Leading terms of relations for standard modules of affine Lie Algebras $C_n^{(1)}$

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Representation Theory XIV; Dubrovnik 22.-27.6.2015.

(supported by CSF - grant 2634)

Introduction:

The generalized Verma $\tilde{\mathfrak{g}}$-module $N(k\Lambda_0)$ is reducible, and we denote by $N^1(k\Lambda_0)$ its maximal $\tilde{\mathfrak{g}}$-submodule. The submodule $N^1(k\Lambda_0)$ is generated by the singular vector $x_\theta(-1)^{k+1}\mathbf{1}$. Set

$$R = U(\mathfrak{g})x_\theta(-1)^{k+1}\mathbf{1}, \qquad \bar{R} = \mathbb{C}\text{-span}\{r_m \mid r \in R, m \in \mathbb{Z}\}.$$

Then $R \subset N^1(k\Lambda_0)$ is an irreducible $\mathfrak{g}$-module, and $\bar{R}$ is the corresponding loop $\tilde{\mathfrak{g}}$-module for the adjoint action.

Theorem

M is a standard module $\Leftrightarrow$ $\bar{R}$ annihilates M.

This theorem implies that for a dominant integral weight Λ of level $\Lambda(c) = k$ we have

$$\bar{R}M(\Lambda) = M^1(\Lambda),$$

where $M^1(\Lambda)$ denotes the maximal submodule of the Verma $\tilde{\mathfrak{g}}$-module $M(\Lambda)$.

Introduction:

Furthermore, since R generates the vertex algebra ideal $N^1(k\Lambda_0) \subset N(k\Lambda_0)$, the vertex operators $Y(v,z)$, $v \in N^1(k\Lambda_0)$, annihilate all standard $\tilde{\mathfrak{g}}$-modules

$$L(\Lambda) = M(\Lambda)/M^1(\Lambda)$$

of level k. We shall call the elements $r_m \in \bar{R}$ *relations* (for standard modules), and $Y(v,z)$, $v \in N^1(k\Lambda_0)$, *annihilating fields* (of standard modules). The field

$$Y(x_\theta(-1)^{k+1}\mathbf{1}, z) = x_\theta(z)^{k+1}$$

generates all annihilating fields.

Introduction:

- $M^1(\Lambda) = \mathcal{U}(\tilde{\mathfrak{g}})\overline{R}v_\Lambda \rightsquigarrow \overline{R}$ *Relations*
- by PBW $\rightsquigarrow$ for $v_\Lambda \in M(\Lambda)$

$$rx_1x_2\cdots x_nv_\Lambda,\ \ r \in \overline{R},\ \ x_i \in \tilde{\mathfrak{g}}$$

  is a spanning set
- The set of the vectors

$$u(\pi)v_\Lambda, \quad \pi \in \mathcal{P}(\tilde{B}_-) \setminus (\mathcal{LT}(\overline{R}v_\Lambda))$$

  is a basis of the standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$.

(narrow)**Framework digression:**

- [A. Meurman and M. Primc], *Annihilating fields of standard modules of* $\mathfrak{sl}(2,\mathbb{C})^{\sim}$ *and combinatorial identities* Memoirs of the Amer. Math. Soc.**137**, No. 652 (1999).
- [A. Meurman and M. Primc], *A basis of the basic* $\mathfrak{sl}(3,\mathbb{C})^{\sim}$*-module* Commun. Contemp. Math. **3** (2001), 593-614.
- [I. Siladić], *Twisted* $\mathfrak{sl}(3,\mathbb{C})^{\sim}$*-modules and combinatorial identities*, arXiv:math/0204042.
- [G. Trupčević], *Combinatorial bases of Feigin-Stoyanovsky's type subspaces of higher-level standard* $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$*-modules* J. Algebra **322** (2009), 3744–3774.
- [M. Primc and T. Šikić], *arXiv:1506.05026/ QA and CO*

(narrow)**Framework digression:**

Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ (DD is of type A_1). Denote by $\{x, h, y\}$ the standard basis of $\mathfrak{sl}_2$, and the corresponding Poincaré-Birkhoff-Witt monomial spanning set of level k standard $\widehat{\mathfrak{sl}}_2$-module $L(k\Lambda_0)$

$$y(-s)^{c_s} \dots y(-2)^{c_2} h(-2)^{b_2} x(-2)^{a_2} y(-1)^{c_1} h(-1)^{b_1} x(-1)^{a_1} v_0, \quad s \geq 0, \tag{1}$$

with $a_j, b_j, c_j \geq 0$.

