Explicit realization of certain affine and superconformal vertex algebras

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Affine Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot,\cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$.

The affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$$

where K is the canonical central element and the Lie algebra structure is given by

$$[x \otimes t^n, y \otimes t^m] = [x,y] \otimes t^{n+m} + n(x,y)\delta_{n+m,0}K.$$

We will say that M is a $\hat{\mathfrak{g}}$–module of level k if the central element K acts on M as a multiplication with k.

Affine vertex algebras

Set $x(n) = x \otimes t^n$, for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, and identify $\mathfrak{g}$ as the subalgebra $\mathfrak{g} \otimes t^0$.

Define the field $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$ which acts on restricted $\hat{\mathfrak{g}}$–modules of level k.

Let $V^k(\mathfrak{g})$ be the universal vertex algebra generated by fields $x(z)$, $x \in \mathfrak{g}$.

As a $\hat{\mathfrak{g}}$–module, $V^k(\mathfrak{g})$ can be realized as a generalized Verma module.

For every $k \in \mathbb{C}$, the irreducible $\hat{\mathfrak{g}}$–module $L_k(\mathfrak{g})$ carries the structure of a simple vertex algebra.

Let now $\mathfrak{g} = sl_2(\mathbb{C})$
with generators e, f, h
and relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.
The corresponding affine Lie algebra $\hat{\mathfrak{g}}$ is of type $A_1^{(1)}$.

The level $k = -2$ is called **critical level.**

$N = 2$ superconformal algebra

$N = 2$ superconformal algebra (SCA) is the infinite-dimensional Lie superalgebra with basis $\mathcal{L}(n), \mathcal{H}(n), \mathcal{G}^{\pm}(r), C$, $n \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$ and (anti)commutation relations given by

$$
\begin{aligned}
&[\mathcal{L}(m), \mathcal{L}(n)] = (m-n)\mathcal{L}(m+n) + \tfrac{C}{12}(m^3 - m)\delta_{m+n,0},\\
&[\mathcal{H}(m), \mathcal{H}(n)] = \tfrac{C}{3} m \delta_{m+n,0}, \quad [\mathcal{L}(m), \mathcal{G}^{\pm}(r)] = (\tfrac{1}{2}m - r)\mathcal{G}^{\pm}(m+r),\\
&[\mathcal{L}(m), \mathcal{H}(n)] = -n\mathcal{H}(n+m), \quad [\mathcal{H}(m), \mathcal{G}^{\pm}(r)] = \pm\mathcal{G}^{\pm}(m+r),\\
&\{\mathcal{G}^{+}(r), \mathcal{G}^{-}(s)\} = 2\mathcal{L}(r+s) + (r-s)\mathcal{H}(r+s) + \tfrac{C}{3}(r^2 - \tfrac{1}{4})\delta_{r+s,0},\\
&[\mathcal{L}(m), C] = [\mathcal{H}(n), C] = [\mathcal{G}^{\pm}(r), C] = 0,\\
&\{\mathcal{G}^{+}(r), \mathcal{G}^{+}(s)\} = \{\mathcal{G}^{-}(r), \mathcal{G}^{-}(s)\} = 0
\end{aligned}
$$

for all $m, n \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$.

Correspondence

When $k \neq -2$, the representation theory of the affine Lie algebra $A_1^{(1)}$ is related with the representation theory of the $N = 2$ superconformal algebra.

The correspondence is given by Kazama-Suzuki mappings.

We shall extend this correspondence to representations at the critical level by introducing a new infinite-dimensional Lie superalgebra $\mathcal{A}$.

Clifford vertex superalgebras

The Clifford vertex superalgebra F is generated by fields

$$\Psi^{+}(z) = \sum_{n\in\mathbb{Z}} \Psi^{+}(n+\tfrac{1}{2})z^{-n-1}, \quad \Psi^{-}(z) = \sum_{n\in\mathbb{Z}} \Psi^{-}(n+\tfrac{1}{2})z^{-n-1}.$$

whose components satisfy the (anti)commutation relations for the infinite dimensional Clifford algebra CL:

$$\{\Psi^{\pm}(r), \Psi^{\mp}(s)\} = \delta_{r+s,0}; \quad \{\Psi^{\pm}(r), \Psi^{\pm}(s)\} = 0$$

where $r, s \in \frac{1}{2} + \mathbb{Z}$.

