Representations of some affine vertex algebras at negative integer levels

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(joint work with Dražen Adamović)

supported by CSF, grant 2634

References:

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D. Adamović, O. P., *Fusion rules and complete reducibility of certain modules for affine Lie algebras*, Journal of Algebra and Its Applications 13 (2014) 18pp.

Vertex operator algebras associated to affine Lie algebras

For a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and $k \in \mathbb{C}$, $k \neq -h^\vee$, denote by $N_\mathfrak{g}(k,0)$ the universal affine vertex operator algebra of level k and by $L_\mathfrak{g}(k,0)$ the associated simple quotient.

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In particular, we will consider the case of orthogonal Lie algebra of type D.

Vertex operator algebra associated to D_ℓ of level $n-\ell+1$

Theorem

Vector

$$v_n = \Big(\sum_{i=2}^{\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1)\Big)^n \mathbf{1}$$

is a singular vector in $N_{D_\ell}(n-\ell+1,0)$*, for any* $n \in \mathbb{Z}_{>0}$.

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In the case $n=1$, we obtain the singular vector

$$v = \sum_{i=2}^{\ell} e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1)\mathbf{1}$$

in $N_{D_\ell}(-\ell+2, 0)$.

Vertex operator algebra associated to D_ℓ of level $-\ell + 2$

We will consider representations of quotient vertex operator algebra

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Using Zhu's theory, we obtain the following classification result:

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The set

$$\{L_{D_\ell}(-\ell+2, t\omega_{\ell-1}), L_{D_\ell}(-\ell+2, t\omega_\ell) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides the complete list of irreducible ordinary $\mathcal{V}_{D_\ell}(-\ell+2,0)$*–modules.*

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Remark: More generally, we can obtain the classification of irreducible weak $\mathcal{V}_{D_\ell}(-\ell+2,0)$–modules from the category $\mathcal{O}$.

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Natural questions:

1. Is $< v >$ a maximal submodule in $N_{D_\ell}(-\ell+2,0)$?
2. Representation theory of $L_{D_\ell}(-\ell+2,0)$?

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We give an answer in some special cases.

Case $\ell = 4$

Denote by θ the automorphism of $N_{D_4}(-2,0)$ induced by the automorphism of the Dynkin diagram of D_4 of order three.

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is a singular vector in $N_{D_4}(-2,0)$, it follows that ($\theta(v) =$)

$$(e_{\epsilon_3-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1)-e_{\epsilon_2-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1)+e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_1-\epsilon_4}(-1))\mathbf{1},$$

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and $(\theta^2(v) =)$

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are also singular vectors in $N_{D_4}(-2,0)$. We consider the associated quotient vertex operator algebra

$$\widetilde{L}_{D_4}(-2,0) = \frac{N_{D_4}(-2,0)}{< v, \theta(v), \theta^2(v) >}.$$

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Theorem

The set

$$\{L_{D_4}(-2,0), L_{D_4}(-2,-2\omega_1), L_{D_4}(-2,-2\omega_3),$$
$$L_{D_4}(-2,-2\omega_4), L_{D_4}(-2,-\omega_2)\}$$

provides a complete list of irreducible weak $\widetilde{L}_{D_4}(-2,0)$*–modules from the category* $\mathcal{O}$*.*
In particular, $L_{D_4}(-2,0)$ *is the unique irreducible ordinary module for* $\widetilde{L}_{D_4}(-2,0)$*.*

Case $\ell = 4$

It follows immediately that:

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Theorem

Vertex operator algebra $\widetilde{L}_{D_4}(-2,0)$ *is simple, i.e.*

$$L_{D_4}(-2,0) = \frac{N_{D_4}(-2,0)}{< v, \theta(v), \theta^2(v) >}.$$

These results were recently generalized by Arakawa and Moreau.

Case ℓ odd

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Let

$$U_{D_\ell}(t) := \begin{cases} V_{D_\ell}(t\omega_{\ell-1}), & \text{for } t \geq 0 \\ V_{D_\ell}(-t\omega_\ell), & \text{for } t < 0. \end{cases}$$

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Tensor products of these modules have been described by S. Okada:

Theorem

Assume that $\ell \geq 3$ is an odd natural number. Assume that $r, s \in \mathbb{Z}$. Then $U_{D_\ell}(t)$ appears in the tensor product $U_{D_\ell}(r) \otimes U_{D_\ell}(s)$ if and only if $t = r + s$. The multiplicity is one.

