Free field realization and weight representations of the twisted Heisenberg-Virasoro algebra

Croatian Science Fundation grant 2634

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Jun 2015

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- D. Adamović, G. R. Free fields realization of the twisted Heisenberg-Virasoro algebra at level zero and its applications, JPAA (2015)

The twisted Heisenberg-Virasoro algebra

$\mathcal{H}$ is a complex Lie algebra with a basis $\{L(n), I(n), C_L, C_I, C_{L,I} : n \in \mathbb{Z}\}$ and a Lie bracket

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m}\frac{n^3-n}{12}C_L,$$
$$[L(n), I(m)] = -mI(n+m) - \delta_{n,-m}(n^2+n)C_{LI},$$
$$[I(n), I(m)] = n\delta_{n,-m}C_I,$$
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$\{I(n), C_I : n \in \mathbb{Z}\}$ spans a copy of the Heisenberg algebra.

The Verma module

- $V(c_L, c_I, c_{L,I}, h, h_I)$ - the **Verma module** with highest weight (h, h_I) and central charge $(c_L, c_I, c_{L,I})$.

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- Appears in the representation theory of toroidal Lie algebras.

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Theorem (Y. Billig)

Assume that $c_I = 0$ and $c_{LI} \neq 0$.

(i) If $\frac{h_I}{c_{LI}} - 1 \notin \mathbb{Z}^*$, then $V(c_L, 0, c_{LI}, h, h_I)$ is irreducible.

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(i) *If* $\frac{h_I}{c_{LI}} - 1 \notin \mathbb{Z}^*$, *then* $V(c_L, 0, c_{LI}, h, h_I)$ *is irreducible.*

(ii) *If* $\frac{h_I}{c_{LI}} - 1 \in \mathbb{Z}^*$, *then* $V(c_L, 0, c_{LI}, h, h_I)$ *has a singular vector* u *at level* $p = |\frac{h_I}{c_{LI}} - 1|$.

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The quotient module

$$L(c_L, 0, c_{L,I}, h, h_I) = V(c_L, 0, c_{L,I}, h, h_I) / U(\mathcal{H})u$$

is irreducible.

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$$\begin{aligned} L(n)\, v_m &= -(m+\alpha+\beta+n\beta)\, v_{m+n}, \\ I(n)\, v_m &= F v_{m+n}, \\ C_L v_m &= C_I v_m = C_{L,I} v_m = 0. \end{aligned}$$

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- $V_{\alpha,\beta,F}$ is reducible if and only if $\alpha \in \mathbb{Z}$, $\beta \in \{0,1\}$ and $F = 0$,
- $V'_{0,0,0} := V/\mathbb{C}v_0$, $V'_{0,1,0} := \bigoplus_{n\neq -1} \mathbb{C}v_n$ and $V'_{\alpha,\beta,F} := V_{\alpha,\beta,F}$ otherwise.

Irreducible Harish-Chandra modules

Theorem (Lu, R., Zhao, K.)

An irreducible weight $\mathcal{H}$–module with finite-dimensional weight spaces is isomorphic either to a highest (or lowest) weight module, or to $V'_{\alpha,\beta,F}$ for some $\alpha, \beta, F \in \mathbb{C}$.

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What about modules with infinite-dimensional weight spaces?

Tensor product modules

Consider $V'_{\alpha,\beta,F} \otimes L\left(c_L, 0, c_{L,I}, h, h_I\right)$ module:

$$\begin{aligned} L(n)(v_k \otimes x) &= L(n)v_k \otimes x + v_k \otimes L(n)x, \\ I(m)(v_k \otimes x) &= Fv_k \otimes x + v_k \otimes I(m)x, \\ C_L(v_k \otimes x) &= c_L(v_k \otimes x), \\ C_I(v_k \otimes x) &= 0 \\ C_{L,I}(v_k \otimes x) &= c_{L,I}(v_k \otimes x). \end{aligned}$$

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To classify irreducible modules $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ we need more detailed formulas for singular vectors.

