Integer and rational variants of a problem of Diophantus and Euler

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**Diophantus:** Find numbers such that the product of any two of them increased by the sum of these two gives a square.

\[
\{4, 9, 28\} \quad \text{and} \quad \left\{ \frac{3}{10}, \frac{7}{10}, \frac{21}{5} \right\}
\]

\[
4 \cdot 9 + 4 + 9 = 7^2, \quad 4 \cdot 28 + 4 + 28 = 12^2, \quad 9 \cdot 28 + 9 + 28 = 17^2
\]

**Euler:** \( \left\{ \frac{5}{2}, \frac{9}{56}, \frac{9}{224}, \frac{65}{224} \right\} \)

Such sets are called *Eulerian tuples*.

**Questions:** Is there any Eulerian

1. *quintuple* consisting of *rationals*?

2. *quintuple* consisting of *positive rationals*?

3. *quadruple* consisting of *integers*?

4. *quadruple* consisting of *positive integers*?
Answers:

1. **YES** (D. 1999)
   \[
   \left\{-\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40}\right\}
   \]

2. **YES** (D. 2002)
   
   based on the fact that there are infinitely many rational points on the curve
   \[y^2 = -(x^2 - x - 3)(x^2 + 2x - 12)\].

4. **NO** (D. & C. Fuchs 2005)

   - connection with *Diophantine* \(m\)-tuples: If \(\{a_1, \ldots, a_m\}\) is an Eulerian \(m\)-tuple, then \(\{a_1 + 1, \ldots, a_m + 1\}\) is a \(D(-1)\)-\(m\)-tuple, i.e. \((a_i + 1)(a_j + 1) - 1 = a_ia_j + a_i + a_j\) is a perfect square.
3. - There is no Eulerian quintuple consisting of integers [D. & C. Fuchs (2005)].

- If there is an Eulerian quadruple consisting of integers, then it necessarily contains 0 or $-2$ [D. & C. Fuchs (2005)].

- There exist at most finitely many Eulerian quadruples consisting of integers. If \{a, b, c, d\} is an Eulerian quadruple, then $\max\{|a|, |b|, |c|, |d|\} < 10^{10^{23}}$ [D. & A. Filipin & C. Fuchs (2007)].
Construction of an infinite family of Eulerian quintuples consisting of positive rationals

Equivalent problem: Find rational $D(-1)$-quintuples with elements $> 1$.

Idea: Interpret the Eulerian quintuple
$$\left\{ -\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40} \right\}$$
as a point on an elliptic curve.

This Eulerian quintuple corresponds to the $D(-1600)$-quintuple
$$\{13, 125, 148, 400, 533\}.$$  \hspace{1cm} (1)

Simple fact: If $B \cdot C + n = k^2$, then $\{B, C, B + C \pm 2k\}$ are $D(n)$-triples.
Quintuple (1) has the form

\[ \{A, B, C, D, z^2\}, \quad (2) \]

where \( A = B + C - 2k, \) \( D = B + C + 2k. \) If \( A = a^2 - \alpha, \) \( B = b^2 - \alpha, \) \( C = c^2 - \alpha, \) \( D = d^2 - \alpha, \) then (2) will be a \( D(\alpha z^2) \)-quintuples iff

\[
(b^2 - \alpha)(c^2 - \alpha) + \alpha x^2 = k^2, \\
(a^2 - \alpha)(d^2 - \alpha) + \alpha x^2 = y^2.
\]

Parametric solution: the set

\[
\left\{ \frac{1}{3}(x^2 + 6x - 18)(-x^2 + 2x + 2), \right. \\
\frac{1}{3}x^2(x + 5)(-x + 3), (x - 2)(5x + 6), \\
\left. \frac{1}{3}(x^2 + 4x - 6)(-x^2 + 4x + 6), 4x^2 \right\}
\]

is a \( D\left(\frac{16}{9}x^2(x^2-x-3)(x^2+2x-12)\right) \)-quintuple.

\( x = \frac{5}{2} \quad \longrightarrow \quad (1). \)
Consider the quartic curve

\[ Q : \quad y^2 = -(x^2 - x - 3)(x^2 + 2x - 12), \]

with a rational point \((\frac{5}{2}, \frac{3}{4})\).

