that \( F_n \) respects the actions of \( B(\Gamma_0(n)) \) that we have defined on \( \mathcal{L}_n \) and on \( \mathcal{C}_n^{\text{an}} \), respectively.

**Proposition 4.** For all \( \gamma \in B(\Gamma_0(n)) \) and all \( (L, L') \in \mathcal{L}_n \), we have

\[
F_n(\gamma(L, L')) = \gamma(F_n(L, L')).
\]

**Proof.** Let \( (L, L') \in \mathcal{L}_n \), and let \( \psi \in \mathcal{L}_\psi \) be such that \( \pi_n(\psi) = (L, L') \), so that \( L = L_\psi \) and \( L' = L_{\gamma_\psi} \psi \). In \( \mathcal{C}_n^{\text{an}} \), we then have

\[
F_n(L, L') = (\mathbb{C}/L_\psi, L_{\gamma_\psi} \psi/L_\psi, 2\pi i dz).
\]

We first assume that \( \gamma \) is a scalar matrix \( (a_0 \ a \ bd) \) with \( a \in \mathbb{Q}^\times \). Then (4) implies

\[
L_{\gamma_\psi} = a^{-1}L_\psi = a^{-1}L \quad \text{and} \quad L_{\gamma_\psi} = a^{-1}L_{\gamma_\psi} = a^{-1}L'.
\]

This implies

\[
F_n(\gamma(L, L')) = F_n(a^{-1}L, a^{-1}L')
\]

\[
= (\mathbb{C}/a^{-1}L, a^{-1}L/a^{-1}L', 2\pi i dz).
\]

On the other hand, we have

\[
\gamma(F_n(L, L')) = (\mathbb{C}/L, L/L', a^{-1} \cdot 2\pi i dz).
\]

These two objects are clearly isomorphic.

Next we assume that \( \gamma \) is of the form \( (a_0 \ b \ cd) \) with \( a, b, c, d \in \mathbb{Z} \) and \( \delta = ad - bc > 0 \). Then the lattices \( L = L_\psi \) and \( L_{\gamma_\psi} \) satisfy \( L_{\gamma_\psi} \supseteq L_\psi \) and \( L_{\gamma_\psi}/L_\psi \) has order \( \delta \). We define a group isomorphism

\[
\phi : (\mathbb{Z}/n\delta\mathbb{Z})^2 \rightarrow (\mathbb{C}/L)[n\delta] = \frac{1}{n\delta}L/L
\]

\[
(x \mod n\delta, y \mod n\delta) \mapsto \psi(x/n\delta, y/n\delta) \mod L.
\]

A straightforward computation shows that both \( \gamma(F_n(L, L')) \) and \( F_n(\gamma(L, L')) \) are equal to \( (\mathbb{C}/L_{\gamma_\psi}, L_{\gamma_\psi}/L_{\gamma_\psi}, 2\pi i dz) \) in \( \mathcal{C}_n^{\text{an}} \). This concludes the proof.

### 3.4. Moduli interpretation of the hyperelliptic involution, and number of elliptic curves in an isogeny class

We will now choose \( \gamma \in B(\Gamma_0(n)) \) such that the hyperelliptic involution of \( X_0(n) \) is the automorphism induced by \( \gamma \) in the above way.

Suppose first that the hyperelliptic involution of \( X_0(n) \) equals the Atkin–Lehner involution \( w_d \) for some \( d \mid n \) with \( \gcd(d,n/d) = 1 \). This is the case for all \( n \) we consider except \( n = 40 \) and \( n = 48 \). For the matrix \( \gamma \) we can take any \( \left( \begin{array}{cc} ad & b \\ \alpha & d \end{array} \right) \) with \( a, b \in \mathbb{Z} \) and \( ad - b(n/d) = 1 \).

In [23, Section 4] it is proved that the hyperelliptic involution on \( X_0(40) \) is induced by the action of the matrix \( \beta_{40} = \left( \begin{array}{cc} -10 & 1 \\ -120 & 10 \end{array} \right) \in B(\Gamma_0(40)) \) of determinant 20,
and that the hyperelliptic involution on \( X_0(48) \) is induced by the matrix \( \beta_{48} = \begin{pmatrix} -5 & 1 \\ -1 & 0 \end{pmatrix} \in B(\Gamma_0(48)) \) of determinant 12.

We now give an application of this fact to isogeny classes of elliptic curves over quadratic fields. Kenku proved in [17] that there are at most 8 elliptic curves in an isogeny class over \( \mathbb{Q} \). In [2] the authors find isogeny classes with 10 elliptic curves over \( \mathbb{Q}(\sqrt{5}) \). We now show that the largest size of isogeny classes of which there are infinitely many is 16, coming from points on \( X_0(48) \). When counting isogeny classes, we count up to \( \mathbb{Q} \)-isomorphism.

**Theorem 5.** There are infinitely many isogeny classes of elliptic curves containing 16 elliptic curves. For any \( n > 16 \), there are finitely many (up to \( \mathbb{Q} \)-isomorphism), if any, isogeny classes of elliptic curves over quadratic fields with more than 16 elliptic curves.

**Proof.** Let \( E_1/K \) correspond to a (quadratic) point on \( X_0(48) \), and let \( E_5 \) be an elliptic curve 16-isogenous over \( K \) to \( E_1 \). Then the 16 isogeny \( f_{1,5} \) factors as \( f_{4,5} \circ f_{3,4} \circ f_{2,3} \circ f_{1,2} \), where \( f_{i,i+1} \) is a 2-isogeny from \( E_i \) to \( E_{i+1} \) and \( E_2, E_3 \) and \( E_4 \) are elliptic curves over \( K \).

Now we note that \( E_2, E_3 \) and \( E_4 \) all have 2 distinct 2-isogenies over \( K \). Each 2-isogeny over \( K \) corresponds to a \( K \)-rational point of order 2 and hence it follows that \( E_2(K), E_3(K) \) and \( E_4(K) \) all have full 2-torsion. But as \( E_i[2] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) has 3 subgroups of order 2, it follows that each of these curves has a third 2-isogeny, say to \( E_6, E_7 \) and \( E_8 \), respectively. We obtain the following isogeny diagram:

\[
\begin{array}{ccccc}
E_1 & \longrightarrow & E_2 & \longrightarrow & E_3 & \longrightarrow & E_4 & \longrightarrow & E_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
E_6 & & E_7 & & E_8 & & & & \\
\end{array}
\]

Each of the \( E_i \) is 3-isogenous (over \( K \)) to an elliptic curve \( E'_i \) and hence we get 16 elliptic curves in this isogeny class. Since \( X_0(48) \) is hyperelliptic it has infinitely many quadratic points.

