Free field realization and weight representations of the twisted Heisenberg-Virasoro algebra Croatian Science Fundation grant 2634

Gordan Radobolja

Faculty of Science, University of Split, Croatia

Jun 2015

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

iliterillediate serie

Tensor product:

realization

Heisenberg-Virasoro VOA

Heisenberg VOA Realization of HV Singular vectors

Eusion rules au ensor product modules

Irreducibility of a tensor product

Using Ω Using Λ More fusion rule

Free-field realization

4 D N 4 D N 4 D N 1 D N

► The twisted Heisenberg-Virasoro Lie algebra  $\mathcal{H}$ , weight representations.

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate series

Tensor product

realization
Heisenberg-Virasoro

Heisenberg VOA
Realization of HV
Singular vectors

usion rules and ensor product

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

4 D L 4 D L 4 E L 4 E L 5 00 Q

- ► The twisted Heisenberg-Virasoro Lie algebra H, weight representations.
- ▶ Irreducibility problem of  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ .

#### Introduction

The twisted Heisenberg-Virasoro algebra

modules

.....

ensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA

Realization of HV

usion rules an insor product odules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

4 D L 4 D L 4 E L 4 E L 500 C

- ▶ The twisted Heisenberg-Virasoro Lie algebra  $\mathcal{H}$ , weight representations.
- ▶ Irreducibility problem of  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ .
- Free-field realization of Heisenberg-Virasoro vertex-algebra.

#### Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

ensor products

realization

Heisenberg-Virasoro VOA

eisenberg VOA ealization of HV ngular vectors

usion rules and ensor product odules

Irreducibility of a tensor product
Using Ω

Using 11 Using A More fusion rul

Free-field realization

4 D L 4 D L 4 E L 4 E L 500 C

- ▶ The twisted Heisenberg-Virasoro Lie algebra  $\mathcal{H}$ , weight representations.
- ▶ Irreducibility problem of  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ .
- Free-field realization of Heisenberg-Virasoro vertex-algebra.
- ► Explicit formulas for singular vectors.

#### Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization

Heisenberg-Virasoro VOA

eisenberg VO) ealization of H ngular vectors

usion rules and ensor product nodules

Irreducibility of a tensor product

Using A More fusion rul

Free-field realization

4 D N 4 D N 4 D N 1 D N

- ▶ The twisted Heisenberg-Virasoro Lie algebra  $\mathcal{H}$ , weight representations.
- ▶ Irreducibiliy problem of  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ .
- Free-field realization of Heisenberg-Virasoro vertex-algebra.
- Explicit formulas for singular vectors.
- ► Irreduciblity of  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$  solved. Fusion rules.

#### Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

Free field realization

Heisenberg-Virasoro VOA

leisenberg VOA ealization of H ingular vectors

usion rules ar ensor product rodules

Irreducibility of a tensor product

Using  $\Omega$ Using  $\Lambda$ More fusion re

Free-field realization (

4 D N 4 D N 4 E N 4 E N 0 O

- ▶ The twisted Heisenberg-Virasoro Lie algebra  $\mathcal{H}$ , weight representations.
- ▶ Irreducibiliy problem of  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ .
- Free-field realization of Heisenberg-Virasoro vertex-algebra.
- Explicit formulas for singular vectors.
- ▶ Irreduciblity of  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$  solved. Fusion rules.
- W(2,2)-structure on  $\mathcal{H}$ -modules.

#### Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

Free field realization

Heisenberg-Virasoro VOA

> ealization of HV ngular vectors

usion rules an ensor product nodules

Irreducibility of a tensor product

Using  $\Omega$ Using  $\Lambda$ More fusion

Free-field realization ( W(2.2)

4 D N 4 D N 4 E N 4 E N 0 C

- ▶ The twisted Heisenberg-Virasoro Lie algebra  $\mathcal{H}$ , weight representations.
- ▶ Irreducibiliy problem of  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ .
- Free-field realization of Heisenberg-Virasoro vertex-algebra.
- Explicit formulas for singular vectors.
- ▶ Irreduciblity of  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$  solved. Fusion rules.
- W(2,2)-structure on  $\mathcal{H}$ -modules.
- D. Adamović, G. R. Free fields realization of the twisted Heisenberg-Virasoro algebra at level zero and its applications, JPAA (2015)

#### Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Tensor products

realization

VOA Heisenberg VOA Realization of HV

Fusion rules and

modules

Irreducibility of a tensor product

Using Ω Using Λ More fusion r

Free-field realization o

# The twisted Heisenberg-Virasoro algebra

 $\mathcal{H}$  is a complex Lie algebra with a basis  $\{L\left(n\right), I\left(n\right), C_{L}, C_{I}, C_{L,I}: n \in \mathbb{Z}\}$  and a Lie bracket

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m} \frac{n^3 - n}{12} C_L,$$

$$[L(n), I(m)] = -mI(n+m) - \delta_{n,-m} (n^2 + n) C_{LI},$$

$$[I(n), I(m)] = n\delta_{n,-m} C_I,$$

$$[\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0.$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

\_ .

ensor products

realization
Heisenberg-Virasoro

VOA
Heisenberg VOA
Realization of HV

ısion rules an nsor product odules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization

# The twisted Heisenberg-Virasoro algebra

 $\mathcal{H}$  is a complex Lie algebra with a basis  $\{L(n), I(n), C_L, C_I, C_{L,I} : n \in \mathbb{Z}\}$  and a Lie bracket

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m} \frac{n^3 - n}{12} C_L,$$

$$[L(n), I(m)] = -mI(n+m) - \delta_{n,-m} (n^2 + n) C_{LI},$$

$$[I(n), I(m)] = n\delta_{n,-m} C_I,$$

$$[\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0.$$

 $\{L\left(n
ight), \mathit{C}_{L}, : n \in \mathbb{Z}\}$  spans a copy of the Virasoro algebra.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

\_ .

ensor products

realization
Heisenberg-Virasoro

Heisenberg VOA
Realization of HV
Singular vectors

usion rules and ensor product nodules

rreducibility of a tensor product Using Ω Using Λ

Free-field realization (

# The twisted Heisenberg-Virasoro algebra

 $\mathcal{H}$  is a complex Lie algebra with a basis  $\{L(n), I(n), C_L, C_I, C_{L,I} : n \in \mathbb{Z}\}$  and a Lie bracket

$$[L(n), L(m)] = (n-m)L(n+m) + \delta_{n,-m} \frac{n^3 - n}{12} C_L,$$

$$[L(n), I(m)] = -mI(n+m) - \delta_{n,-m} (n^2 + n) C_{LI},$$

$$[I(n), I(m)] = n\delta_{n,-m} C_I,$$

$$[\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0.$$

 $\{L\left(n
ight),\,\mathcal{C}_{L},:n\in\mathbb{Z}\}$  spans a copy of the Virasoro algebra.

 $\{I\left(n
ight),\,C_{I}:n\in\mathbb{Z}\}$  spans a copy of the Heisenberg algebra.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

\_ . .

ensor products

realization
Heisenberg-Virasoro

Heisenberg VOA
Realization of HV
Singular vectors

usion rules and ensor product nodules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

401481471717

▶  $V(c_L, c_I, c_{L,I}, h, h_I)$  - the **Verma module** with highest weight  $(h, h_I)$  and central charge  $(c_L, c_I, c_{L,I})$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Tensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

usion rules an nsor product

Irreducibility of a tensor product Using  $\Omega$ 

Free-field realization

- ▶  $V(c_L, c_I, c_{L,I}, h, h_I)$  the **Verma module** with highest weight  $(h, h_I)$  and central charge  $(c_L, c_I, c_{L,I})$ .
- We study representation at level zero ( $c_l = 0$ ).

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

ensor products

realization

Heisenberg-Virasoro VOA Heisenberg VOA

Reisenberg VOA Realization of HV ingular vectors

usion rules ar ensor product nodules

Irreducibility of a tensor product Using  $\Omega$ 

Free-field realization

- ▶  $V(c_L, c_I, c_{L,I}, h, h_I)$  the **Verma module** with highest weight  $(h, h_I)$  and central charge  $(c_L, c_I, c_{L,I})$ .
- We study representation at level zero ( $c_I = 0$ ).
- Y. Billig, Representations of the twisted Heisenberg-Virasoro algebra at level zero, Canadian Math. Bulletin, 46 (2003)

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

ensor products

realization

Heisenberg-Virasoro VOA

leisenberg VOA Realization of HV ingular vectors

ensor product odules

rreducibility of a ensor product

Using  $\Omega$ Using  $\Lambda$ More fusion  $r_{\rm L}$ 

Free-field realization W(2.2)

- ▶  $V(c_L, c_I, c_{L,I}, h, h_I)$  the **Verma module** with highest weight  $(h, h_I)$  and central charge  $(c_L, c_I, c_{L,I})$ .
- We study representation at level zero  $(c_I = 0)$ .
- Y. Billig, Representations of the twisted Heisenberg-Virasoro algebra at level zero, Canadian Math. Bulletin, 46 (2003)
- ► Appears in the representation theory of toroidal Lie algebras.

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

ensor products

realization
Heisenberg-Virasoro

Heisenberg-Virasoro VOA Heisenberg VOA

Singular vectors usion rules and

reducibility of a

tensor product
Using Ω
Using Λ

Free-field realization W(2-2)

### Theorem (Y. Billig)

Assume that  $c_I = 0$  and  $c_{LI} \neq 0$ .

(i) If  $\frac{h_I}{c_{LI}} - 1 \notin \mathbb{Z}^*$ , then  $V(c_L, 0, c_{LI}, h, h_I)$  is irreducible.

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate serie

Tensor product

realization
Heisenberg-Virasoro

Heisenberg VOA Realization of HV Singular vectors

usion rules an

Irreducibility of a tensor product Using Ω Using Λ More fusion rules

Free-field realization

4 D L 4 D L 4 E L 4 E L 500 C

## Theorem (Y. Billig)

Assume that  $c_I = 0$  and  $c_{LI} \neq 0$ .

(i) If  $\frac{h_I}{c_{LI}} - 1 \notin \mathbb{Z}^*$ , then  $V(c_L, 0, c_{LI}, h, h_I)$  is irreducible.

(ii) If  $\frac{h_l}{c_{Ll}} - 1 \in \mathbb{Z}^*$ , then  $V(c_L, 0, c_{Ll}, h, h_l)$  has a singular vector u at level  $p = \lfloor \frac{h_l}{c_{Ll}} - 1 \rfloor$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Tensor products

realization

Heisenberg-Virasoro

70A Heisenberg VOA Realization of HV Singular vectors

usion rules and Insor product Odules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

## Theorem (Y. Billig)

Assume that  $c_I = 0$  and  $c_{LI} \neq 0$ .

