RAD HRVATSKE AKADEMIJE ZNANOSTI I UMJETNOSTI
MATEMATIČKE ZNANOSTI

H. Nath

The pod function and its connection with other partition function

Manuscript accepted for publication

THE POD FUNCTION AND ITS CONNECTION WITH OTHER PARTITION FUNCTIONS

HEMJYOTI NATH

ABSTRACT. The number of partitions of n wherein odd parts are distinct and even parts are unrestricted, often denoted by $pod(n)$. In this paper, we provide linear recurrence relations for $pod(n)$ and the connections of $pod(n)$ with other partition functions.

1. INTRODUCTION

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. For example, $n = 4$ has five partitions, namely,

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

If $p(n)$ denotes the number of partitions of n, then $p(4) = 5$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, here and throughout the paper

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q| < 1.$$

In 2010, Hirschhorn and Sellers [9] defined the partition function $pod(n)$, which counts the number of partitions of n into distinct odd parts where even

---

2020 *Mathematics Subject Classification.* 11P81, 11P83.
Key words and phrases. integer partitions, recurrence relations.

parts maybe repeated. For example, $pod(4) = 3$ with the relevent partitions being

$$4, \quad 3+1, \quad 2+2.$$

The generating function for $pod(n)$ is given by

$$\sum_{n=0}^{\infty} pod(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{(q^2;q^4)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^4;q^4)_\infty}.$$

Hirschhorn and Sellers proved the Ramanujan-type congruences

$$pod\left(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod 3, \quad \text{for all} \quad \alpha \geq 0, \quad n \geq 0$$

using some q-series identities. Radu and Sellers [13] later found congruences for $pod(n)$ modulo 5 and 7 using the theory of modular forms.

For nonnegative integers n and k, let $r_k(n)$ (resp. $t_k(n)$) denote the number of representations of n as sum of k squares (resp. triangular numbers). In 2011, based on the generating function of $pod(3n+2)$ found in [9], Lovejoy and Osburn discovered the following arithmetic relation

$$pod(3n+2) \equiv (-1)^n r_5(8n+5) \pmod 3.$$

Recently, Ballantine and Merca [3], obtained new properties for $pod(n)$ using the connections with 4-regular partitions and, for fixed $k \in \{0, 2\}$, partitions into distinct parts not congruent to k modulo 4.

In a very recent work, Ballantine and Welch [2] proved a recurrence relation for $pod(n)$ combinatorially.

Motivated from their work, we establish new recurrence relations for $pod(n)$ that involve triangular numbers and generalized pentagonal numbers i.e.,

$$T_k = \frac{k(k+1)}{2}, \quad k \in \mathbb{N}_0$$

and

$$G_k = \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3k+1}{2} \right\rceil, \quad k \in \mathbb{N}_0$$

respectively, and its connections with other partition functions.

2. Preliminaries

We require the following definitions, lemmas and theorems to prove the main results in the next two sections. The Jacobi triple product identity [9], is given by

$$(q^2;q^2)_\infty(-qx;q^2)_\infty(-q/x;q^2)_\infty = \sum_{n=-\infty}^{\infty} x^n q^{n^2}, \qquad |q| < 1, \quad x \neq 0. \tag{2.1}$$

For $|qx| < 1$, Ramanujan's general theta function $f(q, x)$ is defined as

$$f(q,x) := \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} x^{n(n-1)/2}. \tag{2.2}$$

Using (2.1), (2.2) takes the shape

$$f(q,x) = (-q, qx)_\infty(-x, qx)_\infty(qx, qx)_\infty. \tag{2.3}$$

The special cases of $f(q, x)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{2.4}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \tag{2.5}$$

$$\varphi(-q) := f(-q,-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}, \tag{2.6}$$

$$\psi(-q) := f(-q,-q^3) = \sum_{n=0}^{\infty} (-1)^{n(n+1)/2} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(-q;-q)_\infty}. \tag{2.7}$$

Theorem 2.1 (Euler's Pentagonal Number Theorem [9]). *We have*

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty. \tag{2.8}$$

Theorem 2.2 (Jacobi's Identity [9]). *We have*

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} = (q;q)_\infty^3. \tag{2.9}$$

Lemma 2.3 (Hirschhorn [7]). *We have the following identities due to Ramanujan,*

$$\frac{(q;q)_\infty^5}{(q^2;q^2)_\infty^2} = \sum_{n=-\infty}^{\infty} (6n+1) q^{n(3n+1)/2}, \tag{2.10}$$

$$\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty} = \sum_{n=-\infty}^{\infty} (3n+1) q^{3n^2+2n}. \tag{2.11}$$

Lemma 2.4 (Baruah [12]). *We have*

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} = \frac{(-q;q)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}. \tag{2.12}$$

3. Recurrence relations

In this section, we prove some recurrence relations for the partition function $pod(n)$ that involve T_k and G_k for $k \in \mathbb{N}_0$.