The second claim follows from results of Bars [3, Theorem 4.3]. \( \square \)

**Remark 3.** The elliptic curve \( E_1 \) is 12-isogenous to a twist of \( E_1^\tau \), so \( E_1^\tau \) is a twist of either \( E_5 \) or \( E_6 \). It follows from the moduli interpretation of \( \beta_{48} \) that the kernel of this isogeny is not in the kernel of a 48-isogeny, ruling out \( E_3' \) so we conclude that \( E_1^\tau \) is a twist of \( E_6' \).

4. **Modular forms**

4.1. **Katz modular forms.** We recall that a modular form of weight \( k \) for \( \Gamma_0(n) \) can be viewed as a rule that to every object \( (E \to S, C, \omega) \) of \( \mathcal{C}_n \), where we may restrict to the case where \( S \) is an affine scheme \( \text{Spec} A \), assigns a global section \( F(E \to S, C, \omega) \in A \) that satisfies \( F(E \to S, C, \omega) = \lambda^{-k} F(E \to S, C, \omega) \) for all \( \lambda \in A^\times \), is compatible with morphisms in \( \mathcal{C}_n \) as above, and is regular at the cusps in a suitable sense; see Katz [12, Chapter 1] for details.

There is also a notion of \( q \)-expansions, and if \( E \) is the Tate curve over \( \mathbb{C}((q)) \) with parameter \( \zeta \), equipped with the canonical subgroup \( C = \mu_n \), and the canonical differential \( \omega = \frac{dq}{\zeta} \), then \( F(E, C, \omega) \in \mathbb{C}((q)) \) is the usual \( q \)-expansion \( F(q) \) of \( F \). We recall that the space of modular forms of weight 2 for \( \Gamma_0(n) \) is isomorphic to the
space of logarithmic differentials on $X_0(n)$; see for example Katz [12, Appendix 1]. If $F_\alpha$ is the modular form of weight 2 attached to a logarithmic differential $\alpha$, then the relation between $\alpha$ and $F_\alpha$ can be expressed in terms of the $q$-expansion of $F_\alpha$ as $\alpha = F_\alpha(q) \frac{dq}{q}$.

The group $\tilde{B}(\Gamma_0(n))$ acts from the right on modular forms of weight $k$ for $\Gamma_0(n)$ via

$$(F \mid \gamma)(E, C, \omega) = F(\gamma(E, C, \omega)).$$

Furthermore, if $\alpha$ is a logarithmic differential on $X_0(n)$ and $\iota$ is the automorphism of $X_0(n)$ induced by $\gamma$, we have

$$(5) \quad F_\alpha \mid \iota \gamma = (\det \gamma) F_{\iota \cdot \alpha}.$$

As usual, we write $\sigma_k(m)$ for the sum of the $k$-th powers of the positive divisors of $m$. Let $E_4$ and $E_6$ denote the standard Eisenstein series, normalized so that their $q$-expansions are

$$E_4(q) = \frac{1}{240} + \sum_{m=1}^{\infty} \sigma_3(m) q^m,$$

$$E_6(q) = -\frac{1}{504} + \sum_{m=1}^{\infty} \sigma_5(m) q^m.$$

If $E$ is elliptic curve given by a Weierstrass equation in $x$ and $y$, equipped with the standard differential $\omega = \frac{dx}{2y}$, these Eisenstein series are related to the usual coefficients $c_4$ and $c_6$ by

$$c_4 = 240 E_4(E, \omega) \quad \text{and} \quad c_6 = 504 E_6(E, \omega).$$

We view $E_4$ and $E_6$ as modular forms for $\Gamma_0(n)$. Furthermore, for every divisor $d$ of $n$, let $E_2^{(d)}$ be the modular form of weight 2 for $\Gamma_0(n)$ given by the $q$-expansion

$$E_2^{(d)}(q) = E_2(q) - d E_2(q^d),$$

where

$$E_2(q) = -\frac{1}{24} + \sum_{m=1}^{\infty} \sigma_1(m) q^m = -\frac{1}{24} + \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)^2}.$$

Over $\mathbb{C}$, a Katz modular form $F$ corresponds to a classical modular form $f$ (viewed as a function on the upper half-plane $\mathbb{H}$) as follows: if $(E, C, \omega) = (\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), (1/n), 2\pi i dz)$, then $F(E, C, \omega) = f(\tau)$. Given a modular form $f(\tau)$, the logarithmic differential corresponding to it is $f(\tau) \frac{dq}{q}$, where $q = \exp(2\pi i \tau)$.

4.2. The modular form $A_{\gamma}$. Let $n$ be a positive integer, and let $\gamma \in B(\Gamma_0(n))$. We will construct a modular form $A_{\gamma}$ of weight 2 on the modular curve $X_0(n)$, based on the moduli interpretation of $\gamma$ described in the previous section.

Let $(E \rightarrow S, C, \omega)$ be an object of $\mathcal{C}_n$. Let $C_\gamma$ be the cyclic subgroup of $E$ determined by the moduli interpretation of $\gamma$ as in the previous section. We may assume that $S$ is an affine scheme $\text{Spec} \, R$ and that $E$ is given by a short Weierstrass equation

$$E: y^2 = x^3 + a_4 x + a_6 \quad (a_4, a_6 \in R, 4a_4^3 + 27a_6^2 \neq 0)$$

$$E: y^2 = x^3 + a_4 x + a_6 \quad (a_4, a_6 \in R, 4a_4^3 + 27a_6^2 \neq 0)$$
Proposition 6.

The solution of series, where $\gamma$ is the matrix inducing the hyperelliptic involution fixed in [REF]. If $n$ is not 40 or 48, the hyperelliptic involution of $X_0(n)$ equals the Atkin–Lehner involution $w_d$ for some $d \mid n$ with $\gcd(d, n/d) = 1$. One can show that

$$A_\gamma = -2d E_2^{(d)},$$

see Elkies [7, Section 3].