(i) If  $\frac{h_l}{c_{Ll}} - 1 \notin \mathbb{Z}^*$ , then  $V(c_L, 0, c_{Ll}, h, h_l)$  is irreducible.

(ii) If  $\frac{h_l}{c_{Ll}} - 1 \in \mathbb{Z}^*$ , then  $V(c_L, 0, c_{Ll}, h, h_l)$  has a singular vector u at level  $p = \lfloor \frac{h_l}{c_{Ll}} - 1 \rfloor$ .

The quotient module

$$L(c_L, 0, c_{L,I}, h, h_I) = V(c_L, 0, c_{L,I}, h, h_I) / U(\mathcal{H})u$$

is irreducible.

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Tensor products

realization

/OA Heisenberg VOA Realization of HV

Realization of H Singular vectors

usion rules ar ensor product nodules

Irreducibility of a tensor product
Using Ω

Free-field

realization W(2,2)

Define an  $\mathcal{H}$ -module structure on Virasoro intermediate series:

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate series

Fensor product

realization Heisenberg-Virasoro VOA Heisenberg VOA

Heisenberg VOA Realization of HV Singular vectors

odules

tensor product
Using Ω
Using Λ
More fusion rules

Free-field realization

4 D L 4 D L 4 E L 4 E L 500 C

Define an  $\mathcal{H}$ -module structure on Virasoro intermediate series:

Let  $\alpha, \beta, F \in \mathbb{C}$  define  $V_{\alpha,\beta,F} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} v_n$  with Lie bracket

$$L(n) v_m = -(m + \alpha + \beta + n\beta) v_{m+n},$$
  
 $I(n) v_m = Fv_{m+n},$   
 $C_L v_m = C_I v_m = C_{L,I} v_m = 0.$ 

Introduction

The twisted Heisenberg-Virasoro algebra

modules
Intermediate series

Tau-au ---- d...-t-

rensor product

realization
Heisenberg-Virasoro

Heisenberg VOA Realization of HV Singular vectors

usion rules an nsor product odules

Irreducibility of a tensor product
Using Ω

Free-field realization

4 D N 4 D N 4 D N 1 D N

Define an  $\mathcal{H}$ -module structure on Virasoro intermediate series:

Let  $\alpha, \beta, F \in \mathbb{C}$  define  $V_{\alpha,\beta,F} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} v_n$  with Lie bracket

$$L(n) v_m = -(m + \alpha + \beta + n\beta) v_{m+n},$$
  
 $I(n) v_m = Fv_{m+n},$   
 $C_L v_m = C_I v_m = C_{L,I} v_m = 0.$ 

As usual,

 $V_{\alpha,\beta,F}\cong V_{\alpha+k,\beta,F}$  for  $k\in\mathbb{Z}$ ,

Introduction

The twisted Heisenberg-Virasoro algebra

modules
Intermediate series

\_ . .

Free field

Heisenberg-Virasoro VOA

Realization of HV Singular vectors

-usion rules ai ensor product nodules

Irreducibility of a tensor product
Using Ω
Using Λ

Free-field realization

4 D N 4 D N 4 D N 1 D N

Define an  $\mathcal{H}$ -module structure on Virasoro intermediate series:

Let  $\alpha, \beta, F \in \mathbb{C}$  define  $V_{\alpha,\beta,F} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} v_n$  with Lie bracket

$$L(n) v_m = -(m + \alpha + \beta + n\beta) v_{m+n},$$
  
 $I(n) v_m = Fv_{m+n},$   
 $C_L v_m = C_I v_m = C_{L,I} v_m = 0.$ 

As usual,

- $V_{\alpha,\beta,F}\cong V_{\alpha+k,\beta,F}$  for  $k\in\mathbb{Z}$ ,
- $V_{\alpha,\beta,F}$  is reducible if and only if  $\alpha \in \mathbb{Z}$ ,  $\beta \in \{0,1\}$  and F=0,

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Intermediate series

Tau-au ---- d...-t-

rensor product

realization
Heisenberg-Virasoro
VOA

VOA Heisenberg VOA Realization of HV Singular vectors

Fusion rules and tensor product modules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization (

4 D N 4 D N 4 E N 4 E N 0 C

Define an  $\mathcal{H}$ -module structure on Virasoro intermediate series:

Let  $\alpha, \beta, F \in \mathbb{C}$  define  $V_{\alpha,\beta,F} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} v_n$  with Lie bracket

$$L(n) v_m = -(m + \alpha + \beta + n\beta) v_{m+n},$$
  
 $I(n) v_m = Fv_{m+n},$   
 $C_L v_m = C_I v_m = C_{L,I} v_m = 0.$ 

As usual,

- $V_{\alpha,\beta,F}\cong V_{\alpha+k,\beta,F}$  for  $k\in\mathbb{Z}$ ,
- $V_{\alpha,\beta,F}$  is reducible if and only if  $\alpha \in \mathbb{Z}$ ,  $\beta \in \{0,1\}$  and F=0,
- $lacksymbol{V}'_{0,0,0}:=V/\mathbb{C} v_0,\ V'_{0,1,0}:=igoplus_{n
  eq-1}\mathbb{C} v_n$  and  $V'_{lpha,eta,F}:=V_{lpha,eta,F}$  otherwise.

Introduction

The twisted Heisenberg-Virasoro algebra

modules
Intermediate series

\_ . .

Tensor products

realization Heisenberg-Virasoro VOA

Heisenberg VOA
Realization of HV
Singular vectors

Fusion rules and tensor product modules

Irreducibility of a tensor product
Using Ω
Using Λ

realization (W(2,2)

### Irreducible Harish-Chandra modules

## Theorem (Lu, R., Zhao, K.)

An irreducible weight  $\mathcal{H}$ -module with finite-dimensional weight spaces is isomorphic either to a highest (or lowest) weight module, or to  $V'_{\alpha,\beta,F}$  for some  $\alpha,\beta,F\in\mathbb{C}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate series

Tensor products

Free field

ealization Heisenberg-Virasc

VOA Heisenberg VOA

Realization of HV ingular vectors

ısion rules an nsor product odules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization

4 D N 4 D N 4 D N 1 D N

#### Irreducible Harish-Chandra modules

Theorem (Lu, R., Zhao, K.)

An irreducible weight  $\mathcal{H}$ -module with finite-dimensional weight spaces is isomorphic either to a highest (or lowest) weight module, or to  $V'_{\alpha,\beta,F}$  for some  $\alpha,\beta,F\in\mathbb{C}$ .

What about modules with infinite-dimensional weight spaces?

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate series

Tensor products

realization

Heisenberg-Virasoro VOA

> leisenberg VOA lealization of HV ingular vectors

usion rules and ensor product

rreducibility of a

Using Ω Using Λ More fusion

Free-field realization

## Tensor product modules

Consider  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$  module:

$$L(n)(v_k \otimes x) = L(n)v_k \otimes x + v_k \otimes L(n)x,$$

$$I(m)(v_k \otimes x) = Fv_k \otimes x + v_k \otimes I(m)x,$$

$$C_L(v_k \otimes x) = c_L(v_k \otimes x),$$

$$C_I(v_k \otimes x) = 0$$

$$C_{L,I}(v_k \otimes x) = c_{L,I}(v_k \otimes x).$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

iliterillediate serie

#### Tensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

ingular vectors

Ision rules and

rreducibility of a

Using Ω
Using Λ
More fusion rule

Free-field realization W(2.2)

## Tensor product modules

Consider  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$  module:

$$L(n)(v_k \otimes x) = L(n)v_k \otimes x + v_k \otimes L(n)x,$$

$$I(m)(v_k \otimes x) = Fv_k \otimes x + v_k \otimes I(m)x,$$

$$C_L(v_k \otimes x) = c_L(v_k \otimes x),$$

$$C_I(v_k \otimes x) = 0$$

$$C_{L,I}(v_k \otimes x) = c_{L,I}(v_k \otimes x).$$

To classify irreducible modules  $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$  we need more detailed formulas for singular vectors.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

micermodiate serie

Tensor products

realization

Heisenberg-Virasoro VOA Heisenberg VOA

Realization of HV Singular vectors

usion rules an insor product odules

Irreducibility of a tensor product

Using Ω Using Λ

Free-field realization

Irreducible  $\mathcal{H}$ -module  $L(c_L, 0, c_{L,I}, 0, 0)$  has the structure of vertex operator algebra which we denote by  $L^{\mathcal{H}}(c_L, c_{L,I})$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

iliterillediate seri

ensor products

ree field ealization

Heisenberg-Virasoro VOA Heisenberg VOA

Realization of HV

nsor product odules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

4 D L 4 D L 4 E L 4 E L 500 C

Irreducible  $\mathcal{H}$ -module  $L(c_L, 0, c_{L,I}, 0, 0)$  has the structure of vertex operator algebra which we denote by  $L^{\mathcal{H}}(c_L, c_{L,I})$ .

## Theorem (Y. Billig)

Let  $c_{L,I} \neq 0$ . Then  $L^{\mathcal{H}}(c_L, c_{L,I})$  is a simpe VOA, and  $V(c_L, 0, c_{L,I}, h, h_I)$  and  $L(c_L, 0, c_{L,I}, h, h_I)$  are  $L^{\mathcal{H}}(c_L, c_{L,I})$ -modules.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

intermediate serie

Tensor products

realization

Heisenberg-Virasoro

OA Heisenberg VOA Realization of HV

ingular vectors usion rules and nsor product

Irreducibility of a tensor product Using  $\Omega$ 

Free-field realization

40.40.45.45. 5 000

Irreducible  $\mathcal{H}$ -module  $L(c_L, 0, c_{L,I}, 0, 0)$  has the structure of vertex operator algebra which we denote by  $L^{\mathcal{H}}(c_L, c_{L,I})$ .

## Theorem (Y. Billig)

Let  $c_{L,I} \neq 0$ . Then  $L^{\mathcal{H}}(c_L, c_{L,I})$  is a simpe VOA, and  $V(c_L, 0, c_{L,I}, h, h_I)$  and  $L(c_L, 0, c_{L,I}, h, h_I)$  are  $L^{\mathcal{H}}(c_L, c_{L,I})$ -modules.

▶  $L^{\mathcal{H}}(c_L, c_{L,I})$  can be realized as a subalgebra of the Heisenberg vertex algebra M(1).