Theorem 3.1. *For $n \geq 0$, we have*

$$\sum_{j=0}^{\infty}(-1)^{\lceil j/2\rceil} pod(n-T_j) = \begin{cases} 1, & if\ n=0,\\ 0, & otherwise. \end{cases}$$

Proof. Replacing q by q^2 and x by $-q$ in (2.1), we have the relation

$$(q^4;q^4)_\infty(q^3;q^4)_\infty(q;q^4)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}, \tag{3.13}$$

Multiplying the above equation by $(q^2;q^2)_\infty/(q;q)_\infty(q^4;q^4)_\infty$, we get

$$1 = \frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^4;q^4)_\infty}\sum_{n=0}^{\infty}(-1)^{\lceil n/2\rceil} q^{n(n+1)/2}. \tag{3.14}$$

This can be written as

$$\left(\sum_{n=0}^{\infty} pod(n)q^n\right)\left(\sum_{n=0}^{\infty}(-1)^{\lceil n/2\rceil} q^{n(n+1)/2}\right) = 1.$$

The proof follows easily applying the well known Cauchy multiplication of two power series on the left hand side of the above equation. □

Corollary 3.2. *For $n \geq 0$, we have*

$$\sum_{j=0}^{\infty} pod(n-T_j) \equiv \begin{cases} 1 \pmod 2, & if\ \ n=0,\\ 0 \pmod 2, & otherwise. \end{cases}$$

The proof is immediate from the above theorem.

Theorem 3.3. *For $n \geq 0$, we have*

$$\sum_{j=0}^{\infty}(-1)^j(2j+1)pod(n-2T_j) = \begin{cases} \displaystyle\sum_{j=0}^{\infty}(-1)^{\lceil k/2\rceil+\lceil j/2\rceil+G_k-2G_j}, & if\ n=G_k,\\ 0, & otherwise. \end{cases}$$

Proof. We can write the generating function of $pod(n)$ as

$$\sum_{n=0}^{\infty} pod(n)q^n = \frac{(-q;-q)_\infty}{(q^2;q^2)^2_\infty}.$$

Multiplying both sides by $(q^2;q^2)^3_\infty$ in the above equation, we arrive at

$$(q^2;q^2)^3_\infty \sum_{n=0}^{\infty} pod(n)q^n = (q^2;q^2)_\infty(-q;-q)_\infty$$

Using (2.8) and (2.9), we get

$$\left(\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{T_n}\right)\left(\sum_{n=0}^{\infty}pod(n)q^n\right)=\left(\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}q^{2G_k}\right)\left(\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil+G_k}q^{G_k}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. □

THEOREM 3.4. *For $n \geq 0$,*

$$\sum_{j=-\infty}^{\infty}(3j+1)pod(n-3j^2-2j)=\begin{cases}\sum_{j=0}^{\infty}(-1)^{\lceil k/2\rceil+\lceil j/2\rceil-4G_j}, & if\ n=G_k,\\ 0, & otherwise.\end{cases}$$

PROOF. From (2.11), we have

$$\frac{(q;q)_\infty^2(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}=\sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n},$$

Multiplying the above equation by $(q^2;q^2)_\infty/(q;q)_\infty(q^4;q^4)_\infty$, to get

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^4;q^4)_\infty}\left(\sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n}\right)=(q;q)_\infty(q^4;q^4)_\infty$$

On simplification, we get

(3.15)

$$\left(\sum_{n=0}^{\infty}pod(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n}\right)=\left(\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}q^{G_k}\right)\left(\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}q^{4G_k}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. □

COROLLARY 3.5. *For $n \geq 0$, we have*

$$\sum_{j=-\infty}^{\infty}(3j+1)pod(n-3j^2-2j)=\begin{cases}\sum_{j=0}^{\infty}(-1)^{\lceil k/2\rceil+\lceil j/2\rceil-2G_j}, & if\ n=T_k,\\ 0, & otherwise.\end{cases}$$