As a vector space,

$$F \cong \bigwedge \left(\Psi^{\pm}(-n - \tfrac{1}{2}) \mid n \leq 0\right)$$

$N = 2$ superconformal vertex algebra

Let $\mathfrak{g} = sl_2$. Consider the vertex superalgebra $V^k(\mathfrak{g}) \otimes F$. Define

$$\tau^+ = e(-1)\mathbf{1} \otimes \Psi^+(-\tfrac{1}{2}), \quad \tau^- = f(-1)\mathbf{1} \otimes \Psi^-(-\tfrac{1}{2}).$$

Then the vertex subalgebra of $V^k(\mathfrak{g}) \otimes F$

generated by τ^+ and τ^- carries the structure of a highest weight module for of the $N = 2$ SCA:

$\mathcal{G}^{\pm}(z) = \sqrt{\frac{2}{k+2}} Y(\tau^{\pm}, z) = \sum_{n \in \mathbb{Z}} \mathcal{G}^{\pm}(n + \frac{1}{2}) z^{-n-2}$

Kazama-Suzuki and "anti" Kazama-Suzuki mappings

Introduced by Fegin, Semikhatov and Tipunin (1997)

Assume that M is a $V^k(\mathfrak{g})$-module. Then $M \otimes F$ is a module for $N = 2$ supereconformal algebra.

Let F_{-1} be the lattice vertex superalgebra associated to the lattice $\mathbb{Z}\sqrt{-1}$.

Assume that N is a (restricted) module for the $N = 2$ SCA. Then $N \otimes F_{-1}$ is a module for the affine Lie algebra $A_1^{(1)}$.

This enables a classification of irreducible modules for simple vertex superalgebras associated to N=2 SCA (D.Adamović, IMRN (1998))

Lie superalgebra $\mathcal{A}$

$\mathcal{A}$ is infinite-dimensional Lie superalgebra with generators $S(n), T(n), G^{\pm}(r), C$, $n \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$, which satisfy the following relations

$$\begin{aligned}
& S(n), T(n), C \quad \text{are in the center of } \mathcal{A}, \\
& \{G^{+}(r), G^{-}(s)\} = 2S(r+s) + (r-s)T(r+s) + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \\
& \{G^{+}(r), G^{+}(s)\} = \{G^{-}(r), G^{-}(s)\} = 0
\end{aligned}$$

for all $n \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$.

Then the vertex superalgebra structure on $\mathcal{V}$ is strongly generated by the fields

$$G^{\pm}(z) = Y(\tau^{\pm}, z) = \sum_{n\in\mathbb{Z}} G^{\pm}(n+\tfrac{1}{2})z^{-n-2},$$

$$S(z) = Y(\nu, z) = \sum_{n\in\mathbb{Z}} S(n)z^{-n-2},$$

$$T(z) = Y(j, z) = \sum_{n\in\mathbb{Z}} T(n)z^{-n-1}.$$

The components of these fields satisfy the (anti)commutation relations for the Lie superalgebra $\mathcal{A}$.

Theorem (A, CMP 2007)

Assume that U is an irreducible $\mathcal{V}$–module such that U admits the following $\mathbb{Z}$–gradation

$$U = \bigoplus_{j \in \mathbb{Z}} U^j, \quad \mathcal{V}^i . U^j \subset U^{i+j}.$$

Let F_{-1} be the vertex superalgebra associated to lattice $\mathbb{Z}\sqrt{-1}$. Then

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U), \quad \textit{where } \ \mathcal{L}_s(U) := \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i}$$

and for every $s \in \mathbb{Z}$ $\mathcal{L}_s(U)$ is an irreducible $A_1^{(1)}$–module at the critical level.

Weyl vertex algebra

The Weyl vertex algebra W is generated by the fields

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n},$$

whose components satisfy the commutation relations for infinite-dimensional Weyl algebra

$$[a(n), a(m)] = [a^*(n), a^*(m)] = 0, \quad [a(n), a^*(m)] = \delta_{n+m,0}.$$

Wakimoto modules

Assume that $\chi(z) \in \mathbb{C}((z))$.