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

ensor products

realization Heisenberg-Virasoro

Heisenberg-Virasoro VOA Heisenberg VOA

ealization of HV ngular vectors

nsor product odules

rreducibility of a ensor product Using Ω Using Λ

Free-field realization

Irreducible  $\mathcal{H}$ -module  $L(c_L, 0, c_{L,I}, 0, 0)$  has the structure of vertex operator algebra which we denote by  $L^{\mathcal{H}}(c_L, c_{L,I})$ .

## Theorem (Y. Billig)

Let  $c_{L,I} \neq 0$ . Then  $L^{\mathcal{H}}(c_L, c_{L,I})$  is a simpe VOA, and  $V(c_L, 0, c_{L,I}, h, h_I)$  and  $L(c_L, 0, c_{L,I}, h, h_I)$  are  $L^{\mathcal{H}}(c_L, c_{L,I})$ -modules.

- ▶  $L^{\mathcal{H}}(c_L, c_{L,I})$  can be realized as a subalgebra of the Heisenberg vertex algebra M(1).
- ▶ M(1)-modules  $M(1, \gamma)$  become  $L^{\mathcal{H}}(c_L, c_{L,I})$ -modules, and also  $\mathcal{H}$ -modules.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization Heisenberg-Virasoro

Heisenberg VOA Realization of H\ Singular vectors

usion rules and ensor product

rreducibility of a ensor product

Using Ω Using Λ More fusion

Free-field realization W(2.2)

Irreducible  $\mathcal{H}$ -module  $L(c_L, 0, c_{L,I}, 0, 0)$  has the structure of vertex operator algebra which we denote by  $L^{\mathcal{H}}(c_L, c_{L,I})$ .

## Theorem (Y. Billig)

Let  $c_{L,I} \neq 0$ . Then  $L^{\mathcal{H}}(c_L, c_{L,I})$  is a simpe VOA, and  $V(c_L, 0, c_{L,I}, h, h_I)$  and  $L(c_L, 0, c_{L,I}, h, h_I)$  are  $L^{\mathcal{H}}(c_L, c_{L,I})$ -modules.

- ▶  $L^{\mathcal{H}}(c_L, c_{L,I})$  can be realized as a subalgebra of the Heisenberg vertex algebra M(1).
- ▶ M(1)-modules  $M(1, \gamma)$  become  $L^{\mathcal{H}}(c_L, c_{L,I})$ -modules, and also  $\mathcal{H}$ -modules.
- Construction of a screening operator will give us realization of certain irr weight modules.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization Heisenberg-Virasoro

Heisenberg VOA Realization of HV Singular vectors

usion rules and ensor product

rreducibility of a

Using Ω
Using Λ
More fusion

Free-field realization o

▶ Let  $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$  be a hyperbolic lattice such that  $\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1$ ,  $\langle \alpha, \beta \rangle = 0$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate series

ensor product

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA
Realization of HV

Singular vectors usion rules an ensor product

reducibility of a ensor product Using Ω Using Λ

Free-field realization (

4 D L 4 D L 4 E L 4 E L 500 C

- ▶ Let  $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$  be a hyperbolic lattice such that  $\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1$ ,  $\langle \alpha, \beta \rangle = 0$ .
- Let  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$  be an abelian Lie algebra and  $\widehat{\mathfrak{h}} = \mathbb{C} \left[ t, t^{-1} \right] \otimes \mathfrak{h} \oplus \mathbb{C} c$  its affinization.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

intermediate seri

Tensor product

Free field realization Heisenberg-Virasoro

Heisenberg-Virasoro VOA Heisenberg VOA Realization of HV Singular vectors

usion rules a

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

4 D L 4 D L 4 E L 4 E L 500 C

- Let  $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$  be a hyperbolic lattice such that  $\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1$ ,  $\langle \alpha, \beta \rangle = 0$ .
- ▶ Let  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$  be an abelian Lie algebra and  $\widehat{\mathfrak{h}} = \mathbb{C} \left[ t, t^{-1} \right] \otimes \mathfrak{h} \oplus \mathbb{C} c$  its affinization.
- For  $\gamma \in \mathfrak{h}$  consider  $\widehat{\mathfrak{h}}$ -module

$$M\left(1,\gamma\right):=U(\widehat{\mathfrak{h}})\otimes_{U(\mathbb{C}[t]\otimes\mathfrak{h}\oplus\mathbb{C}c)}\mathbb{C}$$

where  $t\mathbb{C}[t] \otimes \mathfrak{h}$  acts trivially on  $\mathbb{C}$ ,  $\delta \in \mathfrak{h}$  acts by  $\langle \delta, \gamma \rangle$  and c acts as 1.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

<u>.</u> . .

ensor product

realization Heisenberg-Virasoro VOA

Heisenberg VOA
Realization of HV
Singular vectors

ision rules and nsor product odules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization (

- Let  $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$  be a hyperbolic lattice such that  $\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1, \ \langle \alpha, \beta \rangle = 0.$
- Let  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$  be an abelian Lie algebra and  $\widehat{\mathfrak{h}} = \mathbb{C} \left[ t, t^{-1} \right] \otimes \mathfrak{h} \oplus \mathbb{C} c$  its affinization.
- For  $\gamma \in \mathfrak{h}$  consider  $\widehat{\mathfrak{h}}$ -module

$$M\left(1,\gamma\right):=U(\widehat{\mathfrak{h}})\otimes_{U(\mathbb{C}[t]\otimes\mathfrak{h}\oplus\mathbb{C}c)}\mathbb{C}$$

where  $t\mathbb{C}[t] \otimes \mathfrak{h}$  acts trivially on  $\mathbb{C}$ ,  $\delta \in \mathfrak{h}$  acts by  $\langle \delta, \gamma \rangle$  and c acts as 1.

▶ Denote by  $e^{\gamma}$  a highest weight vector in  $M(1, \gamma)$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization
Heisenberg-Virasoro
VOA

Heisenberg VOA
Realization of HV
Singular vectors

usion rules and ensor product rodules

Irreducibility of a tensor product Using Ω Using Λ

realization ( W(2,2)

# Heisenberg vertex-algebra

- Let  $L = \mathbb{Z}\alpha + \mathbb{Z}\beta$  be a hyperbolic lattice such that  $\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1, \ \langle \alpha, \beta \rangle = 0.$
- Let  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$  be an abelian Lie algebra and  $\widehat{\mathfrak{h}} = \mathbb{C} \left[ t, t^{-1} \right] \otimes \mathfrak{h} \oplus \mathbb{C} c$  its affinization.
- For  $\gamma \in \mathfrak{h}$  consider  $\widehat{\mathfrak{h}}$ -module

$$M(1,\gamma) := U(\widehat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}c)} \mathbb{C}$$

where  $t\mathbb{C}[t] \otimes \mathfrak{h}$  acts trivially on  $\mathbb{C}$ ,  $\delta \in \mathfrak{h}$  acts by  $\langle \delta, \gamma \rangle$  and c acts as 1.

- ▶ Denote by  $e^{\gamma}$  a highest weight vector in  $M(1, \gamma)$ .
- ightharpoonup M(1) := M(1,0) is a vertex-algebra:

$$h(n) = t^n \otimes h, \quad \text{for } h \in \mathfrak{h},$$
  
 $h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}$ 

and  $M(1, \gamma)$  for  $\gamma \in \mathfrak{h}$ , are irreducible M(1)-modules.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA

Realization of HV Singular vectors Fusion rules and

rreducibility of a

Using Ω
Using Λ
More fusion

Free-field realization ( W(2.2)

# Realization of the Heisenberg-Virasoro vertex algebra

▶ Define a Heisenberg vector

$$I = \alpha(-1) + \beta(-1) \in M(1)$$

and a Virasoro vector

$$\omega = \frac{1}{2}\alpha(-1)^2 - \frac{1}{2}\beta(-1)^2 + \lambda\alpha(-2) + \mu\beta(-2) \in M(1)$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization Heisenberg-Virasoro VOA Heisenberg VOA Realization of HV

> usion rules and Insor product

Irreducibility of a tensor product Using  $\Omega$ Using  $\Lambda$ 

Free-field realization (

# Realization of the Heisenberg-Virasoro vertex algebra

▶ Define a Heisenberg vector

$$I = \alpha(-1) + \beta(-1) \in M(1)$$

and a Virasoro vector

$$\omega = \frac{1}{2}\alpha(-1)^2 - \frac{1}{2}\beta(-1)^2 + \lambda\alpha(-2) + \mu\beta(-2) \in M(1)$$

Then

$$I(z) = Y(I,z) = \sum_{n \in \mathbb{Z}} I(n) z^{-n-1}$$
 and 
$$L(z) = Y(\omega,z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$$

4 D N 4 D N 4 D N 1 D N

generate the Heisenberg-Virasoro vertex algebra  $L^{\mathcal{H}}(c_{l}, c_{l-l})$  in M(1).