By substitutions

\[
\begin{align*}
    x &= \frac{63s + 10t + 2619}{18s + 4t + 2403}, \\
    y &= \frac{24s^3 - 6777s^2 - 12t^2 - 34749t + 54898479}{(18s + 4t + 2403)^2},
\end{align*}
\]

\(Q\) is birationally equivalent to the elliptic curve

\[ E : \quad t^2 = s^3 - 18981s - 1001700 \]
\[ = (s - 159)(s + 75)(s + 84). \]

\[ E(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \]
\[ E(\mathbb{Q})_{\text{tors}} = \{O, T_1, T_2, T_3\}, \]
\[ T_1 = (159, 0), \quad T_2 = (-75, 0), \quad T_3 = (-84, 0) \}
\[ \text{rank } E(\mathbb{Q}) = 1, \]
\[ E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}} = \langle P \rangle, \quad P = (2103, -96228). \]
5 additional conditions:

\[
\frac{(x^2+6x-18)(-x^2+2x+2)}{4xy} > 1, \\
x(x+5)(-x+3) > 1, \quad \frac{3(x-2)(5x+6)}{4xy} > 1, \\
\frac{(x^2+4x-6)(-x^2+4x+6)}{4xy} > 1, \quad \frac{3x}{y} > 1
\]

Solutions:

\[
x \in \langle 2.303, 2.306 \rangle \cup \langle 2.602, 2.605 \rangle, \quad y > 0,
\]

\[
x \in \langle -4.605, -4.482 \rangle \cup \langle -1.338, -1.303 \rangle, \quad y < 0.
\]

In terms of elliptic curve \(E\):

\[
x \in \langle -79.22, -76.85 \rangle \cup \langle 458.64, 937.54 \rangle, \quad t > 0,
\]

\[
x \in \langle -82.09, -79.69 \rangle \cup \langle 232.03, 348.77 \rangle, \quad t < 0.
\]
Which points of the form $mP$, $T_1 + mP$, $T_2 + mP$, $T_3 + mP$ satisfy these inequalities?

parametrization by Weierstrass function $\wp$:
$s = \wp(z)$, $t = \frac{1}{2}\wp'(z)$

For points $mP$ the condition becomes:

$$\{m \cdot 0.2145\ldots\} \in \langle 0.5362, 0.6782 \rangle.$$  

$\alpha$ irrational $\Rightarrow$ fractional parts $\{m \cdot \alpha\}$ are dense in $\langle 0, 1 \rangle$

$\Rightarrow$ infinitely many rational Eulerian quintuples with positive elements

E.g. $m = -2, -3, -11, 12, -16, 17, -25, 26, \ldots$
<table>
<thead>
<tr>
<th>point on $E$</th>
<th>Eulerian quintuple</th>
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| $-2P$       | \[
\left\{ \frac{12253738824071768160902809331272805381}{13356284738726537361337339615814680856}, \right. \\
\left. \frac{40228062558134597846809398333}{2027377666049252712575626072}, \right. \\
\left. \frac{90410203607675775632231738735}{2640165528414654368852526998}, \right. \\
\left. \frac{1459249660815833141719920182753327588589}{13356284738726537361337339615814680856}, \right. \\
\left. \frac{16463478877068761615}{200378051669604563} \right\} 
\]
| $T_3 - 2P$  | \[
\left\{ \frac{24384004810826647895250908584025016017}{1226018751971657626989240363062470220}, \right. \\
\left. \frac{11174534572531880776077845373}{1225575724730803312553801852}, \right. \\
\left. \frac{200408761263308135110463918}{200450485329612350005456055}, \right. \\
\left. \frac{2876707800134532926186517692138532777}{1226018751971657626989240363062470220}, \right. \\
\left. \frac{1329253988561517422}{200378051669604563} \right\} 
\]
Theorem: (D. & Fuchs (2005)) There does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ with $2 \leq a < b < c < d$.

Corollaries:

- There does not exist an Eulerian quadruple consisting of positive integers.
- There does not exist a $D(-1)$-quintuple.
- If $\{a, b, c, d\}$ is a $D(-1)$-quadruple with $0 < a < b < c < d$, then $a = 1$ and $b \geq 5$.
- If $Q$ is an Eulerian quadruple consisting of integers, then $0 \in Q$ if $-2 \in Q$. 
Previous results - the following $D(-1)$-triples cannot be extended to $D(-1)$-quadruples:

- Mohanty & Ramasamy (1984): $\{1, 5, 10\}$

- Brown (1985): $\{1, 2, 5\}$, $\{17, 26, 37\}$

- Kedlaya (1998): $\{1, 2, 145\}$, $\{1, 2, 4901\}$, $\{1, 5, 65\}$, $\{1, 5, 20737\}$, $\{1, 10, 17\}$, $\{1, 26, 37\}$

- Dujella (1998): $\{1, 2, c\}$

- Filipin (2005): $\{1, 5, c\}$, $\{1, 10, c\}$

- Fujita (2006): $\{1, 17, c\}$, $\{1, 26, c\}$, $\{1, 37, c\}$, $\{1, 50, c\}$
Theorem: (D. & Filipin & Fuchs (2007)) Let \( \{1, b, c\} \) be a \( D(-1) \)-triple. Then:

(i) If \( c > b^9 \), then there does not exist an extension to a \( D(-1) \)-quadruple \( \{1, b, c, d\} \) such that \( d > c \).

(ii) If \( 11b^6 \leq c \leq b^9 \), then there does not exist an extension to a \( D(-1) \)-quadruple.

Assume that \( \{1, b, c, d\} \) with \( 1 < b < c < d \) is an extension to a \( D(-1) \)-quadruple.