PROOF. From (3.14), we have

$$(q;q)_\infty(q^4;q^4)_\infty=(q^2;q^2)_\infty\sum_{n=0}^{\infty}(-1)^{\lceil n/2\rceil}q^{n(n+1)/2}.$$

And, from (3.15) we find that

$$\left(\sum_{n=0}^{\infty}pod(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n}\right)=(q^2;q^2)_\infty\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}q^{k(k+1)/2},$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} pod(n)q^n\right)\left(\sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n}\right)=\left(\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}q^{2G_k}\right)\left(\sum_{k=0}^{\infty}(-1)^{\lceil k/2\rceil}q^{T_k}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. □

4. Connections of $pod(n)$ with other partition functions

In this section, we will see how $pod(n)$ is related to various other partition functions.

4.1. *Connection with the partition function $ped(n)$.* The partition function $ped(n)$ enumerates the number of partitions of n where the even parts are distinct and odd parts maybe repeated. For example, $ped(6) = 9$ and the partitions are

6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+1+1+1+1, 1+1+1+1+1+1.

The generating function for $ped(n)$ is given by

$$\sum_{n=0}^{\infty} ped(n)q^n=\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}=\frac{(q^4;q^4)_\infty}{(q;q)_\infty}. \tag{4.16}$$

In this section, we shall prove that the partition function $pod(n)$ can be expressed in terms of the partition function $ped(n)$.

Theorem 4.1. *For any nonnegative integer n, we have*

$$pod(n)=\sum_{j=0}^{\infty}(-1)^j pod\left(\frac{j}{2}\right) ped(n-j).$$

with $pod(x) = 0$ if x is not an integer.

Proof. It is known that

$$\sum_{n=0}^{\infty} pod(n)q^n=\frac{1}{\psi(-q)}.$$

We can write the generating function of $pod(n)$ as

$$\sum_{n=0}^{\infty} pod(n)q^n=\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^4;q^4)_\infty},$$

which after some manipulations takes the shape

$$\sum_{n=0}^{\infty} pod(n)q^n=\frac{1}{\psi(q^2)}\sum_{n=0}^{\infty} ped(n)q^n,$$

which is equivalent to

$$\sum_{n=0}^{\infty} pod(n)q^n = \left(\sum_{n=0}^{\infty}(-1)^n pod(n)q^{2n}\right)\left(\sum_{n=0}^{\infty} ped(n)q^n\right).$$

Finally, equating the coefficient of q^n on each side gives the result. ∎

4.2. *Connection with the partitions into distinct odd parts.* The number of partitions of n into distinct parts is usually denoted by $q(n)$. The number of partitions of n into distinct odd parts is denoted by $q_{odd}(n)$. The generating functions for $q(n)$ and $q_{odd}(n)$ are given by

$$\sum_{n=0}^{\infty} q(n)q^n = (-q;q)_\infty,$$

and

$$\sum_{n=0}^{\infty} q_{odd}(n)q^n = (-q;q^2)_\infty.$$

THEOREM 4.2. *For any nonnegative integer n, the partition functions $p(n)$, $q_{odd}(n)$ and $pod(n)$ are related by*

$$pod(n) = \sum_{j=0}^{\infty} q_{odd}(j) p\left(\frac{n}{2} - \frac{j}{2}\right),$$

with $p(x) = 0$ if x is not an integer.

PROOF. Considering the generating function of $pod(n)$, we can write

$$\sum_{n=0}^{\infty} pod(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

which is equivalent to

$$\sum_{n=0}^{\infty} pod(n)q^n = \left(\sum_{n=0}^{\infty} p(n)q^{2n}\right)\left(\sum_{n=0}^{\infty} q_{odd}q^n\right).$$

Finally, equating the coefficient of q^n on each side gives the result. ∎

THEOREM 4.3. *For any nonnegative integer n, the partition functions $q_{odd}(n)$ and $pod(n)$ are related by*

$$q_{odd}(n) = \sum_{j=-\infty}^{\infty} (-1)^j pod\left(\frac{n}{2} - \frac{j(3j+1)}{2}\right).$$

with $pod(x) = 0$ if x is not an integer.

Proof. We can write

$$(q^2;q^2)_\infty \sum_{n=0}^{\infty} pod(n)q^n = (-q;q^2)_\infty,$$

which is equivalent to

$$\left(\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)}\right)\left(\sum_{n=0}^{\infty} pod(n)q^n\right) = \sum_{n=0}^{\infty} q_{odd}(n)q^n.$$

Finally, equating the coefficient of q^n on each side gives the result. □

We denote the difference of the number of partitions of n into an even number of parts and partitions of n into an odd number of parts by $p_{e-o}(n)$. The generating function of $p_{e-o}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{e-o}(n)q^n = \frac{1}{(-q;q)_\infty}.$$

We have the following result.