Introduction

The twisted Heisenberg-Virasoro algebra

modules

T. . . . . . .

Tensor products

realization

Heisenberg-Virasoro
VOA

Heisenberg VOA

Realization of HV

Singular vectors

ensor product odules reducibility of a

ensor product

Jsing Ω

Jsing Λ

More fusion rules

Free-field realization

We get the twisted Heisenberg-Virasoro Lie algebra  ${\mathcal H}$  such that

$$c_L = 2 - 12(\lambda^2 - \mu^2), \quad c_{L,l} = \lambda - \mu$$

i.e.

$$\lambda = \frac{2 - c_L}{24c_{L,I}} + \frac{1}{2}c_{L,I}, \quad \mu = \frac{2 - c_L}{24c_{L,I}} - \frac{1}{2}c_{L,I}.$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

T. . . . . . .

Tensor product

realization

Heisenberg-Virasoro
VOA

Heisenberg VOA

Realization of HV

ingular vectors Ision rules and nsor product

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization

We get the twisted Heisenberg-Virasoro Lie algebra  ${\mathcal H}$  such that

$$c_L = 2 - 12(\lambda^2 - \mu^2), \quad c_{L,I} = \lambda - \mu$$

i.e.

$$\lambda = \frac{2 - c_L}{24c_{L,I}} + \frac{1}{2}c_{L,I}, \quad \mu = \frac{2 - c_L}{24c_{L,I}} - \frac{1}{2}c_{L,I}.$$

Now we may use representation theory of  $M\left(1\right)$  in rep theory of  $\mathcal{H}!$ 

Introduction

The twisted Heisenberg-Virasoro algebra

modules

T. . . . . . .

ensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV Singular vectors

nsor product odules

Irreducibility of a tensor product
Using Ω

Free-field realization

4 D N 4 D N 4 D N 1 D N 0 C

► For every  $r, s \in \mathbb{C}$ ,  $e^{r\alpha+s\beta}$  is a  $\mathcal{H}$ -singular vector and  $U(\mathcal{H})e^{r\alpha+s\beta}$  is a highest weight module with the highest weight

$$h = \Delta_{r,s} = \frac{1}{2}r^2 - \frac{1}{2}s^2 - \lambda r + \mu s, \quad h_l = r - s$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

\_ . .

ensor products

realization Heisenberg-Virasoro VOA Heisenberg VOA

Heisenberg VOA Realization of HV Singular vectors

nsor product odules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

► For every  $r, s \in \mathbb{C}$ ,  $e^{r\alpha+s\beta}$  is a  $\mathcal{H}$ -singular vector and  $U(\mathcal{H})e^{r\alpha+s\beta}$  is a highest weight module with the highest weight

$$h = \Delta_{r,s} = \frac{1}{2}r^2 - \frac{1}{2}s^2 - \lambda r + \mu s, \quad h_l = r - s$$

### Proposition

(i) Let  $(h, h_I) \in \mathbb{C}^2$ ,  $h_I \neq c_{L,I}$ . Then there exist unique  $r, s \in \mathbb{C}$  such that  $e^{r\alpha + s\beta}$  is a highest weight vector of the highest weight  $(h, h_I)$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

\_

Tensor products

realization

Heisenberg-Virasoro
VOA

Heisenberg VOA

Realization of HV

Singular vectors

nsor product odules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization (

► For every  $r, s \in \mathbb{C}$ ,  $e^{r\alpha+s\beta}$  is a  $\mathcal{H}$ -singular vector and  $U(\mathcal{H})e^{r\alpha+s\beta}$  is a highest weight module with the highest weight

$$h = \Delta_{r,s} = \frac{1}{2}r^2 - \frac{1}{2}s^2 - \lambda r + \mu s, \quad h_l = r - s$$

### Proposition

(i) Let  $(h, h_I) \in \mathbb{C}^2$ ,  $h_I \neq c_{L,I}$ . Then there exist unique  $r, s \in \mathbb{C}$  such that  $e^{r\alpha + s\beta}$  is a highest weight vector of the highest weight  $(h, h_I)$ .

(ii) For every  $r, s \in \mathbb{C}$  such that  $r - s = \lambda - \mu = c_{L,I}$ ,  $e^{r\alpha + s\beta}$  is a highest weight vector of weight

$$(h, h_I) = (\frac{c_L - 2}{24}, c_{L,I}).$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization

Heisenberg-Virasoro
VOA

Heisenberg VOA

Realization of HV

Singular vectors

usion rules a ensor produc iodules

Irreducibility of a tensor product

Using A More fusion

Free-field realization ( W(2,2)

▶ Denote by  $\mathcal{F}_{r,s}$  the M(1)-module generated by  $e^{r\alpha+s\beta}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

iliterillediate seri

ensor products

realization Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

usion rules ar ensor product

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

4 D L 4 D L 4 E L 4 E L 500 C

- ▶ Denote by  $\mathcal{F}_{r,s}$  the M(1)-module generated by  $e^{r\alpha+s\beta}$ .
- It is also an  $L^{\mathcal{H}}(c_L, c_{L,I})$ -module, therefore an  $\mathcal{H}$ -module.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

intermediate seri

Tensor products

realization

VOA
Heisenberg VOA
Realization of HV

usion rules ar ensor product

Irreducibility of a tensor product Using  $\Omega$ 

More fusion r

realization W(2.2)

- ▶ Denote by  $\mathcal{F}_{r,s}$  the M(1)-module generated by  $e^{r\alpha+s\beta}$ .
- It is also an  $L^{\mathcal{H}}(c_L, c_{L,I})$ -module, therefore an  $\mathcal{H}$ -module.
- ▶ There is a surjective  $\mathcal{H}$ -homomorphism

$$\Phi: V(c_L, 0, c_{L,I}, h, h_I) \rightarrow U(\mathcal{H})e^{r\alpha+s\beta}$$

such that  $\Phi(v_{h,h_l}) = e^{r\alpha + s\beta}$ .

#### Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Tensor products

Free field realization

Heisenberg-Virasord VOA Heisenberg VOA Realization of HV

usion rules ar ensor product nodules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization W(2.2)

40.49.41.41.1 1 000

- ▶ Denote by  $\mathcal{F}_{r,s}$  the M(1)-module generated by  $e^{r\alpha+s\beta}$ .
- It is also an  $L^{\mathcal{H}}(c_L, c_{L,I})$ -module, therefore an  $\mathcal{H}$ -module.
- ▶ There is a surjective  $\mathcal{H}$ -homomorphism

$$\Phi: V(c_L, 0, c_{L,I}, h, h_I) \rightarrow U(\mathcal{H})e^{r\alpha+s\beta}$$

such that  $\Phi(v_{h,h_I}) = e^{r\alpha + s\beta}$ .

### Proposition

Assume that  $\frac{h_I}{c_{L,I}} - 1 \notin -\mathbb{Z}_{>0}$ . Then  $\mathcal{F}_{r,s} \cong V(c_L, 0, c_{L,I}, h, h_I)$  as  $L^{\mathcal{H}}(c_L, c_{L,I})$ -modules.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization
Heisenberg-Virasoro

VOA
Heisenberg VOA
Realization of HV
Singular vectors

usion rules ar ensor product nodules

Irreducibility of a tensor product

Jsing Ω Jsing Λ More fusion

Free-field realization



•  $u=e^{-rac{lpha+eta}{c_{L,l}}}$  is a highest weight vector of weight (1,0) .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

intermediate sent

Tensor product

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA
Realization of HV

usion rules ai insor product

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

4 D L 4 D L 4 E L 4 E L 5 00 C

- lacksquare  $u=e^{-rac{lpha+eta}{c_{L,I}}}$  is a highest weight vector of weight (1,0) .
- ▶ Let  $Q = \operatorname{Res}_{z} Y(u, z) = u_0$  (well defined on  $M(1, \gamma)$ ).

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate serie

Tensor products

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA

Heisenberg VOA
Realization of HV
Singular vectors

usion rules ar ensor product

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

- $u=e^{-rac{lpha+eta}{c_{L,l}}}$  is a highest weight vector of weight (1,0) .
- ▶ Let  $Q = \operatorname{Res}_{z} Y(u, z) = u_{0}$  (well defined on  $M(1, \gamma)$ ).
- ▶ Screening operator Q commutes with L(n) and I(n).

Introduction

The twisted Heisenberg-Virasoro algebra

modules

iliterillediate seri

Tensor products

realization

Heisenberg-Virasoro

VOA

VOA Heisenberg VOA Realization of HV Singular vectors

usion rules an ensor product 10dules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization

- $u=e^{-rac{lpha+eta}{c_{L,l}}}$  is a highest weight vector of weight (1,0) .
- ▶ Let  $Q = \operatorname{Res}_{z} Y(u, z) = u_{0}$  (well defined on  $M(1, \gamma)$ ).
- ▶ Screening operator Q commutes with L(n) and I(n).
- ▶ So  $Q^j e^{r\alpha + s\beta}$  is either 0 or a singular vector.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization

Heisenberg-Virasord VOA Heisenberg VOA Realization of HV

> ısion rules an nsor product odules

Irreducibility of a tensor product Using Ω Using A

realization W(2,2)

- $u=e^{-rac{lpha+eta}{c_{L,l}}}$  is a highest weight vector of weight (1,0) .
- ▶ Let  $Q = \operatorname{Res}_{z} Y(u, z) = u_{0}$  (well defined on  $M(1, \gamma)$ ).
- ▶ Screening operator Q commutes with L(n) and I(n).
- ▶ So  $Q^j e^{r\alpha + s\beta}$  is either 0 or a singular vector.

### Proposition

Assume that  $\frac{h_l}{c_{L,l}}-1=-p\in -\mathbb{Z}_{>0}$ . As a  $L^{\mathcal{H}}(c_L,c_{L,l})$ -module  $\mathcal{F}_{r,s}$  is generated by  $\mathrm{e}^{r\alpha+s\beta}$  and a family of subsingular vectors  $\{v_{n,p}:n\geq 1\}$  such that

$$Q^n v_{n,p} = e^{r\alpha + s\beta - n\frac{\alpha + \beta}{c_{L,I}}}.$$

In particular  $L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I) = \ker_{\mathcal{F}_{r,s}} Q$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA
Realization of HV

usion rules au ensor product odules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization o



- $u=e^{-rac{lpha+eta}{c_{L,l}}}$  is a highest weight vector of weight (1,0) .
- ▶ Let  $Q = \operatorname{Res}_{z} Y(u, z) = u_{0}$  (well defined on  $M(1, \gamma)$ ).
- ▶ Screening operator Q commutes with L(n) and I(n).
- ▶ So  $Q^j e^{r\alpha + s\beta}$  is either 0 or a singular vector.

### Proposition

Assume that  $\frac{h_l}{c_{L,l}}-1=-p\in -\mathbb{Z}_{>0}$ . As a  $L^{\mathcal{H}}(c_L,c_{L,l})$ -module  $\mathcal{F}_{r,s}$  is generated by  $\mathrm{e}^{r\alpha+s\beta}$  and a family of subsingular vectors  $\{v_{n,p}:n\geq 1\}$  such that

$$Q^n v_{n,p} = e^{r\alpha + s\beta - n\frac{\alpha + \beta}{c_{L,I}}}.$$

In particular  $L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I) = \ker_{\mathcal{F}_{r,s}} Q$ .