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Theorem

Assume that $\ell \geq 3$ is an odd natural number and that $\widetilde{\pi}_r$, $r \in \mathbb{Z}$ are $\mathbb{Z}_{\geq 0}$–graded $\mathcal{V}_{D_\ell}((-\ell+2)\Lambda_0)$-modules such that top component of $\widetilde{\pi}_r$ is isomorphic to the irreducible $\mathfrak{g}_{D_\ell}$–module $U_{D_\ell}(r)$. Let π_r denote the associated simple quotient. Assume that there is a non-trivial intertwining operator of type

$$\begin{pmatrix} & \pi_t & \\ \widetilde{\pi}_r & & \pi_s \end{pmatrix}.$$

Then $t = r + s$.

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provides the complete list of irreducible ordinary $L_{D_\ell}(-\ell+2, 0)$–modules.

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Question: Can we construct nontrivial intertwining operators?

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provides the complete list of irreducible ordinary $L_{D_\ell}(-\ell+2, 0)$–modules.

Question: Can we construct nontrivial intertwining operators?
We give an answer in the case $\ell = 5$.

Conformal embedding of $L_{D_5}(-3,0) \otimes M(1)$ into $L_{E_6}(-3,0)$

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Let $\mathfrak{g}_{E_6}$ be the simple Lie algebra of type E_6. The subalgebra of $\mathfrak{g}_{E_6}$ generated by positive root vectors

$$e_{(5)} = e_{\frac{1}{2}(\epsilon_8-\epsilon_7-\epsilon_6+\epsilon_5-\epsilon_4-\epsilon_3-\epsilon_2-\epsilon_1)},\, e_{\alpha_2} = e_{\epsilon_2+\epsilon_1},$$
$$e_{\alpha_4} = e_{\epsilon_3-\epsilon_2},\, e_{\alpha_3} = e_{\epsilon_2-\epsilon_1},\, e_{\alpha_5} = e_{\epsilon_4-\epsilon_3}$$

and associated negative root vectors is a simple Lie algebra $\mathfrak{g}_{D_5}$ of type D_5.

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$$e_{\alpha_4} = e_{\epsilon_3-\epsilon_2}, e_{\alpha_3} = e_{\epsilon_2-\epsilon_1}, e_{\alpha_5} = e_{\epsilon_4-\epsilon_3}$$

and associated negative root vectors is a simple Lie algebra $\mathfrak{g}_{D_5}$ of type D_5. Thus, $\mathfrak{g}_{E_6}$ has a reductive subalgebra $\mathfrak{g}_{D_5} \oplus \mathfrak{h}$, where $\mathfrak{h} = \mathbb{C}H$, and

$$H = \frac{1}{3}(h_8 - h_7 - h_6 - 3h_5)$$

(where h_i are determined by $\epsilon_i(h_j) = \delta_{ij}$).

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It follows that we have an embedding $N_{D_5}(-3,0) \otimes M(1)$ into $N_{E_6}(-3,0)$, where $M(1)$ denotes the Heisenberg vertex subalgebra generated by H. Moreover, the singular vector in this copy of $N_{D_5}(-3,0)$:

$$v = (e_{(5)}(-1)e_{(12345)}(-1) + e_{(125)}(-1)e_{(345)}(-1) + e_{(135)}(-1)e_{(245)}(-1) + e_{(235)}(-1)e_{(145)}(-1))\mathbf{1}$$

is also a singular vector for $\hat{\mathfrak{g}}_{E_6}$ in $N_{E_6}(-3,0)$.

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Denote by $\widetilde{L}_{D_5}(-3,0)$ the subalgebra of $L_{E_6}(-3,0)$ generated by $\mathfrak{g}_{D_5}$.

Conformal embedding of $L_{D_5}(-3,0) \otimes M(1)$ into $L_{E_6}(-3,0)$

The criterion for conformal embeddings from D. Adamović, O. P., Algebr. Represent. Theory (2013) gives:

Theorem

Vertex operator algebra $\widetilde{L}_{D_5^{(1)}}(-3,0) \otimes M(1)$ *is conformally embedded in* $L_{E_6^{(1)}}(-3,0)$.

Conformal embedding of $L_{D_5}(-3,0) \otimes M(1)$ into $L_{E_6}(-3,0)$

Now, the results on fusion rules give the following decomposition:

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We have:

$$L_{E_6}(-3,0) \cong \bigoplus_{t \in \mathbb{Z}_{\geq 0}} L_{D_5}(-3, t\omega_4) \otimes M(1,t)$$

$$\oplus \bigoplus_{t \in \mathbb{Z}_{<0}} L_{D_5}(-3, -t\omega_5) \otimes M(1,t).$$

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Corollary

As a consequence of this decomposition, one also obtains the following result.

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Corollary

We have:

$$L_{F_4}(-3,0) \cong L_{B_4}(-3,0) \otimes M(1)^+ \oplus L_{B_4}(-3,\omega_1) \otimes M(1)^- \oplus \bigoplus_{t \in \mathbb{Z}_{>0}} L_{B_4}(-3,t\omega_4) \otimes M(1,t).$$