Corollary 4.4. *For any nonnegative integer n, the partition functions $p_{e-o}(n)$ and $pod(n)$ are related by*

$$p_{e-o}(n) = (-1)^n \sum_{j=-\infty}^{\infty} (-1)^j pod\left(\frac{n}{2} - \frac{j(3j+1)}{2}\right).$$

with $pod(x) = 0$ if x is not an integer.

The proof is immediate as

$$q_{odd}(n) = (-1)^n p_{e-o}(n).$$

4.3. *Connection with the overpartition and the partition function $A(n)$.* An overpartition of a nonnegative integer n is a non-increasing sequence of natural numbers whose sum is n, where the first occurence (or equivalently, the last occurence) of a number may be overlined. The eight overpartitions of 3 are

$$3,\quad \overline{3},\quad 2+1,\quad \overline{2}+1,\quad 2+\overline{1},\quad \overline{2}+\overline{1},\quad 1+1+1,\quad \overline{1}+1+1.$$

The number of overpartitions of n is denoted by $\overline{p}(n)$ and its generating function is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.$$

Recently, Merca [11] defined the following functions.

Definition 4.5. *For a positive integer n, let*

1. *$a_e(n)$ be the number of partitions of n into an even number of parts in which the even parts can appear in two colours.*

2. *$a_o(n)$ be the number of partitions of n into an odd number of parts in which the even parts can appear in two colours.*
3. *$A(n) = a_e(n) - a_o(n)$.*

He found that the generating function of $A(n)$ is

$$\sum_{n=0}^{\infty} A(n)q^n = \frac{1}{(-q;q)_\infty(-q^2;q^2)_\infty} = (q;q^2)_\infty(q^2;q^4)_\infty = \frac{(q;q)_\infty}{(q^4;q^4)_\infty}.$$

THEOREM 4.6. *For any nonnegative integer n, we have*

$$pod(n) = \sum_{j=0}^{\infty} A(j)\overline{p}(n-j).$$

PROOF. We start with the generating function of $\overline{p}(n)$,

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.$$

Multiplying the above equation by $(q;q)_\infty/(q^4;q^4)_\infty$, to get

$$\frac{(q;q)_\infty}{(q^4;q^4)_\infty}\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^4;q^4)_\infty},$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} A(n)q^n\right)\left(\sum_{n=0}^{\infty} \overline{p}(n)q^n\right) = \sum_{n=0}^{\infty} pod(n)q^n.$$

Finally, equating the coefficient of q^n on each side gives the result. ∎

4.4. *Connection with the overpartitions into odd parts.* We denote by $\overline{p}_o(n)$ the number of overpartitions of n into odd parts. The generating function for $\overline{p}_o(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}_o(n)q^n = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2(q^4;q^4)_\infty}. \tag{4.17}$$

In this section, we shall prove that the partition function $\overline{p}_o(n)$ can be expressed in terms of the partition function $pod(n)$.

THEOREM 4.7. *For any nonnegative integer n, we have*

$$pod(n) = \sum_{j=0}^{\infty} (-1)^j pod(j)\overline{p}_o(n-j).$$

PROOF. We can write

$$\sum_{n=0}^{\infty} pod(n)q^n = \frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\sum_{n=0}^{\infty} \overline{p}_o(n)q^n,$$

which is equivalent to

$$\sum_{n=0}^{\infty} pod(n)q^n = \left(\sum_{n=0}^{\infty}(-1)^n pod(n)q^n\right)\left(\sum_{n=0}^{\infty}\overline{p}_o(n)q^n\right).$$

Finally, equating the coefficient of q^n on each side gives the result. ∎

THEOREM 4.8. *For any nonnegative integer n, we have*

$$\overline{p}_o(n) = \sum_{j=0}^{\infty} pod\left(n - \frac{j(j+1)}{2}\right).$$

PROOF. Replacing x by 1 in (2.2) and (2.3), we have the relation

$$(-q;q)_\infty(-1;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2},$$

which after simplification takes the shape

$$2(-q;q)_\infty^2(q;q)_\infty = 2\sum_{n=0}^{\infty} q^{n(n+1)/2},$$

which is equivalent to

$$\frac{(q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty^2} = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Further simplification of the above equation gives the identity