(iii) If \( b^3 < c < 11b^6 \), then \( c < 10^{238} \), \( d < 10^{1023} \),

(iv) if \( b^{1.1} < c \leq b^3 \), then \( c < 10^{491} \), \( d < 10^{1023} \),

(v) if \( 3b < c \leq b^{1.1} \), then \( c < 10^{94} \), \( d < 10^{1021} \),

(vi) if \( c = 1 + b + 2\sqrt{b - 1} \), then \( c < 10^{74} \), \( d < 10^{1021} \).
Corollaries:

- There are only finitely many $D(-1)$-quadruples.

- There are only finitely many Eulerian quadruples consisting of integers.

- If $\{a, b, c, d\}$ is a $D(-1)$-quadruple, then $\max\{a, b, c, d\} < 10^{1023}$.

- The number of $D(-1)$-quadruples is bounded by $10^{903}$.

These are first nontrivial results (i.e. for integers $\not\equiv 2 \pmod{4}$) related to the following conjecture:

**Conjecture:** If $n$ is not a perfect square, then there exist only finitely many $D(n)$-quadruples.

Since all elements of a $D(-4)$-quadruple are even, our result implies that the conjecture is valid for $n = -1$ and $n = -4$. 
Let \( \{1, b, c\} \), where \( 1 < b < c \), be a \( D(-1) \)-triple and let \( r, s, t \) be positive integers defined by

\[
\begin{align*}
b - 1 &= r^2, \\
c - 1 &= s^2, \\
bc - 1 &= t^2.
\end{align*}
\]

Assume that there exists a positive integer \( d > c \) such that \( \{1, b, c, d\} \) is a \( D(-1) \)-quadruple. We have

\[
\begin{align*}
d - 1 &= x^2, \\
bd - 1 &= y^2, \\
\cd - 1 &= z^2,
\end{align*}
\]

with integers \( x, y, z \). Eliminating \( d \), we obtain the following system of Pellian equations

\[
\begin{align*}
z^2 - cx^2 &= c - 1, \\
bfz^2 - cy^2 &= c - b.
\end{align*}
\]
The system of Pellian equations can be transformed to finitely many equations of the form \( z = v_m = w_n \), where the sequences \((v_m)\) and \((w_n)\) are given by

\[
\begin{align*}
v_0 &= z_0, \\
v_1 &= (2c - 1)z_0 + 2scx_0, \\
v_{m+2} &= (4c - 2)v_{m+1} - v_m, \\
w_0 &= z_1, \\
w_1 &= (2bc - 1)z_1 + 2tcy_1, \\
w_{m+2} &= (4bc - 2)w_{n+1} - w_n,
\end{align*}
\]

and fundamental solutions satisfy the following inequalities:

\[
|x_0| < s, \ 0 < z_0 < c, \ |y_1| < t, \ 0 < z_1 < c.
\]

Remark: If \( c \leq b^9 \), then \( z_0 = z_1 = s, \ x_0 = 0, \ y_1 = \pm r \).
Congruence relations:

\[
\begin{align*}
  v_m &\equiv (-1)^m z_0 \pmod{2c}, \\
  w_n &\equiv (-1)^n z_1 \pmod{2c}, \\
  v_m &\equiv (-1)^m (z_0 - 2acm^2 z_0 - 2csmx_0) \pmod{8c^2}, \\
  w_n &\equiv (-1)^n (z_1 - 2bcn^2 z_1 - 2ctny_1) \pmod{8c^2}.
\end{align*}
\]

congruence relations \(\Rightarrow\) lower bounds for non-trivial solutions

E.g.

If \(v_m = w_n\), \(n \neq 0, 1\) and \(c \geq 11b^6\), then \(n > \frac{1}{c^6}\).

If \(v_m = w_n\), \(n \neq 0, 1\), \(b^{1.1} \leq c < b^3\) and \(c > 10^{100}\), then \(n \geq c^{0.04}\).
Solutions of our system of Pellian equations induce very good rational approximations to the numbers $\theta_1 = \sqrt{1 + \frac{1-b}{N}}$ and $\theta_2 = \sqrt{1 + \frac{1}{N}}$:

$$\max \left\{ \left| \frac{\theta_1 - \frac{bsx}{ty}}{ \frac{bsx}{ty}} \right|, \left| \frac{\theta_2 - \frac{bz}{ty}}{ \frac{bz}{ty}} \right| \right\} < \frac{b - 1}{y^2}.$$ 

If $c \geq 11b^6$, then we can apply Bennett’s theorem (a modification due to Fujita) on simultaneous rational approximations of square roots which are close to 1.

For $c < 11b^6$, we transform the exponential equation $v_m = w_n$ into a logarithmic inequality and apply Baker’s theory of linear forms in logarithms of algebraic numbers (Matveev’s theorem).

Diophantine approximations $\Rightarrow$ upper bounds for solutions

lower and upper bounds for solutions $\Rightarrow$ contradiction (for sufficiently large $c$)