High rank elliptic curves with prescribed torsion

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Let T be an admissible torsion group for an elliptic curve over the rationals. Define

$$B(T) = \sup \{\text{rank}(E(\mathbf{Q})) : \text{torsion group of elliptic curve } E \text{ over } \mathbf{Q} \text{ is } T\}.$$

The conjecture is that $B(T)$ is unbounded for all T. In the following table we give the best known lower bounds for $B(T)$.

T	$B(T)>=$	Author(s)
0	28	Elkies (2006)
Z/2**Z**	19	Elkies (2009)
Z/3**Z**	13	Eroshkin (2007,2008,2009)
Z/4**Z**	12	Elkies (2006), Dujella - Peral (2014)
Z/5**Z**	8	Dujella - Lecacheux (2009), Eroshkin (2009)
Z/6**Z**	8	Eroshkin (2008), Dujella - Eroshkin (2008), Elkies (2008), Dujella (2008), Dujella - Peral (2012), Dujella - Peral - Tadic (2014,2015)
Z/7**Z**	5	Dujella - Kulesz (2001), Elkies (2006), Eroshkin (2009,2011), Dujella - Lecacheux (2009), Dujella - Eroshkin (2009)
Z/8**Z**	6	Elkies (2006), Dujella - MacLeod - Peral (2013)
Z/9**Z**	4	Fisher (2009), van Beek (2015)
Z/10**Z**	4	Dujella (2005,2008), Elkies (2006), Fisher (2016)
Z/12**Z**	4	Fisher (2008)
Z/2**Z** × **Z**/2**Z**	15	Elkies (2009)
Z/2**Z** × **Z**/4**Z**	9	Dujella - Peral (2012)
Z/2**Z** × **Z**/6**Z**	6	Elkies (2006), Dujella - Peral - Tadic (2015)
Z/2**Z** × **Z**/8**Z**	3	Connell (2000), Dujella (2000,2001,2006,2008), Campbell - Goins (2003), Rathbun (2003,2006,2013), Dujella - Rathbun (2006), Flores - Jones - Rollick - Weigandt - Rathbun (2007), Fisher (2009)

Click on rank r to see the corresponding "record" curve(s) with torsion points and independent points $P_1, P_2, \ldots, P_r$ of infinite order.

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Infinite families of elliptic curves with high rank and prescribed torsion

History of elliptic curves rank records

High rank elliptic curves with prescribed torsion over quadratic fields

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