On the vertex algebra W exists the structure of the $A_1^{(1)}$–module at the critical level defined by

$$\begin{aligned} e(z) &= a(z), \\ h(z) &= -2 : a^*(z)a(z) : -\chi(z) \\ f(z) &= - : a^*(z)^2 a(z) : -2\partial_z a^*(z) - a^*(z)\chi(z). \end{aligned}$$

This module is called the Wakimoto module and it is denoted by $W_{-\chi(z)}$.

Theorem (D.A., CMP 2007, Contemp. Math. 2014)

The Wakimoto module $W_{-\chi}$ is irreducible if and only if $\chi(z)$ satisfies one of the following conditions:

(i) *There is $p \in \mathbf{Z}_{>0}$, $p \geq 1$ such that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad \textit{and} \quad \chi_p \neq 0.$$

(ii) $\chi(z) = \sum_{n=0}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad$ *and* $\quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z})$.

(iii) *There is $\ell \in \mathbb{Z}_{\geq 0}$ such that*

$$\chi(z) = \frac{\ell + 1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

and $S_\ell(-\chi) \neq 0$, where $S_\ell(-\chi) = S_\ell(-\chi_{-1}, -\chi_{-2}, \dots)$ is a Schur polynomial.

Whittaker modules for Weyl vertex algebra

Every restricted module for the Weyl algebra is a module for Weyl vertex algebra W.

For $(\lambda, \mu) \in \mathbb{C}^2$ let $M_1(\lambda, \mu)$ be the module for the Weyl algebra generated by the Whittaker vector v_1 such that

$$a(0)v_1 = \lambda v_1, \ a^*(1)v_1 = \mu v_1, \ a(n+1)v_1 = a^*(n+2)v_1 = 0 \ (n \geq 0)$$

$M_1(\lambda, \mu)$ is a W-module.

Theorem (D.A; R. Lu, K. Z., 2014)

For every $\chi(z) \in \mathbb{C}((z))$, $(\lambda, \mu) \in \mathbb{C}^2$, $\lambda \neq 0$ *there exists irreducible* $\widehat{sl_2}$*–module* $\overline{M_{Wak}}(\lambda, \mu, -2, \chi(z))$ *realized on the* W*–module* $M_1(\lambda, \mu)$ *such that*

$$
\begin{aligned}
e(z) &= a(z); \\
h(z) &= -2 : a^*(z)a(z) : +\chi(z); \\
f(z) &= - : a^*(z)^2 a(z) : -2\partial_z a^*(z) + a^*(z)\chi(z)
\end{aligned}
$$

Whittaker vs. Wakimoto modules

- We see that Whittaker modules from previous theorem are realized using same formulas.
- Wakimoto modules are realized on vertex algebra W, but Whittaker are realized on W–module $M_1(\lambda, \mu)$.
- Whitteker modules are always irreducible, but for Wakimoto modules we have non–trivial criteria.

As a special case, the previous theorem provides a realization of degenerate Whittaker modules at the critical level.

But it does not cover non-degenerate Whittaker modules at the critical level.

We need to modify construction of Wakimoto modules.

Method: Use vertex algebra $\Pi(0)$, a localization of Weyl vertex algebra.

The vertex algebra $\Pi(0)$

Let $\Pi(0)$ be the localization of the Weyl vertex algebra with respect to $a(-1)$, $\Pi(0) = M[(a(-1)^{-1}]$. We have the expansion

$$a^{-1}(z) = Y(a^{-1}, z) = \sum_{n \in \mathbb{Z}} a^{-1}(n) z^{-n+1}.$$

$$a^{-1}(z) a(z) = Id.$$

Theorem

Assume that $\lambda \neq 0$. There is a $\Pi(0)$–module Π_λ generated by the cyclic vector w_λ such that

$$a(0) w_\lambda = \lambda w_\lambda, \quad a^{-1}(0) w_\lambda = \frac{1}{\lambda} w_\lambda, \quad a(n) w_\lambda = a^{-1}(n) w_\lambda = 0 (n \geq 1)$$