In the remaining two cases, namely $n = 40$ and $n = 48$, the hyperelliptic involution of $X_0(n)$ is not an Atkin-Lehner involution. In these cases we have $\gamma = \beta_{40}$ and $\gamma = \beta_{48}$, respectively, as defined in [REF].

4.3. Explicit expressions for $A_\gamma$. Next, for all $n$ in the set mentioned in the introduction, we will derive an explicit expression for $A_\gamma$ in terms of Eisenstein series, where $\gamma$ is the matrix inducing the hyperelliptic involution fixed in [REF].

$$\sum_{P \in C_\gamma \setminus \{0\}} x(P) \in R.$$
It is a straightforward but tedious formal computation to show that
\[ A_{\beta_{40}}(q) = -38 E_2(q) + \sum_{n \geq 1} \left( S_{10}(q^n) - S_{10}(q^{n+1/2}) + S_{20}(q^{n+1/2}) \right), \]
where for \( m \geq 1 \) the rational function \( S_m \in \mathbb{Q}(\zeta_m)(x) \) is defined by
\[
S_m = \sum_{k=1}^{m-1} \frac{\zeta_m^k x}{(1 - \zeta_m^k x)^2}.
\]
By expanding in a Taylor series around \( x = 0 \), we obtain
\[
S_m = m^2 \sum_{l \geq 1} l x^{lm} - \sum_{l \geq 1} l x^l \in \mathbb{Q}[[x]],
\]
and hence
\[
S_m = m^2 \frac{x^m}{(1 - x^m)^2} - \frac{x}{(1 - x)^2} \in \mathbb{Q}(x).
\]
By substituting \( x = \exp(t) \) and expanding in a Taylor series around \( t = 0 \), we get
\[
S_m(1) = \frac{1 - m^2}{12}.
\]
Furthermore, it is easy to check that \( S_m \) is invariant under replacing \( x \) by \( 1/x \). We therefore obtain
\[
A_{\beta_{40}} = -38 E_2(q) - \frac{99}{12} + 2 \sum_{n \geq 1} S_{10}(q^n) - 2 \sum_{n \geq 0} S_{10}(q^{n+1/2}) + 2 \sum_{n \geq 0} S_{20}(q^{n+1/2})
\]
\[
= -38 E_2(q) - \frac{99}{12} + 2 \sum_{n \geq 1} \left( 100 \frac{q^{10n}}{(1 - q^{10n})^2} - \frac{q^n}{(1 - q^n)^2} \right)
\]
\[
- 2 \sum_{n \geq 0} \left( 100 \frac{q^{10n+5}}{(1 - q^{10n+5})^2} - \frac{q^{n+1/2}}{(1 - q^{n+1/2})^2} \right)
\]
\[
+ 2 \sum_{n \geq 0} \left( 400 \frac{q^{20n+10}}{(1 - q^{20n+10})^2} - \frac{q^{n+1/2}}{(1 - q^{n+1/2})^2} \right)
\]
\[
= -40 E_2(q) + 200 E_2(q^{10}) - 200 \sum_{n \geq 0} \frac{q^{10n+5}}{(1 - q^{10n+5})^2} + 800 \sum_{n \geq 0} \frac{q^{20n+10}}{(1 - q^{20n+10})^2}
\]
\[
= -40 E_2(q) + 200 E_2(q^{10}) - 200(E_2(q^{5}) - E_2(q^{10})) + 800(E_2(q^{10}) - E_2(q^{20}))
\]
\[
= -40 E_2(q) - 200 E_2(q^{5}) + 1200 E_2(q^{10}) - 800 E_2(q^{20}),
\]
which gives the first claimed equality; the second follows from the definition of \( E_2^{(d)} \).

Similarly, the hyperelliptic involution on \( X_0(48) \) is induced by the action of the matrix \( \beta_{48} = \begin{pmatrix} -6 & 1 \\ 0 & 0 \end{pmatrix} \) of determinant 12 on \( \mathcal{L}_{40} \) and on \( \mathcal{L}_{40} \). A calculation analogous to the one above gives the claimed identity. \( \square \)

5. Moduli interpretation of quadratic points

Let \( n \) be such that \( X_0(n) \) is hyperelliptic of genus \( \geq 2 \) and such that \( J_0(n)(\mathbb{Q}) \) has rank 0. Let \( \iota \) be the hyperelliptic involution on \( X_0(n) \), and let \( \gamma \) be the element of \( B(\Gamma_0(n)) \) chosen in the previous section so that \( \gamma \) induces the hyperelliptic involution \( \iota \). We put
\[
\delta = \det \gamma.
\]
The purpose of this section is to give a “moduli interpretation” of the quadratic points of $X_0(n)$. In particular, we show that if $E$ is an elliptic curve over a quadratic field $K$ admitting a cyclic subgroup of order $n$, then $E$ is isogenous to a quadratic twist of its Galois conjugate, with finitely many exceptions up to twisting.

5.1. The moduli interpretation. We fix a non-zero meromorphic differential $\alpha$ on $X_0(n)$ satisfying $i^* \alpha = -\alpha$.

Let $f_\alpha$ be the cusp form of weight 2 corresponding to $\alpha$ as above. Then we can uniquely express $E_4$ and $E_6$ as

$$E_4 = g_4 f_\alpha^2, \quad E_6 = g_6 f_\alpha^3,$$

where $g_4, g_6$ are meromorphic functions on $X_0(n)$.

**Lemma 7.** The action of $\gamma$ on $E_4$ and $E_6$ is given by

$$E_4|_{i^*\gamma} = (-\delta)^2 (i^*g_4) f_\alpha^2 = (-\delta)^2 \frac{t^*g_4}{g_4} E_4$$

and

$$E_6|_{i^*\gamma} = (-\delta)^3 (i^*g_6) f_\alpha^3 = (-\delta)^3 \frac{t^*g_6}{g_6} E_6.$$

**Proof.** By (5) we have $f_\alpha|_{2\gamma} = \delta f_{i^*\alpha}$; our choice of $\alpha$ implies $f_{i^*\alpha} = f_{-\alpha} = -f_\alpha$.