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{4.18}$$

Considering the identity (4.18) and the generating function of $\overline{p}_o(n)$, we can write

$$\sum_{n=0}^{\infty}\overline{p}_o(n)q^n = \left(\sum_{n=0}^{\infty} pod(n)q^n\right)\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},$$

which is equivalent to

$$\sum_{n=0}^{\infty}\overline{p}_o(n)q^n = \left(\sum_{n=0}^{\infty} pod(n)q^n\right)\left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. ∎

4.5. *Connection with the ordinary partition function and the partition function* $q_{e-o}(n)$. We denote the difference of the number of partitions of n into distinct parts with an even number of odd parts and partitions of n into distinct parts with an odd number of odd parts by $q_{e-o}(n)$. The generating function of $q_{e-o}(n)$ is given by

$$\sum_{n=0}^{\infty} q_{e-o}(n)q^n = \frac{1}{(-q;q^2)_\infty}.$$

THEOREM 4.9. *For any nonnegative integer n, the partition functions $p(n)$, $q_{e-o}(n)$ and $pod(n)$ are related by*

$$p\left(\frac{n}{2}\right) = \sum_{j=0}^{\infty} pod(j)q_{e-o}(n-j),$$

with $p(x) = 0$ if x is not an integer.

PROOF. We can write

$$\frac{1}{(-q;q^2)_\infty}\sum_{n=0}^{\infty} pod(n)q^n = \frac{1}{(q^2;q^2)_\infty},$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} q_{e-o}(n)q^n\right)\left(\sum_{n=0}^{\infty} pod(n)q^n\right) = \sum_{n=0}^{\infty} p(n)q^{2n}.$$

Finally, equating the coefficient of q^n on each side gives the result. □

4.6. *Connection with the partitions into parts congruent to* 2 (mod 4). We denote the number of partitions of n into parts congruent to 2 (mod 4) by $P_2^{'}(n)$. It is clear that the generating function of $P_2^{'}(n)$ is given by

$$\sum_{n=0}^{\infty} P_2^{'}(n) = \frac{1}{(q^2;q^4)_\infty}.$$

In this section, we will prove a relation connecting the partition functions $pod(n)$, $P_2^{'}(n)$ and $p(n)$.

THEOREM 4.10. *For any nonnegative integer n, the partition functions $p(n)$, $P_2^{'}(n)$ and $pod(n)$ are related by*

$$p(n) = \sum_{j=0}^{\infty} pod(j)P_2^{'}(n-j).$$

PROOF. We can write

$$\frac{1}{(q^2;q^4)_\infty}\sum_{n=0}^{\infty} pod(n)q^n = \frac{1}{(q;q)_\infty}$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} pod(n)q^n\right)\left(\sum_{n=0}^{\infty} P_2^{'}(n)q^n\right) = \sum_{n=0}^{\infty} p(n)q^n.$$

Finally, equating the coefficient of q^n on each side gives the result. □

4.7. *Connections with the ordinary partition function.* In this section, we will prove relations between the partition functions $p(n)$ and $pod(n)$.

THEOREM 4.11. *For any nonnegative integer n, the partition functions $p(n)$ and $pod(n)$ are related by*

$$\sum_{j=-\infty}^{\infty} (-1)^j pod(n-3j^2) = \sum_{j=-\infty}^{\infty} p\left(\frac{n}{4} - \frac{j(3j+1)}{8}\right),$$

with $p(x) = 0$ if x is not an integer.

PROOF. Multiplying (2.12) by $(q;q)_\infty(q^3;q^3)_\infty/(q;q)_\infty(q^3;q^3)_\infty$, to obtain

$$\frac{(q^2;q^2)_\infty(q^3;q^3)_\infty^2}{(q;q)_\infty(q^6;q^6)_\infty} = \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}.$$

Multiplying the above equation by $1/(q^4;q^4)_\infty$, to get

$$\frac{(q^3;q^3)_\infty^2}{(q^6;q^6)_\infty}\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^4;q^4)_\infty} = \frac{1}{(q^4;q^4)_\infty}\sum_{j=-\infty}^{\infty} q^{n(3n+1)/2},$$

which is equivalent to

$$\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2}\right)\left(\sum_{n=0}^{\infty} pod(n)q^n\right) = \left(\sum_{n=0}^{\infty} p(n)q^{4n}\right)\left(\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. □

THEOREM 4.12. *For any nonnegative integer n, the partition functions $p(n)$ and $pod(n)$ are related by*

$$\sum_{j=0}^{\infty} pod(j)p\left(\frac{n}{2} - \frac{j}{2}\right) = \sum_{j=0}^{\infty} p(j)p\left(\frac{n}{4} - \frac{j}{4}\right),$$

with $p(x) = 0$ if x is not an integer.