Modified Wakimoto construction

There is an embedding of vertex algebras

$$V^{-2}(sl_2) \to M_T(0) \otimes \Pi(0)$$

such that

$$\begin{aligned}
e &= a, && (1)\\
h &= -2\beta(-1) = -2a^*(0)a(-1)\mathbf{1} && (2)\\
f &= \Big[T(-2) - (\alpha(-1)^2 + \alpha(-2))\Big]\, a^{-1} && (3)\\
&= -a^*(0)^2 a(-1)\mathbf{1} - 2a^*(-1)\mathbf{1} + T(-2)a^{-1} && (4)
\end{aligned}$$

Non-degenerate Whittaker modules at the critical level

For any $\chi(z) = \sum_{n\in\mathbb{Z}} \chi(n) z^{-n-2} \in \mathbb{C}((z))$ let $M_T(\chi(z))$ be 1-dimensional $M_T(0)$–module such that $T(n)$ acts as multiplication with $\chi(n) \in \mathbb{C}$.

Theorem (ALZ)

Let $\lambda \neq 0$. Let

$$\chi(z) = \frac{\lambda\mu}{z^3} + c(z), \quad c(z) = \sum_{n\leq 0} \chi(n) z^{-n-2} \in \mathbb{C}((z)).$$

Then we have:

$$V_{\widehat{sl_2}}(\lambda, \mu, -2, c(z)) \cong M_T(\chi(z)) \otimes \Pi_\lambda.$$

$V_c^{N=4}$ is generated by the Virasoro field L, three primary fields of conformal weight 1, J^0, J^+ and J^- (even part) and four primary fields of conformal weight $\frac{3}{2}$, $G^\pm$ and $\overline{G}^\pm$ (odd part). The remaining (non-vanishing) λ–brackets are

$$\begin{array}{rl}
[J^0_\lambda, J^\pm] = \pm 2J^\pm & [J^0_\lambda J^0] = \dfrac{c}{3} \\
[J^+_\lambda J^-] = J^0 + \frac{c}{6}\lambda & [J^0_\lambda G^\pm] = \pm G^\pm \\
[J^0 \overline{G}^\pm] = \pm \overline{G}^\pm & [J^+_\lambda G^-] = G^+ \\
[J^-_\lambda G^+] = G^- & [J^+_\lambda \overline{G}^-] = -\overline{G}^+ \\
[J^-_\lambda \overline{G}^+] = -\overline{G}^- & [G^\pm_\lambda \overline{G}^\pm] = (T + 2\lambda)J^\pm \\
[G^\pm_\lambda \overline{G}^\mp] = & L \pm \dfrac{1}{2}TJ^0 \pm \lambda J^0 + \dfrac{c}{6}\lambda^2
\end{array}$$

Let $L_c^{N=4}$ be its simple quotient.

We shall present some results from D.Adamović, arXiv:1407.1527. (to appear in Transformation Groups)

Theorem

(i) The simple affine vertex algebra $L_k(sl_2)$ with $k = -3/2$ is conformally embedded into $L_c^{N=4}$ with $c = -9$.
(ii)

$$L_c^{N=4} \cong (M \otimes F)^{int}$$

where $M \otimes F$ is a maximal sl_2–integrable submodule of the Weyl-Clifford vertex algebra $M \otimes F$.

$L_c^{N=4}$ with $c=-9$ as an $\widehat{sl_2}$–module

$L_c^{N=4}$ with $c=-9$ is completely reducible $\widehat{sl_2}$–module and the following decomposition holds:

$$L_c^{N=4} \cong \bigoplus_{m=0}^{\infty} (m+1) L_{A_1}(-(\frac{3}{2}+n)\Lambda_0 + n\Lambda_1).$$

$L_c^{N=4}$ is a completely reducible $sl_2 \times \widehat{sl_2}$–modules. sl_2 action is obtained using screening operators for Wakimoto realization of $\widehat{sl_2}$–modules at level $-3/2$.

The affine vertex algebra $L_k(sl_3)$ with $k = -3/2$.