The claim now follows from the fact that $E_4|_{i^*\gamma} = (i^*g_4)(f_\alpha|_{2\gamma})^2$ and $E_4|_{i^*\gamma} = (i^*g_4)(f_\alpha|_{2\gamma})^2$. □

For the rest of this section, we fix the following data. Let $K$ be a quadratic field, and let $\sigma$ be the non-trivial element of $\text{Gal}(K/Q)$. Let $E$ be an elliptic curve over $K$ together with a cyclic subgroup $C$ of order $n$. Let $P$ be the $K$-rational point of $X_0(n)$ determined by $(E, C)$. We assume that $E$ does not have complex multiplication, and furthermore that the $K$-rational point of $X_0(n)$ defined by the pair $(E, C)$ does not belong to the finite set of exceptional quadratic points on $X_0(n)$.

We fix a non-zero global differential $\omega$ on $E$; this gives rise to a (non-uniquely determined) Weierstrass equation and to the usual $c$-coefficients $c_4$ and $c_6$, which only depend on $\omega$. We define $\mu$ and $\lambda$ in $K^\times$ by

$$\mu = \frac{21 g_6(P)c_4}{10 g_4(P)c_6},$$

$$\lambda = -\delta \frac{\sigma(\mu)}{\mu}.$$

We define an extension $L$ of $K$ (of degree 1 or 2) by

$$L = K(\sqrt{\lambda}).$$

We note that $L$ can also be written as $K(\sqrt{-\delta \text{Norm}_{K/Q}(\mu)})$ and is therefore either $K$ itself or a $V_4$-extension of $Q$. 


Let \( E' \) be an elliptic curve over \( K \), equipped with a non-zero global differential \( \omega' \) (or equivalently given by the Weierstrass equation), such that the \( c \)-coefficients of \( E' \) are
\[
\begin{align*}
c'_4 &= \lambda^2 \sigma(c_4), \\
c'_6 &= \lambda^3 \sigma(c_6).
\end{align*}
\]
We note that \( E' \) is a quadratic twist of the Galois conjugate \( E^\sigma \) of \( E \), namely \( E' = (E^\sigma)^{(\lambda)} \).

**Proposition 8.** In the above situation, there is a normalized \( \delta \)-isogeny \( E \to E' \) with kernel \( C_\gamma \).

**Proof.** Let \( \omega'' \) be the differential induced by \( \omega \) on \( E/C_\gamma \), and let \( c'_4 \) and \( c'_6 \) be the corresponding \( c \)-coefficients. We have to prove that \( (E/C_\gamma, \omega'') \) is isomorphic to \( (E', \omega') \). It is enough to show that \( (c'_4, c'_6) = (c''_4, c''_6) \), or equivalently that each of the two modular forms \( E_4 \) and \( E_6 \) takes the same value on \( (E/C_\gamma, \omega'') \) and \( (E', \omega') \).

There is some cyclic subgroup \( C' \) in \( E/C_\gamma \) such that
\[
\gamma(E, C, \omega) = (E/C_\gamma, C', \omega'').
\]
Using Lemma 7 and the fact that \( E_4 \) only depends on \( (E, \omega) \) and \( g_4 \) only depends on \( P \), we get
\[
E_4(E/C_\gamma, \omega'') = E_4(E/C_\gamma, C', \omega'') = (E_4|_{4\gamma})(E, C, \omega) = (-\delta)^2 \left( \frac{\zeta^* g_4}{g_4} \right)(P) E_4(E, C, \omega) = (-\delta)^2 g_4(\zeta(P)) g_4(P) E_4(E, \omega).
\]
This implies
\[
c''_4 = (-\delta)^2 g_4(\zeta(P)) g_4(P) c_4.
\]
Noting that \( g_4(\zeta(P)) = \sigma(g_4(P)) \), we rewrite this as
\[
c''_4 = (-\delta)^2 \frac{\sigma(g_4(P)/c_4)}{g_4(P)/c_4} \sigma(c_4).
\]
Likewise,
\[
c''_6 = (-\delta)^3 \frac{\sigma(g_6(P)/c_6)}{g_6(P)/c_6} \sigma(c_6).
\]
The fact that \( E \) defines the point \( P \) on \( X_0(n) \) implies that \( E \) is a twist of the curve defined by the Weierstrass equation \( y^2 = x^3 - 5g_4(P)x + \frac{7}{2}g_6(P) \), so there exists \( \mu \in K^\times \) such that
\[
(7) \quad 240g_4(P) = \mu^2 c_4 \quad \text{and} \quad 504g_6(P) = \mu^3 c_6.
\]
This implies
\[
c''_4 = \left( -\delta \frac{\sigma(\mu)}{\mu} \right)^2 \sigma(c_4) \quad \text{and} \quad c''_6 = \left( -\delta \frac{\sigma(\mu)}{\mu} \right)^3 \sigma(c_6).
\]
Furthermore, it follows from (7) that \( \mu \) can be expressed as
\[
\mu = \frac{21g_6(P)c_4}{10g_4(P)c_6},
\]
as claimed. \( \square \)
Corollary 9. 

(1) The curve $E$ is a $Q$-curve that is completely defined over $L$.

(2) The curve $E$ acquires even rank over $L$.

Proof. The first claim follows from Proposition 8 and our assumption that $E$ does not have complex multiplication. To prove the second claim, we note that the curves $E^{(\lambda)}$ and $E^\sigma$ are isogenous, which implies

$$\text{rk } E(L) = \text{rk } E(K) + \text{rk } E^{(\lambda)}(K) = \text{rk } E(K) + \text{rk } E^\sigma(K).$$

It remains to observe that $E(K)$ and $E^\sigma(K)$ have the same rank. \(\square\)

One can in fact show that $E$ acquires “false complex multiplication” over $L$, in the sense of [5]. More precisely, let $M = Q$ if $L = K$, and let $M$ be one of the quadratic subfields of $L$ other than $K$ itself if $[L : K] = 2$. Then there is an action of the ring $\mathbb{Z}[\sqrt{m}]$ on the Weil restriction $A = \text{Res}_{L/M} E$ and hence on the group $A(M) = E(L)$, where $m$ is either $\delta$ or $-\delta$. In particular, since $\delta$ is not a square in any of the cases we consider in this paper, $E(L)$ has even rank. The rank of $E$ will also be even over many extensions of $L$; see [6] for details.

Remark 4. The question over which field the isogenies between a $Q$-curve and its Galois conjugates are defined was studied from a somewhat different perspective by González [8]. We leave it to the interested reader to compare the two approaches.