PROOF. We can write

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty} pod(n)q^n = \frac{1}{(q;q)_\infty(q^4;q^4)_\infty}$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} pod(n)q^n\right)\left(\sum_{n=0}^{\infty} p(n)q^{2n}\right) = \left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{n=0}^{\infty} p(n)q^{4n}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. □

4.8. *Connections with the cubic partition function.* In 2010, Chan [5] introduced the cubic partition function $a(n)$ which counts the number of partitions of n where the even parts can appear in two colors. For example, there are four cubic partitions of 3, namely

$$3, \quad 2_1 + 1, \quad 2_2 + 1, \quad 1 + 1 + 1,$$

where the subscripts 1 and 2 denote the colors. The generating function of $a(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

In this section, we shall prove that the partition function $pod(n)$ can be expressed in terms of the partition function $a(n)$.

THEOREM 4.13. *For any nonnegative integer n, the partition functions $a(n)$ and $pod(n)$ are related by*

$$pod(n) = \sum_{j=-\infty}^{\infty} (-1)^j a(n-2j^2). \tag{4.19}$$

PROOF. Replacing q by $-q^2$ and x by $-q^2$ in (2.1), we have

$$(q^2;q^4)_\infty (q^2;q^4)_\infty (q^4;q^4)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}.$$

Multiplying the above equation by $(q^4;q^4)_\infty/(q;q)_\infty$, to get

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2},$$

which after simplification becomes

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty (q^4;q^4)_\infty} = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2},$$

which is equivalent to

$$\sum_{n=0}^{\infty} pod(n)q^n = \left(\sum_{n=0}^{\infty} a(n)q^n\right)\left(\sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. □

Corollary 4.14. *For any nonnegative integer n, we have*

$$pod(n) \equiv a(n) \pmod 2.$$

Proof. Equation (4.19) can be rewritten as

$$pod(n) = a(n) + 2\sum_{j=1}^{\infty}(-1)^j a(n-2j^2),$$

which under modulo 2, gives the result. □

Theorem 4.15. *For any nonnegative integer n, the partition functions $a(n)$ and $pod(n)$ are related by triangular numbers as*

$$pod(n) = \sum_{j=0}^{\infty} a\left(\frac{n}{2} - \frac{j(j+1)}{4}\right),$$

with $a(x) = 0$ if x is not an integer.

Proof. The generation function of $pod(n)$ can be written as

$$\sum_{n=0}^{\infty} pod(n)q^n = \psi(q)\frac{1}{(q^2;q^2)_\infty (q^4;q^4)_\infty},$$

which is equivalent to

$$\sum_{n=0}^{\infty} pod(n)q^n = \left(\sum_{n=0}^{\infty} q^{T_n}\right)\left(\sum_{n=0}^{\infty} a(n)q^{2n}\right)$$

Finally, equating the coefficient of q^n on each side gives the result. □

4.9. *Connection with the partition function $q_{odd}^{'''}(n)$.* The number of partitions of n into distinct parts is usually denoted by $q(n)$. The number of partitions of n into distinct odd parts is denoted in this paper by $q_{odd}(n)$. The generating functions for $q(n)$ and $q_{odd}(n)$ are given by

$$\sum_{n=0}^{\infty} q(n)q^n = (-q;q)_\infty$$

and

$$\sum_{n=0}^{\infty} q_{odd}(n)q^n = (-q;q^2)_\infty.$$

We denote in this paper, $q_{odd}^{'''}(n)$ to be the number of partitions of n into distinct odd parts in 3 colors. For example, we have $q_{odd}^{'''}(4) = 9$, and the nine partitions are

3_1+1_1, 3_2+1_2, 3_3+1_3, 3_1+1_2, 3_1+1_3, 3_2+1_1, 3_3+1_1, 3_2+1_3, 3_3+1_2, where the subscripts 1, 2 and 3 denote the colors. It is clear that the generating function of $q_{odd}^{'''}(n)$ is given by

$$\sum_{n=0}^{\infty} q_{odd}^{'''}(n) = (-q;q^2)^3.$$

In this section, we shall prove that the partition function $q_{odd}^{'''}(n)$ can be expressed in terms of the partition function $pod(n)$.