Theorem

(i) The simple affine vertex algebra $L_k(sl_3)$ with $k = -3/2$ is realized as a subalgebra of $L_c^{N=4} \otimes F_{-1}$ with $c = -9$. In particular $L_k(sl_3)$ can be realized as subalgebra of

$$M \otimes F \otimes F_{-1}.$$

(ii) $L_c^{N=4} \otimes F_{-1}$ is a completely reducible $A_2^{(1)}$–module at level $k = -3/2$.

- $L_c^{N=4}$ has only one irreducible module in the category of strong modules. Every $\mathbf{Z}_{>0}$–graded $L_c^{N=4}$–module with finite-dimensional weight spaces (with respect to $L(0)$) is semisimple ("Rationality in the category of strong modules")
- $L_c^{N=4}$ has two irreducible module in the category $\mathcal{O}$. There are non-semisimple $L_c^{N=4}$–modules from the category $\mathcal{O}$.
- $L_c^{N=4}$ has infinitely many irreducible modules in the category of weight modules.
- $L_c^{N=4}$ admits logarithmic modules on which $L(0)$ does not act semi-simply.

Theorem (D.A, 2014)

Assume that U is an irreducible $L_c^{N=4}$–module with $c = -9$ such that $U = \bigoplus_{j \in \mathbb{Z}} U^j$ is $\mathbb{Z}$–graded (in a suitable sense).
Let F_{-1} be the vertex superalgebra associated to lattice $\mathbb{Z}\sqrt{-1}$.
Then

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U), \quad \text{where } \mathcal{L}_s(U) := \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i}$$

and for every $s \in \mathbb{Z}$ $\mathcal{L}_s(U)$ is an irreducible $A_2^{(1)}$–module at level $-3/2$.

Connection with C_2–cofinite vertex algebras appearing in LCFT

Drinfeld-Sokolov reduction maps:
$L_c^{N=4}$ to doublet vertex algebra $\mathcal{A}(p)$ and even part $(L_c^{N=4})^{even}$ to triplet vertex algebra $\mathcal{W}(p)$ with $p = 2$ (symplectic-fermion case)

Vacuum space of $L_k(sl_3)$ with $k = -3/2$ contains the vertex algebra $\mathcal{W}_{A_2}(p)$ with $p = 2$ (which is conjecturally C_2–cofinite).

Connection with C_2–cofinite vertex algebras appearing in LCFT:

Vacuum space of $L_k(sl_3)$ with $k = -3/2$ contains the vertex algebra $\mathcal{W}_{A_2}(p)$ with $p = 2$ (which is conjecturally C_2–cofinite).

Affine vertex algebra $L_k(sl_2)$ for $k + 2 = \frac{1}{p}$, $p \geq 2$ can be conformally embedded into the vertex algebra $\mathcal{V}^{(p)}$ generated by $L_k(sl_2)$ and 4 primary vectors $\tau^{\pm}_{(p)}, \overline{\tau}^{\pm}_{(p)}$.

$\mathcal{V}^{(p)} \cong L_c^{N=4}$ for $p = 2$.

Drinfeld-Sokolov reduction maps $\mathcal{V}^{(p)}$ to the doublet vertex algebra $\mathcal{A}(p)$ and even part $(\mathcal{V}^{(p)})^{even}$ to the triplet vertex algebra $\mathcal{W}(p)$. (C_2–cofiniteness and RT of these vertex algebras were obtain in a work of D.A and A. Milas)

The Vertex algebra $\mathcal{W}_{A_2}(p)$: Definition

We consider the lattice

$$\sqrt{p}A_2 = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2, \quad \langle\gamma_1,\gamma_1\rangle = \langle\gamma_2,\gamma_2\rangle = 2p,\ \langle\gamma_1,\gamma_2\rangle = -p.$$

Let $M_{\gamma_1,\gamma_2}(1)$ be the s Heisenberg vertex subalgebra of $V_{\sqrt{p}A_2}$ generated by the Heisenberg fields $\gamma_1(z)$ and $\gamma_2(z)$.

$$\mathcal{W}_{A_2}(p) = \mathrm{Ker}_{V_{\sqrt{p}A_2}} e_0^{-\gamma_1/p} \bigcap \mathrm{Ker}_{V_{\sqrt{p}A_2}} e_0^{-\gamma_2/p}.$$