5.2. An example. The curve $X_0(22)$ has genus 2, and the Small Modular Curve database in Magma gives the equation

$$X_0(22): y^2 - x^3y = -x^4 + 5x^3 - 10x^2 + 12x - 8.$$ 

The hyperelliptic involution $\iota$ equals the Atkin–Lehner involution $w_{11}$ and is induced by the matrix $\gamma = \left( \begin{smallmatrix} 1 & 1 \\ 12 & 11 \end{smallmatrix} \right)$. We fix the differential

$$\alpha = \frac{dx}{(x-2)(2y-x^3)}.$$ 

Then we have

$$E^{(11)}_2 = -\frac{5x^3 + 2x^2 + 12x + 8}{12} f_\alpha.$$ 

Furthermore,

$$E_4 = g_4f_\alpha^2 \quad \text{and} \quad E_6 = g_6f_\alpha^3,$$

where $g_4$ and $g_6$ are the rational functions defined by

$$240g_4 = 120(-x^3 + 6x^2 + 4x + 8)y$$

$$+ 121x^6 - 484x^5 + 604x^4 - 352x^3 - 400x^2 + 2496x - 2240,$$

$$-504g_6 = 36(-37x^6 + 236x^5 - 140x^4 + 1520x^3 + 368x^2 + 1408x - 448)y$$

$$+ 1331x^9 - 7986x^8 + 17304x^7 - 9832x^6 - 49632x^5$$

$$+ 148704x^4 - 174720x^3 + 131712x^2 + 16128x - 179200.$$

We consider the points with $x = -1$. These are defined over the quadratic field $K$ of discriminant $-143$, and the points are $P = (-1, \beta)$ and $\iota(P) = P^\sigma = (-1, -1 - \beta)$, where $\beta^2 + \beta + 36 = 0$. One of the elliptic curves in the family (consisting of quadratic twists) corresponding to the point $P$ is

$$E: y^2 + xy + (1 + \beta)y = x^3 - \frac{1}{2} x^2 + (74 - 28\beta)x + \frac{637 - 281\beta}{2}.$$
The class number of $K$ is 10, and $E$ does not admit a global minimal model. The element $\mu \in K^\times$ happens to be 1, so $E$ and $(E^\sigma)^{(-1)}$, with $\sigma$ the non-trivial automorphism of $K$, are related by an 11-isogeny. Furthermore, if $C \subset E$ is the canonical cyclic subgroup of order 22 and $\omega$ is the standard differential, the modular form $A_\gamma$ takes the value $-77/6$ on $(E, C, \omega)$. Using Elkies’s algorithm, one computes that the kernel $C_\gamma$ of this 11-isogeny is defined by the polynomial

$$P_{C_\gamma} = x^5 + 6x^4 + (285 + 33\beta)x^3 - (1110 + 759\beta)x^2 + (40298 - 12496\beta)x - (13223 + 38324\beta).$$

Using known algorithms, one can also compute the rational functions defining the isogeny, but we will not write these down.

Finally, we note that both $E$ and its quadratic twist by $-11$ have rank 0; this can be shown by a 2-descent. This implies that $E$ has rank 0 over $L = K(\sqrt{-11})$, which is consistent with Corollary 9.

6. Fields of definition of elliptic curves with 28- or 40-isogenies

In this section we prove that all but finitely many $\mathbb{Q}$-isomorphism classes of elliptic curves over quadratic fields with 28- or 40-isogenies are defined over real quadratic fields.

**Theorem 10.** Among all elliptic curves over quadratic fields with a 28- or 40-isogeny, all but finitely many (up to $\mathbb{Q}$-isomorphism) are defined over real quadratic fields.

**Proof.** Let $X = X_0(n)$, where $n = 28$ or 40. Using the notation of section 2 and by (2) and using the fact that $J_0(n)(\mathbb{Q})$, and hence $\phi^{-1}(J_0(n)(\mathbb{Q}))$, is finite, it follows that all but finitely many quadratic points on $X^2(\mathbb{Q})$ are in the set $\phi^{-1}(0)$, and hence are points fixed by the hyperelliptic involution.

Let $\iota$ the hyperelliptic involution of $X$ and $P$ be a quadratic point on $X$ such that $\{P, P\sigma\} \in \phi^{-1}(0)$. Now it follows that there exists a $\mathbb{Q}$-rational point $R$ on $X/\iota$ such that $P$ is the inverse image of $R$ under the quotient map $X \to X/\iota$.

The modular curves $X_0(28)$ and $X_0(40)$ admit the equations

$$X_0(28) : y^2 = f_1(x) = x^6 + 6x^5 + 25x^4 + 60x^3 + 100x^2 + 96x + 64,$$

$$X_0(40) : y^2 = f_2(x) = x^8 + 8x^6 - 2x^4 + 8x^2 + 1.$$

By construction, the point $P = (x, \sqrt{f_1(x)})$ has $x \in \mathbb{Q}$; it is easy to check that $f_1(x) > 0$ and $f_2(x) > 0$. It follows that $P$ is defined over a real quadratic field.

**Remark 5.** The finite list of $\mathbb{Q}$-isomorphism classes of elliptic curves which are exceptions to Theorem 10 (they have a 28- or 40-isogeny and are defined over an imaginary quadratic field) are exactly the curves corresponding to the points in Tables 4 and 10.

7. Exceptional points

7.1. **Notation.** When making isogeny diagrams, the vertices will be points $P$ on $X_0(n)$ which correspond to an elliptic curve $E$ together with a cyclic subgroup $G$ of order $n$. We use this notation as the points $P$ take considerably less space to write down. Furthermore, we do not write down the functions that for a given
point on $X_0(n)$ construct an elliptic curve with an $n$-isogeny, as these functions are implemented in Magma.

Let $P_1, P_2$ be points on the modular curve $X_0(n)$. If the isogeny diagram of the isogeny class is of the form

$$P_1 \xrightarrow{n} P_2,$$

we will say that the isogeny diagram is simple and denote it by $S(P_1, P_2, n)$.