THEOREM 4.16. *For any nonnegative integer n, the partition functions $q_{odd}^{'''}(n)$ and $pod(n)$ are related by*

$$q_{odd}^{'''}(n) = \sum_{j=-\infty}^{\infty} pod(n-j^2). \tag{4.20}$$

PROOF. Replacing x by 1 in (2.1), we have

$$(-q;q^2)_\infty^2 (q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Multiplying the above equation by $(-q;q^2)_\infty/(q^2;q^2)_\infty$, to get

$$(-q;q^2)_\infty^3 = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2},$$

which is equivalent to

$$\sum_{n=0}^{\infty} q_{odd}^{'''}(n) q^n = \left(\sum_{n=0}^{\infty} pod(n) q^n\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. □

COROLLARY 4.17. *For any nonnegative integer n, we have*

$$pod(n) \equiv q_{odd}^{'''}(n) \pmod 2$$

PROOF. Equation (4.20) can be rewritten as

$$q_{\mathrm{odd}}^{'''}(n) = pod(n) + 2\sum_{j=1}^{\infty} pod(n-j^2),$$

which under modulo 2, gives the result. □

4.10. *Relation between the partition functions $pod(n)$, $p(n)$, $a(n)$ and $q_{odd}(n)$.* In this section, we shall prove a relation between the partition functions $pod(n)$, $p(n)$, $a(n)$ and $q_{odd}(n)$.

THEOREM 4.18. *For any nonnegative integer n, the partition functions $pod(n)$, $p(n)$, $a(n)$ and $q_{odd}(n)$ are related by*

$$\sum_{j=0}^{\infty} pod(j)p(n-j) = \sum_{j=0}^{\infty} a(j)q_{odd}(n-j).$$

PROOF. We can write

$$\sum_{n=0}^{\infty} pod(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\frac{(q;q)_\infty(q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty},$$

which on simplification becomes

$$\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} pod(n)q^n = (-q;q^2)_\infty \sum_{n=0}^{\infty} a(n)q^n,$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{n=0}^{\infty} pod(n)q^n\right) = \left(\sum_{n=0}^{\infty} q_{odd}(n)q^n\right)\left(\sum_{n=0}^{\infty} a(n)q^n\right).$$

Finally, equating the coefficient of q^n on each side gives the result. ☐

4.11. *Relation between the partition functions $p(n), \overline{\mathcal{EO}}(n)$ and $pod(n)$.* Andrews [4] introduced the partition function $\mathcal{EO}(n)$ counts the number of partitions of n where every even part is less than each odd part. For example, $\mathcal{EO}(8) = 12$ with the relevant partitions being

8, 6+2, 7+1, 4+4, 4+2+2, 5+3, 5+1+1+1, 2+2+2+2, 3+3+2,

$3+3+1+1$, $3+1+1+1+1+1$, $1+1+1+1+1+1+1+1$.

The generating function for $\mathcal{EO}(n)$ is

$$\sum_{n=0}^{\infty} \mathcal{EO}(n)q^n = \frac{1}{(1-q)(q^2;q^2)_\infty}.$$

The partition function $\overline{\mathcal{EO}}(n)$ is the number of partitions counted by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. For example, $\overline{\mathcal{EO}}(8) = 5$, with the relevent partitions being

8, $4+2+2$, $3+3+2$, $3+3+1+1$, $1+1+1+1+1+1+1+1$.

The generating function of $\overline{\mathcal{EO}}(n)$ is

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^n = \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty^2} = \frac{(q^4;q^4)_\infty^3}{(q^2;q^2)_\infty^2}.$$

In this section, we will prove a relation between the partition functions $p(n), \overline{\mathcal{EO}}(n)$ and $pod(n)$.

THEOREM 4.19. *For any nonnegative integer n, the partition functions $pod(n)$, $p(n)$, and $\overline{\mathcal{EO}}(n)$ are related by*

$$\sum_{j=0}^{\infty} p(n-2T_j) = \sum_{j=0}^{\infty} pod(j)\overline{\mathcal{EO}}(n-j).$$

PROOF. We can write

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^n \frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty} = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}.$$

Multiplying the above equation by $1/(q;q)_\infty$, to get

$$\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^n \frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty(q;q)_\infty} = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty(q;q)_\infty},$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^n\right)\left(\sum_{n=0}^{\infty} pod(n)q^n\right) = \left(\sum_{n=0}^{\infty} p(n)q^n\right)\left(\sum_{n=0}^{\infty} q^{2T_n)}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. ∎

4.12. *Relation between the partition functions $pod(n)$ and $p_3(n)$.* In 2018, Hirschhorn [8] studied the number of partitions of n in three colors, $p_3(n)$, given by the generating function

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{(q;q)_\infty^3}.$$

In this section, we shall prove a relation between partition function $pod(n)$ and partitions in three colors $p_3(n)$.