The Heisenberg-Virasoro vertex-algebra

Irreducible $\mathcal{H}$–module $L(c_L, 0, c_{L,I}, 0, 0)$ has the structure of vertex operator algebra which we denote by $L^{\mathcal{H}}(c_L, c_{L,I})$.

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Let $c_{L,I} \neq 0$. Then $L^{\mathcal{H}}(c_L, c_{L,I})$ is a simpe VOA, and $V(c_L, 0, c_{L,I}, h, h_I)$ and $L(c_L, 0, c_{L,I}, h, h_I)$ are $L^{\mathcal{H}}(c_L, c_{L,I})$–modules.

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- $M(1)$–modules $M(1, \gamma)$ become $L^{\mathcal{H}}(c_L, c_{L,I})$–modules, and also $\mathcal{H}$–modules.
- Construction of a screening operator will give us realization of certain irr weight modules.

Heisenberg vertex-algebra

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- Denote by e^{γ} a highest weight vector in $M(1,\gamma)$.
- $M(1) := M(1,0)$ is a vertex-algebra:

$$\begin{aligned} h(n) &= t^n \otimes h, \qquad \text{for } h \in \mathfrak{h}, \\ h(z) &= \sum_{n\in\mathbb{Z}} h(n)\, z^{-n-1} \end{aligned}$$

  and $M(1,\gamma)$ for $\gamma \in \mathfrak{h}$, are irreducible $M(1)$–modules.

Realization of the Heisenberg-Virasoro vertex algebra

- Define a Heisenberg vector

$$I = \alpha(-1) + \beta(-1) \in M(1)$$

and a Virasoro vector

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- Then

$$\begin{aligned} I(z) &= Y(I,z) = \sum_{n\in\mathbb{Z}} I(n)\, z^{-n-1} \quad \text{and} \\ L(z) &= Y(\omega,z) = \sum_{n\in\mathbb{Z}} L(n)\, z^{-n-2} \end{aligned}$$

generate the Heisenberg-Virasoro vertex algebra $L^{\mathcal{H}}(c_L, c_{L,I})$ in $M(1)$.

Realization of the twisted Heisenberg-Virasoro algebra

We get the twisted Heisenberg-Virasoro Lie algebra $\mathcal{H}$ such that

$$c_L = 2 - 12(\lambda^2 - \mu^2), \quad c_{L,I} = \lambda - \mu$$

i.e.

$$\lambda = \frac{2 - c_L}{24 c_{L,I}} + \frac{1}{2} c_{L,I}, \quad \mu = \frac{2 - c_L}{24 c_{L,I}} - \frac{1}{2} c_{L,I}.$$

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Now we may use representation theory of $M(1)$ in rep theory of $\mathcal{H}$!

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- For every $r, s \in \mathbb{C}$, $e^{r\alpha+s\beta}$ is a $\mathcal{H}$-singular vector and $U(\mathcal{H})e^{r\alpha+s\beta}$ is a highest weight module with the highest weight

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Proposition

(i) *Let* $(h, h_I) \in \mathbb{C}^2$, $h_I \neq c_{L,I}$. *Then there exist unique* $r, s \in \mathbb{C}$ *such that* $e^{r\alpha+s\beta}$ *is a highest weight vector of the highest weight* (h, h_I).

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(ii) *For every* $r, s \in \mathbb{C}$ *such that* $r - s = \lambda - \mu = c_{L,I}$, $e^{r\alpha+s\beta}$ *is a highest weight vector of weight*

$$(h, h_I) = (\frac{c_L - 2}{24}, c_{L,I}).$$

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- There is a surjective $\mathcal{H}$–homomorphism

$$\Phi : V(c_L, 0, c_{L,I}, h, h_I) \to U(\mathcal{H})e^{r\alpha+s\beta}$$

such that $\Phi(v_{h,h_I}) = e^{r\alpha+s\beta}$.