Let $P_1, P_2, P_3, P_4$ be points on the modular curve $X_0(n)$. If the isogeny diagram of the isogeny class is of the form

$$\begin{array}{c}
\begin{array}{cc}
P_1 & a \\
\downarrow b & \\
P_3 & a \\
\end{array}
\end{array}
\begin{array}{cc}
P_2 & b \\
\downarrow b & \\
P_4 & \\
\end{array}$$

where $a$ and $b$ denote the degrees of the isogenies between the curves, and $ab = n$, we say that the isogeny diagram is a square and denote it by $SQ(P_1, P_2, P_3, P_4, a, b)$. 
Model:
\[ y^2 + (-x^3)y = -x^4 + 5x^3 - 10x^2 + 12x - 8 \]

Genus: 3
Hyperelliptic involution: \( w_{11} \)
Group structure:
\[ J_0(22)(\mathbb{Q}) \simeq \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \]

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>( d )</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>-7</td>
<td>( \left( \frac{1}{2}(w - 1), \frac{1}{2}(w + 11) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>-7</td>
<td>( \left( \frac{1}{2}(w - 1), -w - 3 \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_3 )</td>
<td>-7</td>
<td>( \left( \frac{1}{2}(-w + 3), \frac{1}{16}(7w - 13) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_4 )</td>
<td>-7</td>
<td>( \left( \frac{1}{2}(-w + 3), \frac{1}{4}(-3w + 1) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_5 )</td>
<td>-7</td>
<td>( \left( \frac{1}{2}(-w + 1), \frac{1}{2}(w - 5) \right) )</td>
<td>-7</td>
</tr>
<tr>
<td>( P_6 )</td>
<td>-7</td>
<td>( \left( \frac{1}{2}(-w + 1), 0 \right) )</td>
<td>-7</td>
</tr>
<tr>
<td>( P_7 )</td>
<td>33</td>
<td>( \left( \frac{1}{2}(-w - 3), \frac{1}{2}(-3w - 13) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_8 )</td>
<td>33</td>
<td>( \left( \frac{1}{2}(-w - 3), -6w - 34 \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_9 )</td>
<td>-47</td>
<td>( \left( \frac{1}{4}(w + 1), \frac{1}{16}(-7w + 1) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{10} )</td>
<td>-47</td>
<td>( \left( \frac{1}{4}(w + 1), \frac{1}{4}(-w - 9) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{11} )</td>
<td>-47</td>
<td>( \left( \frac{1}{6}(-w + 5), \frac{1}{27}(w - 41) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{12} )</td>
<td>-47</td>
<td>( \left( \frac{1}{6}(-w + 5), \frac{1}{6}(-w - 7) \right) )</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:
\( \text{SQ}(P_1, P_2, P_3, P_4, 2, 11), \text{SQ}(P_7, P_7, P_8, P_8, 2, 11), \text{SQ}(P_9, P_{10}, P_{11}, P_{12}, 2, 11). \)

Remark 6. The points \( P_9 \) and \( P_{10} \) are lifts of a point on \( X_0(22)/w_2 \).
Table 2. $X_0(23)$

Model:
\[ y^2 + (-x^3 - x - 1)y = -2x^5 - 3x^2 + 2x - 2 \]

Genus: 2
Hyperelliptic involution: $w_{23}$
Group structure:
\[ J_0(23)(\mathbb{Q}) \simeq \mathbb{Z}/11\mathbb{Z} \]

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>-5</td>
<td>$\left(\frac{1}{3}(2w - 2), \frac{1}{5}(8w + 70)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_2$</td>
<td>-5</td>
<td>$\left(\frac{1}{3}(2w - 2), \frac{1}{27}(-22w - 89)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_3$</td>
<td>-7</td>
<td>$\left(\frac{1}{4}(-w + 3), \frac{1}{3}(-w - 1)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_4$</td>
<td>-7</td>
<td>$\left(\frac{1}{4}(-w + 3), \frac{1}{16}(-5w + 23)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_5$</td>
<td>-7</td>
<td>$(0, \frac{1}{2}(w + 1))$</td>
<td>-7</td>
</tr>
<tr>
<td>$P_6$</td>
<td>-7</td>
<td>$(2, \frac{1}{2}(5w + 11))$</td>
<td>-28</td>
</tr>
<tr>
<td>$P_7$</td>
<td>-11</td>
<td>$\left(\frac{1}{6}(-w + 1), \frac{1}{54}(-19w + 49)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_8$</td>
<td>-11</td>
<td>$\left(\frac{1}{6}(-w + 1), \frac{1}{3}(2w + 1)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_9$</td>
<td>-11</td>
<td>$(1, \frac{1}{2}(w + 3))$</td>
<td>-11</td>
</tr>
<tr>
<td>$P_{10}$</td>
<td>-15</td>
<td>$\left(\frac{1}{4}(w + 1), \frac{1}{16}(-3w + 5)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_{11}$</td>
<td>-15</td>
<td>$\left(\frac{1}{4}(w + 1), \frac{1}{2}(w + 1)\right)$</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:
\[ S(P_1, P_2, 23), S(P_3, P_4, 23), S(P_7, P_{18}, 23), S(P_{10}, P_{11}, 23). \]
Table 3. $X_0(26)$

Model:
\[ y^2 + (-x^3 - x - 1)y = -2x^5 - 3x^2 + 2x - 2 \]

Genus: 2

Hyperelliptic involution: $w_{26}$

Group structure:
\[ J_0(26)(\mathbb{Q}) \cong \mathbb{Z}/21\mathbb{Z} \]

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$-3$</td>
<td>$\left(\frac{1}{2}w + 1, -1\right)$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$-3$</td>
<td>$\left(\frac{1}{2}w + 1, 1\right)$</td>
<td>$-12$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$-11$</td>
<td>$\left(\frac{1}{2}(w - 1), 7\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$-11$</td>
<td>$\left(\frac{1}{6}(-w - 1), \frac{1}{27}(7w + 28)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_5$</td>
<td>$-11$</td>
<td>$\left(\frac{1}{6}(-w - 1), \frac{1}{5}(-2w + 1)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_6$</td>
<td>$-11$</td>
<td>$\left(\frac{1}{2}(w - 1), -w - 2\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_7$</td>
<td>$-23$</td>
<td>$\left(\frac{1}{6}(-w + 1), \frac{1}{27}(w + 2)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_8$</td>
<td>$-23$</td>
<td>$\left(\frac{1}{4}(w + 1), \frac{1}{18}(-w + 3)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_9$</td>
<td>$-23$</td>
<td>$\left(\frac{1}{4}(w + 1), \frac{1}{3}(-w - 1)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_{10}$</td>
<td>$-23$</td>
<td>$\left(\frac{1}{6}(-w + 1), \frac{1}{18}(w + 11)\right)$</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:

$SQ(P_3, P_4, P_5, P_6, 2, 13)$, $SQ(P_7, P_8, P_9, P_{10}, 2, 13)$. 
Table 4. \( X_0(28) \)

Model:
\[
y^2 + (-2x^3 + 3x^2 - 3x)y = x^4 - 3x^3 + 4x^2 - 3x + 1
\]

Genus: 2
Hyperelliptic involution: \( w_7 \)
Group structure:
\[
J_0(28)(\mathbb{Q}) \simeq \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}
\]

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>( d )</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>-3</td>
<td>( \left( \frac{1}{2}(w + 1), 0 \right) )</td>
<td>-12</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>-3</td>
<td>( \left( \frac{1}{2}(w + 1), 1 \right) )</td>
<td>-12</td>
</tr>
<tr>
<td>( P_3 )</td>
<td>-7</td>
<td>( \left( \frac{1}{2}(-w + 1), \frac{1}{2}(w + 1) \right) )</td>
<td>-7</td>
</tr>
<tr>
<td>( P_4 )</td>
<td>-7</td>
<td>( \left( \frac{1}{2}(w + 1), \frac{1}{8}(w + 5) \right) )</td>
<td>-7</td>
</tr>
<tr>
<td>( P_5 )</td>
<td>-7</td>
<td>( \left( \frac{1}{4}(-w + 3), \frac{1}{8}(-w + 3) \right) )</td>
<td>-28</td>
</tr>
<tr>
<td>( P_6 )</td>
<td>-23</td>
<td>( \left( \frac{1}{4}(w + 1), \frac{1}{8}(-w + 19) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_7 )</td>
<td>-23</td>
<td>( \left( \frac{1}{6}(-w + 1), \frac{1}{54}(w + 29) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_8 )</td>
<td>-23</td>
<td>( \left( \frac{1}{6}(-w + 5), \frac{1}{54}(-3w + 7) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_9 )</td>
<td>-23</td>
<td>( \left( \frac{1}{6}(w + 5), \frac{1}{54}(-w + 25) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{10} )</td>
<td>-23</td>
<td>( \left( \frac{1}{4}(-w + 3), \frac{1}{8}(w - 11) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{11} )</td>
<td>-23</td>
<td>( \left( \frac{1}{8}(w + 3), \frac{1}{54}(3w + 57) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{12} )</td>
<td>-23</td>
<td>( \left( \frac{1}{4}(w + 1), \frac{1}{8}(-w + 3) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{13} )</td>
<td>-23</td>
<td>( \left( \frac{1}{6}(-w + 1), \frac{1}{6}(-w + 7) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{14} )</td>
<td>-23</td>
<td>( \left( \frac{1}{6}(-w + 5), \frac{1}{16}(-w + 13) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{15} )</td>
<td>-23</td>
<td>( \left( \frac{1}{6}(w + 5), \frac{1}{6}(w - 1) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{16} )</td>
<td>-23</td>
<td>( \left( \frac{1}{8}(w + 3), \frac{1}{16}(w + 3) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_{17} )</td>
<td>-23</td>
<td>( \left( \frac{1}{4}(-w + 3), \frac{1}{8}(w + 5) \right) )</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:

In these diagrams the dashed lines represent 7-isogenies, while the full lines represent 4-isogenies.
Table 5. $X_0(29)$

Model:
\[ y^2 + (-x^3 - 1)y = -x^5 - 3x^4 + 2x^2 + 2x - 2 \]

Genus: 2
Hyperelliptic involution: $w_{29}$
Group structure:
\[ J_0(29)(\mathbb{Q}) \simeq \mathbb{Z}/7\mathbb{Z} \]

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$-1$</td>
<td>$(w - 1, 2w + 4)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$-1$</td>
<td>$(w - 1, w - 1)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$-7$</td>
<td>$\left(\frac{1}{4}(w + 1), \frac{1}{16}(-11w - 7)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$-7$</td>
<td>$\left(\frac{1}{4}(w + 1), \frac{1}{8}(5w + 9)\right)$</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:
\[ S(P_1, P_2, 29), S(P_3, P_4, 29). \]
Table 6. $X_0(30)$

Model:
\[ y^2 + (-x^4 - x^3 - x^2)y = 3x^7 + 19x^6 + 60x^5 + 110x^4 + 121x^3 + 79x^2 + 28x + 4 \]
Genus: 3
Hyperelliptic involution: $w_{15}$
Group structure:
\[ J_0(30)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z} \]

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>5</td>
<td>$(-w - 3, 71w + 159)$</td>
<td>$-15$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>5</td>
<td>$(\frac{1}{2}(-w - 3), 4w + 9)$</td>
<td>$-60$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$-7$</td>
<td>$(\frac{1}{2}(-w - 3), w - 3)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$-7$</td>
<td>$(\frac{1}{4}(w - 3), \frac{1}{16}(5w + 9))$</td>
<td>no</td>
</tr>
<tr>
<td>$P_5$</td>
<td>$-7$</td>
<td>$(\frac{1}{4}(-w - 3), \frac{1}{16}(5w - 9))$</td>
<td>no</td>
</tr>
<tr>
<td>$P_6$</td>
<td>$-7$</td>
<td>$(\frac{1}{2}(w - 3), \frac{1}{2}(w - 15))$</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:

In this diagram, the horizontal lines are 5-isogenies, the vertical lines 2-isogenies and the diagonal ones are 3-isogenies.