THEOREM 4.20. *For any nonnegative integer n, the partition functions $pod(n)$ and $p_3(n)$ are related by*

$$\sum_{j=0}^{\infty} pod(j)pod(n-j) = \sum_{j=-\infty}^{\infty} p_3\left(\frac{n}{2}-\frac{j^2}{2}\right), \tag{4.21}$$

with $p_3(x)=0$ if x is not an integer.

PROOF. From (2.4), we have the relation

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2},$$

which can be written as

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \left(\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^4;q^4)_\infty}\right)^2 (q^2;q^2)_\infty^3.$$

Multiplying the above equation by $1/(q^2;q^2)_\infty^3$, to obtain

$$\frac{1}{(q^2;q^2)_\infty^3}\sum_{n=-\infty}^{\infty} q^{n^2} = \left(\sum_{n=0}^{\infty} pod(n)q^n\right)^2$$

which is equivalent to

$$\left(\sum_{n=0}^{\infty} pod(n)q^n\right)^2 = \left(\sum_{n=0}^{\infty} p_3(n)q^{2n}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right).$$

Finally, equating the coefficient of q^n on each side gives the result. □

Corollary 4.21. *For any nonnegative integer n, we have*

$$p_3\left(\frac{n}{2}\right) \equiv \sum_{j=0}^{\infty} pod(j)pod(n-j) \pmod 2,$$

with $p_3(x) = 0$ if x is not an integer.

Proof. Equation (4.21) can be rewritten as

$$\sum_{j=0}^{\infty} pod(j)pod(n-j) = p_3\left(\frac{n}{2}\right) + 2\sum_{j=1}^{\infty} p_3\left(\frac{n}{2}-\frac{j^2}{2}\right).$$

which under modulo 2, gives the result. □

Acknowledgements.

The author wants to thank Prof. N.D. Baruah for taking the time to check this paper. His input and suggestions were really helpful in making this paper better.

References

[1] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, AMS, Providence, 2006.
[2] C. Ballantine and A. Welch, Ped and pod partitions: combinatorial proofs of recurrence relations, *Discrete Math.*, **346** (2023), 113259.
[3] C. Ballantine and M. Merca, 4-regular partitions and the pod function, *Quaestiones Mathematicae*, (2023), 1–25.
[4] G. E. Andrews, Integer partitions with even parts below odd parts and the mock theta functions, *Ann. Comb.*, **22** (2018), 433–445.
[5] H. C. Chan, Ramanujan's cubic continued fraction and an analog of his "most beautiful identity", *Int. J. Number Theory*, **6** (2010), 673–680.
[6] J. Lovejoy and R. Osburn, Quadratic forms and four partition functions modulo 3, *Integers*, **11** (2011), A4.
[7] M. D. Hirschhorn, *The power of q, a personal journey*, Developments in Mathematics, 49. Springer, 2017.
[8] M. D. Hirschhorn, Partition in 3 colours, *Ramanujan J.* **45** (2018), 399–411.
[9] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of partitions with odd parts distinct, *Ramanujan J.* **22** (2010), 273–284.
[10] M. Merca, New relations for the number of partitions with distinct even parts, *J. Number Theory*, **176** (2017), 1–12.

[11] M. Merca, A further look at cubic partitions, *Ramanujan J.* **59** (2022), 253–277.

[12] N. D. Baruah, Divisibility of sums of partition numbers by multiples of 2 and 3, *Bull. Austral. Math. Soc.*, (2023), 1–9.

[13] S. Radu and J. A. Sellers, Congruence properties modulo 5 and 7 for the pod function, *Int. J. Number Theory*, **7** (2011), 2249–2259.

Naslov

Prvi autor, drugi autor i treći autor

Sažetak. Hrvatski prijevod sažetka.

Hemjyoti Nath
Department of Mathematical Sciences
Tezpur University
Napaam, Tezpur, Assam, India
E-mail: msm22017@tezu.ac.in