We also have its subalgebra:

$$\mathcal{W}^0_{A_2}(p) = \mathrm{Ker}_{M_{\gamma_1,\gamma_2}(1)} e_0^{-\gamma_1/p} \bigcap \mathrm{Ker}_{M_{\gamma_1,\gamma_2}(1)} e_0^{-\gamma_2/p}$$

$\mathcal{W}_{A_2}(p)$ and $\mathcal{W}^0_{A_2}(p)$ have vertex subalgebra isomorphic to the simple $\mathcal{W}(2,3)$–algebra with central charge $c_p = 2 - 24\frac{(p-1)^2}{p}$.

The Vertex algebra $\mathcal{W}_{A_2}(p)$: Conjecture

(i) $\mathcal{W}_{A_2}(p)$ is a C_2–cofinite vertex algebra for $p \geq 2$ and that it is a completely reducible $\mathcal{W}(2,3) \times sl_3$–module.

(ii) $\mathcal{W}_{A_2}(p)$ is strongly generated by $\mathcal{W}(2,3)$ generators and by $sl_3.e^{-\gamma_1-\gamma_2}$, so by 8 primary fields for the $\mathcal{W}(2,3)$–algebra.

Note that $\mathcal{W}_{A_2}(p)$ is a generalization of the triplet vertex algebra $\mathcal{W}(p)$ and $\mathcal{W}^0_{A_2}(p)$ is a generalization of the singlet vertex subalgebra of $\mathcal{W}(p)$.

Relation with parafermionic vertex algebras for $p = 2$

(i) Let $K(sl_3, k)$ be the parafermion vertex subalgebra of $L_k(sl_3)$.

(ii) Creutzig-Linshaw proved that generically $K(sl_3, k)$ is $\mathcal{W}$–algebra of type $\mathcal{W}(2^3, 3^5, 4^7, ...)$

(iii) For $k = -3/2$ we have

$$K(sl_3, k) = \mathcal{W}^0_{A_2}(p).$$

Realization of simple W–algebras

Let $F_{-p/2}$ denotes the generalized lattice vertex algebra associated to the lattice $\mathbb{Z}(\frac{p}{2}\varphi)$ such that

$$\langle \varphi, \varphi \rangle = -\frac{2}{p}.$$

Let $\mathcal{R}^{(p)}$ by the subalgebra of $\mathcal{V}^{(p)} \otimes F_{-p/2}$ generated by $x = x(-1)\mathbf{1} \otimes 1$, $x \in \{e, f, h\}$, $1 \otimes \varphi(-1)\mathbf{1}$ and

$$e_{\alpha_1,p} := \frac{1}{\sqrt{2}}\, \tau^{+}_{(p)} \otimes e^{\frac{p}{2}\varphi} \tag{5}$$

$$f_{\alpha_1,p} := \frac{1}{\sqrt{2}}\, \overline{\tau}^{-}_{(p)} \otimes e^{-\frac{p}{2}\varphi} \tag{6}$$

$$e_{\alpha_2,p} := \frac{1}{\sqrt{2}}\, \overline{\tau}^{+}_{(p)} \otimes e^{-\frac{p}{2}\varphi} \tag{7}$$

$$f_{\alpha_2,p} := \frac{1}{\sqrt{2}}\, \tau^{-}_{(p)} \otimes e^{\frac{p}{2}\varphi} \tag{8}$$

Realization of simple W–algebras

$\mathcal{R}^{(2)} \cong L_{A_2}(-\frac{3}{2}\Lambda_0)$.

$\mathcal{R}^{(3)} \cong \mathcal{W}_k(sl_4, f_\theta)$ with $k = -8/3$.

(Conjecture) $\mathcal{R}^{(p)}$ and $\mathcal{V}^{(p)}$ have finitely many irreducible modules in the category $\mathcal{O}$.

$\mathcal{R}^{(p)}$ and $\mathcal{V}^{(p)}$ have infinitely many irreducible modules outside of the category $\mathcal{O}$ and admit logarithmic modules.

Thank you