Remark 7. All the curves in the diagram are $\mathbb{Q}$-curves and are 6-isogenous to their Galois conjugates and arise from rational points on the curve $X_0(30)/w_6$. 
Table 7. $X_0(33)$

Model:
\[ y^2 + (-x^4 - x^2 - 1)y = 2x^6 - 2x^5 + 11x^4 - 10x^3 + 20x^2 - 11x + 8 \]

Genus: 3
Hyperelliptic involution: $w_{11}$
Group structure: \[ J_0(33)(\mathbb{Q}) \cong \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} \]

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>-2</td>
<td>$(-\frac{1}{2}w, \frac{1}{4}(-4w - 5))$</td>
<td>no</td>
</tr>
<tr>
<td>$P_2$</td>
<td>-2</td>
<td>$(w - 1, -5w - 5)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_3$</td>
<td>-2</td>
<td>$(-\frac{1}{2}w, w + 2)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_4$</td>
<td>-2</td>
<td>$(w - 1, 7w - 2)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_5$</td>
<td>-2</td>
<td>$(w, -w + 1)$</td>
<td>-8</td>
</tr>
<tr>
<td>$P_6$</td>
<td>-2</td>
<td>$(w, w + 2)$</td>
<td>-8</td>
</tr>
<tr>
<td>$P_7$</td>
<td>-7</td>
<td>$\left(\frac{1}{4}(-3w + 1), \frac{1}{2}(9w + 93)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_8$</td>
<td>-7</td>
<td>$\left(\frac{1}{2}(w + 1), -1\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_9$</td>
<td>-7</td>
<td>$\left(\frac{1}{4}(-3w + 1), \frac{1}{2}(9w + 33)\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_{10}$</td>
<td>-7</td>
<td>$\left(\frac{1}{2}(w + 1), -w + 1\right)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_{11}$</td>
<td>-11</td>
<td>$\left(\frac{1}{2}(-w + 1), w + 1\right)$</td>
<td>-11</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:
\[ SQ(P_1, P_2, P_3, P_4, 3, 11), SQ(P_7, P_8, P_9, P_{10}, 3, 11). \]

Table 8. $X_0(35)$

Model:
\[ y^2 + (-x^4 - x^2 - 1)y = -x^7 - 2x^6 - 5x^5 - 3x^4 + 3x^3 - 2x^2 + x \]

Genus: 3
Hyperelliptic involution: $w_{35}$
Group structure: \[ J_0(35)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z} \]

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>5</td>
<td>$\left(\frac{1}{2}(-w - 1), w + 3\right)$</td>
<td>-35</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:
Table 9. $X_0(39)$

Model:
$$y^2 + (-x^4 - x^3 - x^2 - x - 1)y = -2x^7 + 2x^5 - 7x^4 + 2x^3 - 2x$$

Genus: 3
Hyperelliptic involution: $w_{39}$
Group structure: $J_0(39)(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/28\mathbb{Z}$
Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$-3$</td>
<td>$(\frac{1}{2}(-w + 1), \frac{1}{2}(-w + 1))$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$-3$</td>
<td>$(\frac{1}{2}(-w + 1), -1)$</td>
<td>$-27$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$-7$</td>
<td>$(\frac{1}{4}(-w + 3), \frac{1}{2}(3w + 7))$</td>
<td>no</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$-7$</td>
<td>$(\frac{1}{4}(w + 3), \frac{1}{6}(3w + 1))$</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:

$SQ(P_3, P_3^\sigma, P_4, P_4^\sigma, 3, 13)$.

Remark 8. The points $P_3$ and $P_4$ come from points on $X_0(39)/w_3$.

Table 10. $X_0(40)$

Model:
$$y^2 + (-x^4 - 1)y = 2x^6 - x^4 + 2x^2$$

Genus: 3
Hyperelliptic involution: induced by $\beta_{40} = \begin{pmatrix} -10 & 1 \\ -120 & 10 \end{pmatrix}$
Group structure: $J_0(40)(\mathbb{Q}) \simeq \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$
Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$-1$</td>
<td>$(w, 2w + 1)$</td>
<td>$-16$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$-1$</td>
<td>$(w, -2w + 1)$</td>
<td>$-16$</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:
Model:
\[ y^2 + (-x^4 - x)y = -x^7 - 2x^6 + 2x^5 + 5x^4 + 2x^3 - 4x^2 - 5x - 2 \]
Genus: 3
Hyperelliptic involution: \( w_{41} \)
Group structure:
\[ J_0(41)(\mathbb{Q}) \cong \mathbb{Z}/10\mathbb{Z} \]
Exceptional conjugacy classes of points:
<table>
<thead>
<tr>
<th>Name</th>
<th>( d )</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>-1</td>
<td>( \left( \frac{1}{2}(-w - 1), \frac{1}{4}(-3w - 4) \right) )</td>
<td>no</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>-1</td>
<td>( \left( \frac{1}{2}(-w - 1), \frac{1}{4}(w + 1) \right) )</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:
\[ S(P_1, P_2, 41). \]

Table 12. \( X_0(48) \)

Model:
\[ y^2 = x^8 + 14x^4 + 1 \]
Genus: 3
Hyperelliptic involution: induced by \( \beta_{48} = \begin{pmatrix} -6 & 1 \\ -48 & 6 \end{pmatrix} \)
Group structure:
\[ J_0(48)(\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \]
Exceptional conjugacy classes of points:
<table>
<thead>
<tr>
<th>Name</th>
<th>( d )</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>-1</td>
<td>( (w, 4) )</td>
<td>-16</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>-1</td>
<td>( (w, -4) )</td>
<td>-16</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:
Table 13. $X_0(50)$

Model:

$$y^2 + (-x^3 - 1)y = -x^5 - 3x^3 - x$$

Genus: 2
Hyperelliptic involution: $w_{50}$
Group structure:

$$J_0(50)(\mathbb{Q}) \simeq \mathbb{Z}/15\mathbb{Z}$$

Exceptional conjugacy classes of points:

<table>
<thead>
<tr>
<th>Name</th>
<th>$d$</th>
<th>Coordinates</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$-1$</td>
<td>$(w, -w)$</td>
<td>$-4$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$-1$</td>
<td>$(w, 1)$</td>
<td>$-16$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$-7$</td>
<td>$(\frac{1}{2}(w - 1), 3)$</td>
<td>no</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$-7$</td>
<td>$(\frac{1}{2}(-w - 1), \frac{1}{16}(3w + 15))$</td>
<td>no</td>
</tr>
<tr>
<td>$P_5$</td>
<td>$-7$</td>
<td>$(\frac{1}{2}(-w - 1), \frac{1}{8}(-w + 3))$</td>
<td>no</td>
</tr>
<tr>
<td>$P_6$</td>
<td>$-7$</td>
<td>$(\frac{1}{2}(w - 1), \frac{1}{2}(-w + 1))$</td>
<td>no</td>
</tr>
</tbody>
</table>

Isogeny diagrams of non-CM points, up to conjugation:

$$SQ(P_3, P_4, P_5, P_6, 2, 25),$$
References

[22] F. Najman, Torsion of rational elliptic curves over cubic fields and sporadic points on $X_1(n)$, to appear.
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