▶ For  $r = s = -\frac{1}{c_{L,l}}$  we get  $L^{\mathcal{H}}(c_L, c_{L,l}) = \ker_{M(1)} Q$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA
Realization of HV
Singular vectors

usion rules au ensor product odules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization of W(2,2)

## Schur polynomials

Schur polynomials  $S_r(x_1, x_2, \cdots)$  in variables  $x_1, x_2, \ldots$  are defined by the following equation:

$$\exp\left(\sum_{n=1}^{\infty}\frac{x_n}{n}y^n\right)=\sum_{r=0}^{\infty}S_r(x_1,x_2,\cdots)y^r.$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor product

realization
Heisenberg-Virasoro

Heisenberg VOA
Realization of HV
Singular vectors

usion rules ar ensor product

Irreducibility of a tensor product Using Ω

Free-field realization

40.49.45.45. 5 900

## Schur polynomials

Schur polynomials  $S_r(x_1, x_2, \cdots)$  in variables  $x_1, x_2, \ldots$ are defined by the following equation:

$$\exp\left(\sum_{n=1}^{\infty}\frac{x_n}{n}y^n\right)=\sum_{r=0}^{\infty}S_r(x_1,x_2,\cdots)y^r.$$

Also

$$S_r(x_1,x_2,\cdots) = \frac{1}{r!} \left| \begin{array}{ccccc} x_1 & x_2 & \cdots & x_r \\ -r+1 & x_1 & x_2 & \cdots & x_{r-1} \\ 0 & -r+2 & x_1 & \cdots & x_{r-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & x_1 \end{array} \right| \left| \begin{array}{cccccc} Free field \\ realization \\ Heisenberg VOA \\ Realization of HV \\ Singular vectors \\ Fusion rules and tensor product modules \\ Irreducibility of a tensor product at tensor pro$$

Structure of Verma

# Schur polynomials

Schur polynomials  $S_r(x_1, x_2, \cdots)$  in variables  $x_1, x_2, \ldots$  are defined by the following equation:

$$\exp\left(\sum_{n=1}^{\infty}\frac{x_n}{n}y^n\right)=\sum_{r=0}^{\infty}S_r(x_1,x_2,\cdots)y^r.$$

Also

$$S_r(x_1, x_2, \cdots) = rac{1}{r!} \left| egin{array}{cccccc} x_1 & x_2 & \cdots & x_r \\ -r+1 & x_1 & x_2 & \cdots & x_{r-1} \\ 0 & -r+2 & x_1 & \cdots & x_{r-2} \\ dots & \ddots & \ddots & dots \\ 0 & \cdots & 0 & -1 & x_1 \end{array} \right|.$$

Schur polynomials naturally appear in formulas for vertex operator for lattice vertex algebras. Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

ensor products

realization
Heisenberg-Virasoro

Heisenberg VOA Realization of HV Singular vectors

-usion rules a ensor produc nodules

Irreducibility of a tensor product

Jsing Ω Jsing Λ More fusion ru

Free-field realization W(2,2)

## Schur polynomials and singular vectors

#### **Theorem**

Assume that  $c_{L,l} \neq 0$  and  $p = \frac{h_l}{c_{L,l}} - 1 \in \mathbb{Z}_{>0}$ . Then  $\Omega v_{h,h_l}$  where

$$\Omega = S_p\left(-\frac{I(-1)}{c_{L,I}}, -\frac{I(-2)}{c_{L,I}}, \ldots, -\frac{I(-p)}{c_{L,I}}\right)$$

is a singular vector of weight p in the Verma module  $V(c_L,0,c_{L,I},h,(1+p)c_{L,I})$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor product

realization
Heisenberg-Virasoro

Heisenberg VOA Realization of HV Singular vectors

usion rules an ensor product

rreducibility of a ensor product Using Ω Using Λ

Free-field realization

# Schur polynomials and singular vectors

#### **Theorem**

Assume that  $c_{L,l} \neq 0$  and  $p = 1 - \frac{h_l}{c_{L,l}} \in \mathbb{Z}_{>0}$ . Then  $\Lambda v_{h,h_l}$  where

$$\Lambda = \sum_{i=0}^{p-1} S_i \left( \frac{I(-1)}{c_{L,I}}, \dots, \frac{I(-i)}{c_{L,I}} \right) L_{i-p} +$$

$$\sum_{i=0}^{p-1} \left( \frac{h}{p} + \frac{c_L - 2}{24} \frac{\left(p-1\right)^2 - pi}{p} \right) S_i \left( \frac{I\left(-1\right)}{c_{L,I}}, \dots, \frac{I\left(-i\right)}{c_{L,I}} \right) \frac{I\left(i - \frac{\text{Heisenbergen}}{c_{L,I}} \right)}{c_{L,I}}$$

is a singular vector of weight p in the Verma module  $V(c_L, 0, c_{L,I}, h, (1-p) c_{L,I})$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization
Heisenberg-Virasoro
VOA

Realization of HV Singular vectors usion rules an

Irreducibility of a

tensor product
Using  $\Omega$ 

Using A More fusion

Free-field realization of



# Intertwining operators and tensor product modules

As with Virasoro algebra, the existence of a nontrivial intertwining operator of the type

$$\begin{pmatrix} L(c_L, 0, c_{L,I}, h'', h''_I) \\ L(c_L, 0, c_{L,I}, h, h_I) & L(c_L, 0, c_{L,I}, h', h'_I) \end{pmatrix}$$

yields a nontrivial  ${\mathcal H}$ -homomorphism

$$\phi: V_{\alpha,\beta,F}' \otimes L(c_L,0,c_{L,I},h',h_I') \rightarrow L(c_L,0,c_{L,I},h'',h_I'')$$

where

$$\alpha = h + h' - h'$$
,  $\beta = 1 - h$ ,  $F = h_I$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

micimicalate sen

Tensor products

realization
Heisenberg-Virasoro

Heisenberg-Virasoro VOA Heisenberg VOA

Singular vectors

Fusion rules and tensor product

modules

Irreducibility of a

tensor product Using Ω Using Λ

Free-field realization

vv (.

# Intertwining operators and tensor product modules

As with Virasoro algebra, the existence of a nontrivial intertwining operator of the type

$$\begin{pmatrix} L(c_L, 0, c_{L,I}, h'', h''_I) \\ L(c_L, 0, c_{L,I}, h, h_I) & L(c_L, 0, c_{L,I}, h', h'_I) \end{pmatrix}$$

yields a nontrivial  ${\mathcal H}$ -homomorphism

$$\varphi: V'_{\alpha,\beta,F} \otimes L(c_{L},0,c_{L,I},h',h'_{I}) \to L(c_{L},0,c_{L,I},h'',h''_{I})$$

where

$$\alpha = h + h' - h'$$
,  $\beta = 1 - h$ ,  $F = h_I$ .

By dimension argument, we get reducibility of  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h',h'_I)$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization Heisenberg-Virasoro

Heisenberg-Virasoro VOA Heisenberg VOA

Fusion rules and tensor product

modules

Irreducibility of a

Using Ω
Using Λ

Free-field realization

From the standard fusion rules result for M(1) we get intertwining operators in the category of  ${\mathcal H} ext{--modules}$ :

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

intermediate serie

Tensor product

realization
Heisenberg-Virasoro

Heisenberg VOA Realization of HV Singular vectors

Fusion rules and tensor product modules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

4 D L 4 D L 4 E L 4 E L 5 00 Q

From the standard fusion rules result for M(1) we get intertwining operators in the category of  $\mathcal{H} ext{--modules}$ :

#### **Theorem**

Let 
$$(h, h_I) = (\Delta_{r_1, s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2, s_2}, r_2 - s_2) \in \mathbb{C}^2$$
 such that  $\frac{h_I}{c_{L,I}} - 1, \frac{h'_I}{c_{L,I}} - 1, \frac{h_I + h'_I}{c_{L,I}} - 1 \notin \mathbb{Z}_{>0}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

.....

Tensor products

realization Heisenberg-Virasoro VOA

eisenberg VOA ealization of HV ingular vectors

Fusion rules and tensor product modules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization

From the standard fusion rules result for M(1) we get intertwining operators in the category of  $\mathcal{H} ext{--modules}$ :

#### **Theorem**

Let  $(h, h_I) = (\Delta_{r_1, s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2, s_2}, r_2 - s_2) \in \mathbb{C}^2$  such that  $\frac{h_I}{c_{L,I}} - 1, \frac{h'_I}{c_{L,I}} - 1, \frac{h_I + h'_I}{c_{L,I}} - 1 \notin \mathbb{Z}_{>0}$ . Then there is a non-trivial intertwining operator of the type

$$\begin{pmatrix} L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h'', h_{I} + h'_{I}) \\ L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, h_{I}) & L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h', h'_{I}) \end{pmatrix}$$

where  $h'' = \Delta_{r_1 + r_2, s_1 + s_2}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

micrimodiate ser

ensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

Fusion rules and tensor product modules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization



From the standard fusion rules result for M(1) we get intertwining operators in the category of  $\mathcal{H} ext{--modules}$ :

#### **Theorem**

Let  $(h,h_I)=(\Delta_{r_1,s_1},r_1-s_1), (h',h'_I)=(\Delta_{r_2,s_2},r_2-s_2)\in\mathbb{C}^2$  such that  $\frac{h_I}{c_{L,I}}-1,\frac{h'_I}{c_{L,I}}-1,\frac{h_I+h'_I}{c_{L,I}}-1\notin\mathbb{Z}_{>0}$ . Then there is a non-trivial intertwining operator of the type

$$\begin{pmatrix} L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h'', h_{I} + h'_{I}) \\ L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, h_{I}) & L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h', h'_{I}) \end{pmatrix}$$

where  $h'' = \Delta_{r_1 + r_2, s_1 + s_2}$ . In particular, the  $\mathcal{H}$ -module  $V'_{\alpha, \beta, F} \otimes L^{\mathcal{H}}(c_L, 0, c_{L,I}, h', h'_I)$  is reducible where

$$\alpha = h + h' - h''$$
,  $\beta = 1 - h$ ,  $F = h_I$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor product

realization

Heisenberg-Virasoro
VOA

leisenberg VOA lealization of HV ingular vectors

Fusion rules and tensor product modules

Irreducibility of a tensor product
Using  $\Omega$ Using  $\Lambda$ More fusion rules

Free-field realization W(2,2)



### Corollary

Let  $(h, h_I) = (\Delta_{r_1, s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2, s_2}, r_2 - s_2) \in \mathbb{C}^2$  and that there are  $p, q \in \mathbb{Z}_{>0}, q \leq p$  such that

$$\frac{h_l}{c_{L,l}} - 1 = -q, \quad \frac{h'_l}{c_{L,l}} - 1 = p.$$

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate series

Tensor products

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA
Realization of HV

Fusion rules and tensor product modules

Irreducibility of a tensor product
Using  $\Omega$ Using  $\Lambda$ More fusion rules