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Proposition

Assume that $\frac{h_I}{c_{L,I}} - 1 \notin -\mathbb{Z}_{>0}$. *Then* $\mathcal{F}_{r,s} \cong V(c_L, 0, c_{L,I}, h, h_I)$ *as* $L^{\mathcal{H}}(c_L, c_{L,I})$*–modules.*

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- Screening operator Q commutes with $L(n)$ and $I(n)$.

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Assume that $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$. *As a* $L^{\mathcal{H}}(c_L, c_{L,I})$*–module* $\mathcal{F}_{r,s}$ *is generated by* $e^{r\alpha+s\beta}$ *and a family of subsingular vectors* $\{v_{n,p} : n \geq 1\}$ *such that*

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In particular $L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I) = \ker_{\mathcal{F}_{r,s}} Q$.

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In particular $L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I) = \ker_{\mathcal{F}_{r,s}} Q$.

- For $r = s = -\frac{1}{c_{L,I}}$ we get $L^{\mathcal{H}}(c_L, c_{L,I}) = \ker_{M(1)} Q$.

Schur polynomials

- Schur polynomials $S_r(x_1, x_2, \cdots)$ in variables $x_1, x_2, \ldots$ are defined by the following equation:

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$$S_r(x_1, x_2, \cdots) = \frac{1}{r!} \begin{vmatrix} x_1 & x_2 & \cdots & & x_r \\ -r+1 & x_1 & x_2 & \cdots & x_{r-1} \\ 0 & -r+2 & x_1 & \cdots & x_{r-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & x_1 \end{vmatrix}$$

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- Schur polynomials naturally appear in formulas for vertex operator for lattice vertex algebras.

Schur polynomials and singular vectors

Theorem

Assume that $c_{L,I} \neq 0$ *and* $p = \frac{h_I}{c_{L,I}} - 1 \in \mathbb{Z}_{>0}$. *Then* $\Omega v_{h,h_I}$ *where*

$$\Omega = S_p\left(-\frac{I(-1)}{c_{L,I}}, -\frac{I(-2)}{c_{L,I}}, \ldots, -\frac{I(-p)}{c_{L,I}}\right)$$

is a singular vector of weight p *in the Verma module* $V(c_L, 0, c_{L,I}, h, (1+p)\, c_{L,I})$.

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Assume that $c_{L,I} \neq 0$ *and* $p = 1 - \frac{h_I}{c_{L,I}} \in \mathbb{Z}_{>0}$. *Then* $\Lambda v_{h,h_I}$ *where*

$$\Lambda = \sum_{i=0}^{p-1} S_i\left(\frac{I(-1)}{c_{L,I}}, \ldots, \frac{I(-i)}{c_{L,I}}\right) L_{i-p} +$$

$$\sum_{i=0}^{p-1}\left(\frac{h}{p} + \frac{c_L - 2}{24}\frac{(p-1)^2 - pi}{p}\right) S_i\left(\frac{I(-1)}{c_{L,I}}, \ldots, \frac{I(-i)}{c_{L,I}}\right)\frac{I(i-p)}{c_{L,I}}$$

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Intertwining operators and tensor product modules

As with Virasoro algebra, the existence of a nontrivial intertwining operator of the type

$$\binom{L(c_L,0,c_{L,I},h'',h_I'')}{L(c_L,0,c_{L,I},h,h_I)\qquad L(c_L,0,c_{L,I},h',h_I')}$$

yields a nontrivial $\mathcal{H}$-homomorphism

$$\varphi : V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h',h_I') \rightarrow L(c_L,0,c_{L,I},h'',h_I'')$$

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By dimension argument, we get reducibility of $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h',h_I')$.

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From the standard fusion rules result for $M(1)$ we get intertwining operators in the category of $\mathcal{H}$–modules:

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Corollary

Let $(h, h_I) = (\Delta_{r_1,s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2,s_2}, r_2 - s_2) \in \mathbb{C}^2$ *and that there are* $p, q \in \mathbb{Z}_{>0}$, $q \leq p$ *such that*

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where $h'' = \Delta_{r_2 - r_1, s_2 - s_1}$.