Free-field realization

### Corollary

Let  $(h, h_l) = (\Delta_{r_1, s_1}, r_1 - s_1), (h', h'_l) = (\Delta_{r_2, s_2}, r_2 - s_2) \in \mathbb{C}^2$  and that there are  $p, q \in \mathbb{Z}_{>0}, q \leq p$  such that

$$\frac{h_l}{c_{L,l}} - 1 = -q, \quad \frac{h'_l}{c_{L,l}} - 1 = p.$$

Then there is a non-trivial intertwining operator of the type

$$\begin{pmatrix} L^{\mathcal{H}}(c_{L},0,c_{L,I},h'',h_{I}+h'_{I}) \\ L^{\mathcal{H}}(c_{L},0,c_{L,I},h,h_{I}) & L^{\mathcal{H}}(c_{L},0,c_{L,I},h',h'_{I}) \end{pmatrix}$$

where  $h'' = \Delta_{r_2 - r_1, s_2 - s_1}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tauran ...............................

lensor products

realization Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV Singular vectors

Fusion rules and tensor product modules

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization

### Corollary

Let  $(h, h_I) = (\Delta_{r_1, s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2, s_2}, r_2 - s_2) \in \mathbb{C}^2$  and that there are  $p, q \in \mathbb{Z}_{>0}, q \leq p$  such that

$$\frac{h_l}{c_{L,l}} - 1 = -q, \quad \frac{h'_l}{c_{L,l}} - 1 = p.$$

Then there is a non-trivial intertwining operator of the type

$$\begin{pmatrix} L^{\mathcal{H}}(c_{L},0,c_{L,I},h'',h_{I}+h'_{I}) \\ L^{\mathcal{H}}(c_{L},0,c_{L,I},h,h_{I}) & L^{\mathcal{H}}(c_{L},0,c_{L,I},h',h'_{I}) \end{pmatrix}$$

where  $h'' = \Delta_{r_2-r_1,s_2-s_1}$ .

In particular, the  $\mathcal{H}$ -module  $V'_{\alpha,\beta,F}\otimes L^{\mathcal{H}}(c_L,0,c_{L,I},h',h''_I)$  is reducible where

$$\alpha = h + h' - h'', \ \beta = 1 - h, \ F = h_I.$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

<u>.</u> . .

Tensor products

realization Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

Fusion rules and tensor product modules

rreducibility of a ensor product Using Ω

Free-field realization

10.14.17.17.7.000

Next we use formulas for  $\Omega$  and  $\Lambda$  to get irreducibility criterion for  $V'_{\alpha,\beta,F}\otimes L\left(c_{L},0,c_{L,I},h,h_{I}\right)$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Intermediate serie

ensor products

-ree field ealization Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

ingular vectors usion rules and nsor product

Irreducibility of a tensor product

Using Ω
Using Λ
More fusion rules

Free-field realization

4 D L 4 D L 4 E L 4 E L 5 00 C

- Next we use formulas for  $\Omega$  and  $\Lambda$  to get irreducibility criterion for  $V'_{\alpha,\beta,F}\otimes L\left(c_{L},0,c_{L,I},h,h_{I}\right)$ .
- R. Lu and K. Zhao introduced a useful criterion:

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

intermediate seri

ensor products

Free field realization

Heisenberg-Virasoro VOA

leisenberg VOA lealization of HV ingular vectors

ısion rules an nsor product odules

Irreducibility of a tensor product

Using A More fusion rul

Free-field realization W(2-2)

- Next we use formulas for  $\Omega$  and  $\Lambda$  to get irreducibility criterion for  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$ .
- R. Lu and K. Zhao introduced a useful criterion:
- ▶ Define a linear map  $\phi_n:U(\mathcal{H}_-) o \mathbb{C}$

$$\begin{aligned} \phi_n(1) &= 1 \\ \phi_n(I(-i)u) &= -F\phi_n(u) \\ \phi_n(L(-i)u) &= (\alpha + \beta + k + i + n - i\beta)\phi_n(u) \end{aligned}$$

for  $u \in U(\mathcal{H}_{-})_{-k}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

intermediate sent

Tensor products

Free field realization

Heisenberg-Virasoro VOA

Realization of H Singular vectors

usion rules and ensor product

Irreducibility of a tensor product

Using  $\Omega$ Using  $\Lambda$ More fusion n

Free-field realization

- Next we use formulas for  $\Omega$  and  $\Lambda$  to get irreducibility criterion for  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$ .
- R. Lu and K. Zhao introduced a useful criterion:
- ▶ Define a linear map  $\phi_n:U(\mathcal{H}_-) o \mathbb{C}$

$$\begin{aligned} \phi_n(1) &= 1 \\ \phi_n(I(-i)u) &= -F\phi_n(u) \\ \phi_n(L(-i)u) &= (\alpha + \beta + k + i + n - i\beta)\phi_n(u) \end{aligned}$$

for  $u \in U(\mathcal{H}_{\underline{\ }})_{-k}$ .

▶  $V'_{\alpha,\beta,F} \otimes L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$  is irreducible if and only if  $\phi_n(\Omega) \neq 0$  (i.e.  $\phi_n(\Lambda) \neq 0$ ) for every  $n \in \mathbb{Z}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Tensor products

realization

VOA Heisenberg VOA

Singular vectors

ensor product nodules

Irreducibility of a tensor product

Using Ω Using Λ More fusion r

Free-field realization



▶ If  $p = \frac{h_l}{c_{l,l}} - 1 \in \mathbb{Z}_{>0}$ , then for every  $n \in \mathbb{Z}$  we have

$$\phi_n(\Omega) = (-1)^p \binom{-\frac{F}{c_{L,l}}}{p}.$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Intermediate series

Tensor product

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA

VOA Heisenberg VOA Realization of HV Singular vectors

ensor product nodules reducibility of a

Irreducibility of a tensor product
Using Ω

Free-field realization (

4 D L 4 D L 4 E L 4 E L 500 C

▶ If  $p = \frac{h_l}{c_{l,l}} - 1 \in \mathbb{Z}_{>0}$ , then for every  $n \in \mathbb{Z}$  we have

$$\phi_n(\Omega) = (-1)^p \begin{pmatrix} -\frac{F}{c_{L,l}} \\ p \end{pmatrix}.$$

#### Theorem

Let  $p = \frac{h_l}{c_{L,l}} - 1 \in \mathbb{Z}_{>0}$ . Module  $V'_{\alpha,\beta,F} \otimes L^{\mathcal{H}}(c_L, 0, c_{L,l}, h, h_l)$  is irreducible if and only if  $F \neq (i-p)c_{L,l}$ , for  $i=1,\ldots,p$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

\_ .

Tensor products

realization
Heisenberg-Virasoro

VOA
Heisenberg VOA
Realization of HV
Singular vectors

-usion ruies an ensor product nodules

Irreducibility of a tensor product Using Ω

▶ If  $p = \frac{h_l}{c_{l,l}} - 1 \in \mathbb{Z}_{>0}$ , then for every  $n \in \mathbb{Z}$  we have

$$\phi_n(\Omega) = (-1)^p \begin{pmatrix} -\frac{F}{c_{L,l}} \\ p \end{pmatrix}.$$

#### Theorem

Let  $p = \frac{h_I}{c_{L,I}} - 1 \in \mathbb{Z}_{>0}$ . Module  $V'_{\alpha,\beta,F} \otimes L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$  is irreducible if and only if  $F \neq (i - p)c_{L,I}$ , for  $i = 1, \ldots, p$ .

► This expands the list of reducible tensor products realized with intertwining operators.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization
Heisenberg-Virasoro

Heisenberg VOA Realization of HV Singular vectors

ensor product nodules

Irreducibility of a tensor product Using Ω

Free-field realization

401451451 5 000

▶ If  $\frac{h_l}{c_{l,l}} - 1 = -p \in -\mathbb{Z}_{>0}$ , then for every  $n \in \mathbb{Z}$  we have

$$\phi_{n}(\Lambda) = (-1)^{p-1} \binom{F/c_{L,l} - 1}{p-1} (\alpha + n + \beta) + (-1)^{p-1} (1 - \beta) \binom{F/c_{L,l} - 2}{p-1} + g_{p}(F)$$

for a certain polynomial  $g_p \in \mathbb{C}[x]$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

....

Tensor products

realization
Heisenberg-Virasoro

VOA
Heisenberg VOA

Realization of HV Singular vectors

ensor product nodules

Irreducibility of a tensor product
Using Ω
Using Λ
More fusion rules



▶ If  $\frac{h_l}{c_{l,l}} - 1 = -p \in -\mathbb{Z}_{>0}$ , then for every  $n \in \mathbb{Z}$  we have

$$\phi_{n}(\Lambda) = (-1)^{p-1} \binom{F/c_{L,I} - 1}{p-1} (\alpha + n + \beta) + (-1)^{p-1} (1 - \beta) \binom{F/c_{L,I} - 2}{p-1} + g_{p}(F)$$

for a certain polynomial  $g_p \in \mathbb{C}[x]$ .

If  $F/c_{L,l} \notin \{1, \ldots, p-1\}$ , then for every  $n \in \mathbb{Z}$  there is a unique  $\alpha := \alpha_n \in \mathbb{C}$  such that  $\phi_n(\Lambda) = 0$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

intermediate sene

Tensor products

realization Heisenberg-Virasoro

VOA
Heisenberg VOA
Realization of HV

Fusion rules and tensor product

Irreducibility of a tensor product Using Ω Using Λ More fusion rules

▶ If  $\frac{h_l}{c_{l,l}} - 1 = -p \in -\mathbb{Z}_{>0}$ , then for every  $n \in \mathbb{Z}$  we have

$$\phi_{n}(\Lambda) = (-1)^{p-1} \binom{F/c_{L,I} - 1}{p-1} (\alpha + n + \beta) + (-1)^{p-1} (1 - \beta) \binom{F/c_{L,I} - 2}{p-1} + g_{p}(F)$$

for a certain polynomial  $g_p \in \mathbb{C}[x]$ .

- ▶ If  $F/c_{L,I} \notin \{1, ..., p-1\}$ , then for every  $n \in \mathbb{Z}$  there is a unique  $\alpha := \alpha_n \in \mathbb{C}$  such that  $\phi_n(\Lambda) = 0$ .
- ► This, along with previous results on existence of intertwining operators result with the following:

Introduction

The twisted Heisenberg-Virasoro algebra

modules

<u>.</u> . . .

Tensor products

realization Heisenberg-Virasoro

VOA
Heisenberg VOA
Realization of HV

Fusion rules and tensor product modules

Irreducibility of a tensor product Using  $\Omega$ Using  $\Lambda$ 



#### **Theorem**

Let  $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ . We write V short for  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$ .

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

intermediate seri

Tensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

Singular vectors usion rules and ensor product

Irreducibility of a tensor product Using Ω Using Λ

Free-field realization

4 D L 4 D L 4 E L 4 E L 500 C

#### **Theorem**

Let  $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ . We write V short for  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$ .

(i) Let  $F/c_{L,l} \notin \{1,\ldots,p-1\}$  and let  $\alpha_0 \in \mathbb{C}$  be such that  $\phi_0(\Lambda) = 0$ . Then V is reducible if and only if  $\alpha \equiv \alpha_0 \mod \mathbb{Z}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

....

ensor products

realization
Heisenberg-Virasoro

Heisenberg-Virasoro VOA Heisenberg VOA

ealization of HV ingular vectors

ensor product odules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$  More fusion rules

#### Theorem

Let  $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ . We write V short for  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$ .

(i) Let  $F/c_{L,I} \notin \{1,\ldots,p-1\}$  and let  $\alpha_0 \in \mathbb{C}$  be such that  $\phi_0(\Lambda)=0$ . Then V is reducible if and only if  $\alpha\equiv\alpha_0$  mod  $\mathbb{Z}$ . In this case  $W^0=U(\mathcal{H})(v_0\otimes v)$  is irreducible submodule of V and  $V/W^0$  is a highest weight  $\mathcal{H}$ -module  $\widetilde{L}(c_L,0,c_{L,I},h'',h''_I)$  (not necessarily irreducible) where

$$h'' = -\alpha_0 + h + (1 - \beta), \qquad h''_l = F + h_l.$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization

Heisenberg-Virasoro VOA Heisenberg VOA

Realization of HV Singular vectors

usion rules an ensor product nodules

Irreducibility of a tensor product Using Ω Using Λ

#### Theorem

Let  $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ . We write V short for  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$ .

(i) Let  $F/c_{L,I} \notin \{1,\ldots,p-1\}$  and let  $\alpha_0 \in \mathbb{C}$  be such that  $\phi_0(\Lambda) = 0$ . Then V is reducible if and only if  $\alpha \equiv \alpha_0 \mod \mathbb{Z}$ . In this case  $W^0 = U(\mathcal{H})(v_0 \otimes v)$  is irreducible submodule of V and  $V/W^0$  is a highest weight  $\mathcal{H}$ -module  $\widetilde{L}(c_L,0,c_{L,I},h'',h''_I)$  (not necessarily irreducible) where

$$h'' = -\alpha_0 + h + (1 - \beta), \qquad h''_I = F + h_I.$$

(ii) Let  $F/c_{L,I} \in \{2, \ldots, p-1\}$ . Then V is reducible.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

....

ensor products

realization

VOA

Heisenberg VOA

Realization of HV Singular vectors

usion rules an ensor product 10dules

Irreducibility of a tensor product

Using Ω
Using Λ
More fusion

#### Theorem

Let  $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ . We write V short for  $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h,h_I)$ .

(i) Let  $F/c_{L,I} \notin \{1,\ldots,p-1\}$  and let  $\alpha_0 \in \mathbb{C}$  be such that  $\phi_0(\Lambda)=0$ . Then V is reducible if and only if  $\alpha\equiv\alpha_0$  mod  $\mathbb{Z}$ . In this case  $W^0=U(\mathcal{H})(v_0\otimes v)$  is irreducible submodule of V and  $V/W^0$  is a highest weight  $\mathcal{H}$ -module  $\widetilde{L}(c_L,0,c_{L,I},h'',h''_I)$  (not necessarily irreducible) where

$$h'' = -\alpha_0 + h + (1 - \beta), \qquad h''_I = F + h_I.$$

(ii) Let  $F/c_{L,l} \in \{2, \ldots, p-1\}$ . Then V is reducible.

(iii) Let p>1 and  $F/c_{L,I}=1$ . Then V is reducible if and only if  $1-\beta=\frac{c_L-2}{24}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization

Heisenberg-Virasoro VOA Heisenberg VOA

Realization of HV Singular vectors

usion rules and ensor product nodules

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization of W(2.2)

40.40.41.41.1.1.000

#### Theorem

Let  $(h, h_I) = (\Delta_{r_1,s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2,s_2}, r_2 - s_2)$  such that

$$rac{h_I}{c_{L,I}}-1=p, \ rac{h_I'}{c_{L,I}}-1=q, \quad p,q\in\mathbb{Z}\setminus\{0\}.$$

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

Intermediate series

Free field realization Heisenberg-Virasoro VOA

Heisenberg-Virasoro /OA Heisenberg VOA Realization of HV Singular vectors

educibility of a

tensor product
Using Ω
Using Λ
More fusion rules

Free-field realization of

4 D N 4 D N 4 D N 1 D N

#### Theorem

Let  $(h,h_I)=(\Delta_{r_1,s_1},r_1-s_1), (h',h_I')=(\Delta_{r_2,s_2},r_2-s_2)$  such that

$$\frac{h_I}{c_{L,I}}-1=p,\ \frac{h_I'}{c_{L,I}}-1=q,\quad p,q\in\mathbb{Z}\setminus\{0\}.$$

Let

$$d = \dim I \begin{pmatrix} L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h'', h''_{I}) \\ L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, h_{I}) & L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h', h'_{I}) \end{pmatrix}.$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

iliterillediate serie

Tensor product

Free field

realization Heisenberg-Virasoro VOA Heisenberg VOA Realization of HV

ingular vectors usion rules and nsor product

Irreducibility of a tensor product Using Ω Using Λ

More fusion rules Free-field realization of

#### Theorem

Let  $(h,h_I)=(\Delta_{r_1,s_1},r_1-s_1), (h',h_I')=(\Delta_{r_2,s_2},r_2-s_2)$  such that

$$\frac{h_I}{c_{L,I}}-1=p,\ \frac{h_I'}{c_{L,I}}-1=q,\quad p,q\in\mathbb{Z}\setminus\{0\}.$$

Let

$$d = \dim I \begin{pmatrix} L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h'', h''_{I}) \\ L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, h_{I}) & L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h', h'_{I}) \end{pmatrix}.$$

Then d = 1 if and only if  $h''_l = h_l + h'_l$  and one of the following holds:

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor product

realization Heisenberg-Virasoro VOA Heisenberg VOA

Realization of HV Singular vectors Fusion rules and

Irreducibility of a tensor product Using  $\Omega$ 

More fusion rules Free-field realization of

#### Theorem

Let  $(h,h_I)=(\Delta_{r_1,s_1},r_1-s_1), (h',h_I')=(\Delta_{r_2,s_2},r_2-s_2)$  such that

$$\frac{h_I}{c_{L,I}}-1=p,\ \frac{h_I'}{c_{L,I}}-1=q,\quad p,q\in\mathbb{Z}\setminus\{0\}.$$

Let

$$d = \dim I \begin{pmatrix} L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h'', h''_{I}) \\ L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, h_{I}) & L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h', h'_{I}) \end{pmatrix}.$$

Then d = 1 if and only if  $h''_l = h_l + h'_l$  and one of the following holds:

(i) 
$$p, q < 0$$
 and  $h'' = \Delta_{r_1 + r_2, s_1 + s_2}$ 

Introduction

The twisted Heisenberg-Virasoro algebra

modules

realization

Heisenberg-Virasoro
VOA

Realization of HV Singular vectors Fusion rules and

Irreducibility of a tensor product Using Ω

More fusion rules

#### Theorem

Let  $(h, h_I) = (\Delta_{r_1,s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2,s_2}, r_2 - s_2)$  such that

$$rac{h_I}{c_{L,I}}-1=p, \ rac{h_I'}{c_{L,I}}-1=q, \quad p,q\in\mathbb{Z}\setminus\{0\}.$$

Let

$$d = \dim I \left( \frac{L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h'', h''_{I})}{L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, h_{I}) - L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h', h'_{I})} \right).$$

Then d = 1 if and only if  $h''_l = h_l + h'_l$  and one of the following holds:

- (i) p, q < 0 and  $h'' = \Delta_{r_1 + r_2, s_1 + s_2}$
- (ii)  $1 \leq -p \leq q$  and  $h'' = \Delta_{r_2-r_1,s_2-s_1}$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor product

realization Heisenberg-Virasoro VOA Heisenberg VOA Realization of HV

usion rules and ensor product

Irreducibility of a tensor product Using Ω Using A More fusion rules

#### Theorem

Let  $(h,h_I)=(\Delta_{r_1,s_1},r_1-s_1), (h',h_I')=(\Delta_{r_2,s_2},r_2-s_2)$  such that

$$\frac{h_I}{c_{L,I}} - 1 = p, \ \frac{h_I'}{c_{L,I}} - 1 = q, \quad p, q \in \mathbb{Z} \setminus \{0\}.$$

Let

$$d = \dim I \begin{pmatrix} L^{\mathcal{H}}(c_L, 0, c_{L,I}, h'', h''_I) \\ L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I) & L^{\mathcal{H}}(c_L, 0, c_{L,I}, h', h'_I) \end{pmatrix}.$$

Then d = 1 if and only if  $h''_l = h_l + h'_l$  and one of the following holds:

- (i) p, q < 0 and  $h'' = \Delta_{r_1 + r_2, s_1 + s_2}$
- (ii)  $1 \leq -p \leq q$  and  $h'' = \Delta_{r_2-r_1,s_2-s_1}$
- (iii)  $1 \le -q \le p$  and  $h'' = \Delta_{r_1 r_2, s_1 s_2}$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor product

realization Heisenberg-Virasoro VOA Heisenberg VOA Realization of HV

usion rules and ensor product modules

Irreducibility of a tensor product Using Ω Using A More fusion rules

#### Theorem

Let  $(h,h_I)=(\Delta_{r_1,s_1},r_1-s_1), (h',h_I')=(\Delta_{r_2,s_2},r_2-s_2)$  such that

$$rac{h_I}{c_{L,I}}-1=p, \ rac{h_I'}{c_{L,I}}-1=q, \quad p,q\in\mathbb{Z}\setminus\{0\}.$$

Let

$$d = \dim I \begin{pmatrix} L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h'', h''_{I}) \\ L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, h_{I}) & L^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h', h'_{I}) \end{pmatrix}.$$

Then d = 1 if and only if  $h''_l = h_l + h'_l$  and one of the following holds:

- (i) p,q < 0 and  $h'' = \Delta_{r_1 + r_2,s_1 + s_2}$
- (ii)  $1 \leq -p \leq q$  and  $h'' = \Delta_{r_2-r_1,s_2-s_1}$
- (iii)  $1 \le -q \le p$  and  $h'' = \Delta_{r_1 r_2, s_1 s_2}$ d = 0 otherwise.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

lensor product

realization Heisenberg-Virasoro VOA Heisenberg VOA

Singular vectors Fusion rules and ensor product

Irreducibility of a tensor product Using Ω Using Λ More fusion rules

## Nontrivial intertwining operators

$$\begin{pmatrix} (\Delta_{r_1+r_2,s_1+s_2}, (1-(p+q-1))c_{L,I}) \\ (\Delta_{r_1,s_1}, (1-p)c_{L,I}) & (\Delta_{r_2,s_2}, (1-q)c_{L,I}) \end{pmatrix}$$
 for  $p, q \ge 1$ 

$$\begin{pmatrix} (\Delta_{r_2-r_1,s_2-s_1}, (1+(q-p+1))c_{L,I}) \\ (\Delta_{r_1,s_1}, (1-p)c_{L,I}) & (\Delta_{r_2,s_2}, (1+q)c_{L,I}) \end{pmatrix}$$
 for  $1 \le -p \le q$ 

$$\begin{pmatrix} (\Delta_{r_1-r_2,s_1-s_2}, (1+(p-q+1))c_{L,I}) \\ (\Delta_{r_1,s_1}, (1+p)c_{L,I}) & (\Delta_{r_2,s_2}, (1-q)c_{L,I}) \end{pmatrix}$$
 for  $1 \le -q \le p$ 

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization
Heisenberg-Virasoro

Heisenberg VOA
Realization of HV
Singular vectors

usion rules and ensor product

Irreducibility of a tensor product Using Ω

More fusion rules

▶ Vertex-algebra  $L^W(c_L, c_W)$  is generated by fields

$$Y(L(-2),z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$
  
 $Y(W(-2),z) = \sum_{n\in\mathbb{Z}} W(n) z^{-n-2}.$ 

Introduction

The twisted Heisenberg-Virasoro algebra

modules

iliterillediate serie

Tensor product

realization
Heisenberg-Virasoro
VOA

VOA Heisenberg VOA Realization of HV Singular vectors

usion rules and

Irreducibility of a tensor product Using  $\Omega$ 

Free-field realization of W(2,2)

40.40.45.45. 5 000

▶ Vertex-algebra  $L^W(c_L, c_W)$  is generated by fields

$$Y(L(-2),z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$
  
 $Y(W(-2),z) = \sum_{n\in\mathbb{Z}} W(n) z^{-n-2}.$ 

▶ Vertex-algebra  $L^{\mathcal{H}}(c_L, c_{L,I})$  is generated by fields

$$Y(L(-2),z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2},$$
  
 $Y(I(-1),z) = \sum_{n \in \mathbb{Z}} I(n) z^{-n-1}.$ 

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor product

realization Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

Fusion rules ar ensor product

Irreducibility of a tensor product

Using Ω Using Λ More fusion ru

#### Theorem

There is a non-trivial homomorphism of vertex algebras

$$\Psi: L^{W}(c_{L}, c_{W}) \to L^{\mathcal{H}}(c_{L}, c_{L,I})$$

$$L(-2) \mapsto L(-2) \mathbf{1}$$

$$W(-2) \mapsto (I^{2}(-1) + 2c_{L,I}I(-2))\mathbf{1}$$

where

$$c_W = -24c_{L,I}^2.$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

\_ . .

Free field

Heisenberg-Virasoro VOA Heisenberg VOA

Heisenberg VOA Realization of HV Singular vectors

Fusion rules and tensor product modules

Irreducibility of a tensor product Using Ω

Free-field realization of W(2,2)

4 D L 4 D L 4 E L 4 E L 5 000

► Every  $L^{\mathcal{H}}(c_L, c_{L,I})$ -module becomes a  $L^{\mathcal{W}}(c_L, c_{\mathcal{W}})$ -module.

Introduction

The twisted Heisenberg-Virasoro algebra

Structure of Verma modules

intermediate sen

Tensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV Singular vectors

Singular vectors usion rules and ensor product

Irreducibility of a tensor product Using  $\Omega$  Using  $\Lambda$ 

Free-field realization of W(2,2)

4 D N 4 D N 4 D N 1 D N

- ► Every  $L^{\mathcal{H}}(c_L, c_{L,I})$ -module becomes a  $L^{W}(c_L, c_W)$ -module.
- ▶  $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$  is a  $L^W(c_L, c_W)$ -module and  $v_{h,h_I}$  is a W(2,2) highest weight vector such that

$$L(0)v_{h,h_I} = hv_{h,h_I}, \quad W(0)v_{h,h_I} = h_Wv_{h,h_I}$$
  
where  $h_W = h_I(h_I - 2c_{I-I}).$ 

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization

Heisenberg-Virasoro VOA Heisenberg VOA

Realization of HV

ensor product nodules

Irreducibility of a tensor product Using Ω Using Λ

- ► Every  $L^{\mathcal{H}}(c_L, c_{L,I})$ -module becomes a  $L^{\mathcal{W}}(c_L, c_{\mathcal{W}})$ -module.
- ▶  $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$  is a  $L^W(c_L, c_W)$ -module and  $v_{h,h_I}$  is a W(2,2) highest weight vector such that

$$L(0)v_{h,h_I} = hv_{h,h_I}, \quad W(0)v_{h,h_I} = h_Wv_{h,h_I}$$

where  $h_W = h_I (h_I - 2c_{L,I})$ .

▶ There is a nontrivial W(2,2)-homomorphism

$$\Psi: V^{W(2,2)}(c, c_W, h, h_W) \to V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization

Heisenberg-Virasoro VOA Heisenberg VOA

Realization of HV Singular vectors

ensor product nodules

Irreducibility of a tensor product Using Ω Using Λ

### Example

Let  $h_W=\frac{1-p^2}{24}c_W=(p^2-1)c_{L,I}^2=h_I(h_I-2c_{L,I})$  as above. Then there are nontrivial W(2,2)-homomorphisms

$$V^{W(2,2)}(c,c_W,h,\frac{1-p^2}{24}c_W)$$

$$V^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, (1+p) c_{L,I}) \qquad V^{\mathcal{H}}(c_{L}, 0, c_{L,I}, h, (1-p) c_{L,I})$$

Introduction

The twisted Heisenberg-Virasoro algebra

modules

intermediate seri

Tensor products

Free field realization Heisenberg-Virasoro

VOA
Heisenberg VOA
Realization of HV
Singular vectors

Fusion rules and tensor product

rreducibility of a ensor product Using Ω Using Λ More fusion rules

#### **Theorem**

(i) Let  $\frac{h_l}{c_{L,l}} - 1 \notin -\mathbb{Z}_{>0}$ . Then  $\Psi$  is an isomorphism of W(2,2)-modules.

Introduction

The twisted Heisenberg-Virasoro algebra

modules

intermediate serie

Tensor product

realization
Heisenberg-Virasoro
VOA
Heisenberg VOA

Heisenberg VOA Realization of HV Singular vectors

odules

Using Ω
Using Λ
More fusion rules

Free-field realization of W(2,2)

4 D N 4 D N 4 D N 1 D N

### Theorem

- (i) Let  $\frac{h_l}{c_{L,l}} 1 \notin -\mathbb{Z}_{>0}$ . Then  $\Psi$  is an isomorphism of W(2,2)-modules.
- (ii) If  $\frac{h_l}{c_{l,l}} 1 = p \in \mathbb{Z}_{>0}$  then

$$\Psi^{-1}\left(S_p\left(-\frac{I(-1)}{c_{L,I}},-\frac{I(-2)}{c_{L,I}},\cdots\right)v_{h,h_I}\right)=u'$$

is a singular vector in  $V^{W(2,2)}(c_L, c_W, h, h_W)_{h+p}$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

Tensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV

sion rules an

Irreducibility of a tensor product Using  $\Omega$ 

### Theorem

- (i) Let  $\frac{h_l}{c_{L,l}} 1 \notin -\mathbb{Z}_{>0}$ . Then  $\Psi$  is an isomorphism of W(2,2)-modules.
- (ii) If  $\frac{h_l}{c_{l,l}}-1=p\in\mathbb{Z}_{>0}$  then

$$\Psi^{-1}\left(S_p\left(-\frac{I(-1)}{c_{L,I}},-\frac{I(-2)}{c_{L,I}},\cdots\right)v_{h,h_I}\right)=u'$$

is a singular vector in  $V^{W(2,2)}(c_L,c_W,h,h_W)_{h+p}$ .

(iii) If 
$$\frac{h_l}{c_{l,l}} - 1 = -p \in -\mathbb{Z}_{>0}$$
 then  $\Psi\left(u'\right) = 0$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization
Heisenberg-Virasoro

VOA Heisenberg VOA Realization of HV Singular vectors

usion rules ar ensor product

Irreducibility of a tensor product
Using Ω

#### Theorem

- (i) Let  $\frac{h_l}{c_{L,l}} 1 \notin -\mathbb{Z}_{>0}$ . Then  $\Psi$  is an isomorphism of W(2,2)-modules.
- (ii) If  $\frac{h_l}{c_{l,l}}-1=p\in\mathbb{Z}_{>0}$  then

$$\Psi^{-1}\left(S_p\left(-\frac{I(-1)}{c_{L,I}},-\frac{I(-2)}{c_{L,I}},\cdots\right)v_{h,h_I}\right)=u'$$

is a singular vector in  $V^{W(2,2)}(c_L, c_W, h, h_W)_{h+p}$ .

(iii) If 
$$\frac{h_l}{c_{l,l}}-1=-p\in -\mathbb{Z}_{>0}$$
 then  $\Psi\left(u'\right)=0$ .

(iv) Let  $\frac{h_l}{c_{l,l}} - 1 = -p \in -\mathbb{Z}_{>0}$  and let u be a subsingular vector in  $V^{W(2,2)}\left(c_{L},c_{W},h_{pq},h_{W}\right)_{h+pq}$ . Then  $\Psi\left(u\right)$  is a singular vector in  $V^{\mathcal{H}}(c_{L},0,c_{L,l},h,(1-p)c_{L,l})$ .

Introduction

The twisted Heisenberg-Virasoro algebra

modules

ensor products

realization

Heisenberg-Virasoro

Heisenberg VOA
Realization of HV
Singular vectors

usion rules and ensor product

Irreducibility of a tensor product

Using Ω
Using Λ
More fusion