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In particular, the $\mathcal{H}$*–module* $V'_{\alpha,\beta,F} \otimes L^{\mathcal{H}}(c_L, 0, c_{L,I}, h', h''_I)$ *is reducible where*

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- Next we use formulas for Ω and Λ to get irreducibility criterion for $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$.

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$$\phi_n(1) = 1$$
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- $V'_{\alpha,\beta,F} \otimes L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ is irreducible if and only if $\phi_n(\Omega) \neq 0$ (i.e. $\phi_n(\Lambda) \neq 0$) for every $n \in \mathbb{Z}$.

Irreducibility criterion

- If $p = \frac{h_I}{c_{L,I}} - 1 \in \mathbb{Z}_{>0}$, then for every $n \in \mathbb{Z}$ we have

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- This expands the list of reducible tensor products realized with intertwining operators.

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- This, along with previous results on existence of intertwining operators result with the following:

Irreducibiliy criterion

Theorem

Let $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$. *We write* V *short for* $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$.

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(i) Let $F/c_{L,I} \notin \{1, \ldots, p-1\}$ and let $\alpha_0 \in \mathbb{C}$ be such that $\phi_0(\Lambda) = 0$. Then V is reducible if and only if $\alpha \equiv \alpha_0 \mod \mathbb{Z}$.

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(ii) Let $F/c_{L,I} \in \{2, \dots, p-1\}$. Then V is reducible.
(iii) Let $p > 1$ and $F/c_{L,I} = 1$. Then V is reducible if and only if $1 - \beta = \frac{c_L - 2}{24}$.

Fusion rules

Theorem

Let $(h, h_I) = (\Delta_{r_1,s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2,s_2}, r_2 - s_2)$ *such that*

$$\frac{h_I}{c_{L,I}} - 1 = p, \ \frac{h'_I}{c_{L,I}} - 1 = q, \quad p, q \in \mathbb{Z} \setminus \{0\}.$$

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Then $d = 1$ *if and only if* $h''_I = h_I + h'_I$ *and one of the following holds:*

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(ii) $1 \leq -p \leq q$ and $h'' = \Delta_{r_2-r_1, s_2-s_1}$

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$$d = \dim I \binom{L^{\mathcal{H}}(c_L, 0, c_{L,I}, h'', h''_I)}{L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I) \quad L^{\mathcal{H}}(c_L, 0, c_{L,I}, h', h'_I)}.$$

Then $d = 1$ *if and only if* $h''_I = h_I + h'_I$ *and one of the following holds:*

(i) $p, q < 0$ and $h'' = \Delta_{r_1+r_2, s_1+s_2}$

(ii) $1 \leq -p \leq q$ and $h'' = \Delta_{r_2-r_1, s_2-s_1}$

(iii) $1 \leq -q \leq p$ and $h'' = \Delta_{r_1-r_2, s_1-s_2}$

Fusion rules

Theorem

Let $(h, h_I) = (\Delta_{r_1,s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2,s_2}, r_2 - s_2)$ *such that*

$$\frac{h_I}{c_{L,I}} - 1 = p, \ \frac{h'_I}{c_{L,I}} - 1 = q, \quad p, q \in \mathbb{Z} \setminus \{0\}.$$

Let

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$d = 0$ otherwise.

Nontrivial intertwining operators

$$\begin{pmatrix} & (\Delta_{r_1+r_2,s_1+s_2}, (1-(p+q-1))c_{L,l}) & \\ (\Delta_{r_1,s_1}, (1-p)c_{L,l}) & & (\Delta_{r_2,s_2}, (1-q)c_{L,l}) \end{pmatrix}$$

for $p, q \geq 1$

$$\begin{pmatrix} & (\Delta_{r_2-r_1,s_2-s_1}, (1+(q-p+1))c_{L,l}) & \\ (\Delta_{r_1,s_1}, (1-p)c_{L,l}) & & (\Delta_{r_2,s_2}, (1+q)c_{L,l}) \end{pmatrix}$$

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Vertex-algebra homomorphism

- Vertex-algebra $L^W(c_L, c_W)$ is generated by fields

$$
\begin{aligned}
Y(L(-2), z) &= \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \\
Y(W(-2), z) &= \sum_{n \in \mathbb{Z}} W(n) z^{-n-2}.
\end{aligned}
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- Vertex-algebra $L^{\mathcal{H}}(c_L, c_{L,I})$ is generated by fields

$$\begin{aligned} Y(L(-2), z) &= \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}, \\ Y(I(-1), z) &= \sum_{n\in\mathbb{Z}} I(n) z^{-n-1}. \end{aligned}$$

Vertex-algebra homomorphism

Theorem

There is a non-trivial homomorphism of vertex algebras

$$\begin{aligned}\Psi : L^{W}(c_L, c_W) &\longrightarrow L^{\mathcal{H}}(c_L, c_{L,I})\\ L(-2) &\mapsto L(-2)\mathbf{1}\\ W(-2) &\mapsto (I^2(-1) + 2c_{L,I}I(-2))\mathbf{1}\end{aligned}$$

where

$$c_W = -24c_{L,I}^2.$$

Vertex-algebra homomorphism

- Every $L^{\mathcal{H}}(c_L, c_{L,I})$–module becomes a $L^{W}(c_L, c_W)$–module.

Vertex-algebra homomorphism

- Every $L^{\mathcal{H}}(c_L, c_{L,I})$–module becomes a $L^W(c_L, c_W)$–module.
- $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ is a $L^W(c_L, c_W)$–module and v_{h,h_I} is a $W(2,2)$ highest weight vector such that

$$L(0)v_{h,h_I} = hv_{h,h_I}, \quad W(0)v_{h,h_I} = h_W v_{h,h_I}$$

where $h_W = h_I(h_I - 2c_{L,I})$.

Vertex-algebra homomorphism

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  where $h_W = h_I(h_I - 2c_{L,I})$.
- There is a nontrivial $W(2,2)$-homomorphism

$$\Psi : V^{W(2,2)}(c, c_W, h, h_W) \to V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$$

Highest weight H–modules as W(2,2)–modules

Example

Let $h_W = \frac{1-p^2}{24} c_W = (p^2-1)c_{L,I}^2 = h_I(h_I - 2c_{L,I})$ as above. Then there are nontrivial $W(2,2)$-homomorphisms

$$\begin{array}{ccc} & V^{W(2,2)}(c, c_W, h, \frac{1-p^2}{24}c_W) & \\ \Psi_+ \swarrow & & \searrow \Psi_- \\ V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, (1+p)\, c_{L,I}) & & V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, (1-p)\, c_{L,I}) \end{array}$$

Highest weight H–modules as W(2,2)–modules

Theorem

(i) *Let* $\frac{h_I}{c_{L,I}} - 1 \notin -\mathbb{Z}_{>0}$. *Then* Ψ *is an isomorphism of* $W(2,2)$*–modules.*

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(ii) *If* $\frac{h_I}{c_{L,I}} - 1 = p \in \mathbb{Z}_{>0}$ *then*

$$\Psi^{-1}\left(S_p\left(-\frac{I(-1)}{c_{L,I}}, -\frac{I(-2)}{c_{L,I}}, \cdots\right) v_{h,h_I}\right) = u'$$

is a singular vector in $V^{W(2,2)}(c_L, c_W, h, h_W)_{h+p}$.

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(iii) If $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ then $\Psi(u') = 0$.

Highest weight H–modules as W(2,2)–modules

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(i) *Let* $\frac{h_I}{c_{L,I}} - 1 \notin -\mathbb{Z}_{>0}$. *Then* Ψ *is an isomorphism of* $W(2,2)$*–modules.*

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is a singular vector in $V^{W(2,2)}(c_L, c_W, h, h_W)_{h+p}$.

(iii) *If* $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ *then* $\Psi(u') = 0$.

(iv) *Let* $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ *and let* u *be a subsingular vector in* $V^{W(2,2)}(c_L, c_W, h_{pq}, h_W)_{h+pq}$. *Then* $\Psi(u)$ *is a singular vector in* $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, (1-p)c_{L,I})$.