The spanning set (1) can be reduced to a smaller spanning set of $L(k\Lambda_0)$ satisfying the difference conditions

$$\begin{aligned} a_{j+1} + b_j + a_j &\leq k, \\ a_{j+1} + c_j + b_j &\leq k, \\ b_{j+1} + a_{j+1} + c_j &\leq k, \\ c_{j+1} + b_{j+1} + c_j &\leq k. \end{aligned} \tag{2}$$

(narrow)**Framework digression:**

In [FKLMM] and [MP] it is proved, by different methods, that this spanning set is a basis of $L(k\Lambda_0)$.
[B. Feigin, R. Kedem, S. Loktev, T. Miwa and E. Mukhin], Combinatorics of the $\widehat{\mathfrak{s}l}_2$ spaces of coinvariants, Transformation Groups **6** *(2001), 25–52.*
The degree of monomial vector (1) satisfying the difference conditions (2) is

$$-m = -\sum_{j\geq 1} ja_j - \sum_{j\geq 1} jb_j - \sum_{j\geq 1} jc_j,$$

so we are naturally led to interpret monomial basis vectors (1) in terms of colored partitions with parts j in three colors: x, h and y

Simple Lie algebra of type C_n:

- root system:

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j = 1, ..., n\} \setminus \{\ominus\} \ .$$

- simple roots:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_1 - \varepsilon_2, \cdots, \alpha_{n-1}\varepsilon_1 - \varepsilon_2, \alpha_n = 2\varepsilon_n$$

- For a root vector X_α we shall use following notation

$$\begin{array}{llll} X_{ij} \text{ or just } ij & \text{if} & \alpha = \varepsilon_i + \varepsilon_j \ , & i \leq j \\ X_{i\underline{j}} \text{ or just } i\underline{j} & \text{if} & \alpha = \varepsilon_i - \varepsilon_j \ , & i \neq j \\ X_{\underline{i}\underline{j}} \text{ or just } \underline{i}\underline{j} & \text{if} & \alpha = -\varepsilon_i - \varepsilon_j \ , & i \geq j \end{array}$$

and for $i = j$ we shall write

$$X_{i\underline{i}} = \alpha_i^\vee \text{ or just } i\underline{i} \ .$$

Simple Lie algebra of type C_n:

These vectors form a basis B of $\mathfrak{g}$ which we shall write in a triangular scheme, e.g. for $n = 3$ the basis B is

$$\begin{array}{llllll}
11 & & & & & \\
12 & 22 & & & & \\
13 & 23 & 33 & & & \\
1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{33} & & \\
1\underline{2} & 2\underline{2} & 3\underline{2} & \underline{32} & \underline{22} & \\
1\underline{1} & 2\underline{1} & 3\underline{1} & \underline{31} & \underline{21} & \underline{11}
\end{array}$$

Ordered basis of C_n:

- In general for the set of indices we use order

$$1 \succ 2 \succ \cdots \succ n-1 \succ n \succ \underline{n} \succ \underline{n-1} \succ \cdots \succ \underline{2} \succ \underline{1}$$

and a basis element X_{ab} we write in a^{th} column and b^{th} row,

$$B = \{X_{ab} \mid b \in \{1, 2, \cdots, n, \underline{n}, \cdots, \underline{2}, \underline{1}\},\ a \in \{1, \cdots, b\}\}\ .$$

- on B the corresponding reverse lexicographical order, i.e.

$$X_{ab} \succ X_{a'b'} \textit{ if } b \succ b' \textit{ or } b = b' \textit{ and } a \succ a'\ .$$

- In other words, X_{ab} is larger than $X_{a'b'}$ if $X_{a'b'}$ lies in a row b' below the row b, or X_{ab} and $X_{a'b'}$ are in the same row $b = b'$, but $X_{a'b'}$ (lies in a column b' which) is to the right of $X_{a'b'}$ (a column b)

Order on the set of colored partitions

With this ordered basis b of $\mathfrak{g}$ we define the set of colored partitions $\mathcal{P}$, i.e. monomial basis of $\mathcal{S} \cong \mathcal{S}(\bar{\mathfrak{g}})$.
For instance, for colored partitions with same shape we compare their colors with reverse lexicographical order

$$X_{11}(-3)^2 X_{1\underline{1}}(-2)^2 X_{11}(-2) \prec X_{\underline{11}}(-3) X_{11}(-3) X_{11}(-2)^3 \ .$$

These two colored partitions have the same shape $(-3)^2(-2)^3$ with colors

$$11\ 11; 1\underline{1}\ 1\underline{1}\ 11 \ \textit{and} \ \underline{11}\ 11; 11\ 11\ 11$$

and comparing from the right we se $11 = 11$, $1\underline{1} \prec 11$.

Cascade $\mathcal{C}$ in the base B

- Definition
  The sequence of basis elements $(X_{a_1 b_1}, X_{a_2 b_2}, \cdots, X_{a_s b_s})$ is a cascade $\mathcal{C}$ in the base B if
  1. for each $i \in \{1, 2, \cdots, s-1\}$ we have $b_{i+1} \prec b_i$ or $b_{i+1} = b_i$ and $a_{i+1} \succ a_i$
  2. for each $X_{a_i b_i}$ is given some multiplicity $n_{a_i b_i} \in \mathbb{Z}_{\geq 0}$.

- We can visualize a cascade $\mathcal{C}$ in the basis B as a staircase in the triangle B going downwards from the right to the left, or as a sequence of waterfalls flowing from the right to the left.
- Sometimes we shall think of a cascade $\mathcal{C}$ as a set of points in the basis B and write $\mathcal{C} \subset B$.
- We shall also write a cascade with multiplicities $\mathcal{C}$ in the basis B as a monomial

Cascade $\mathcal{C}$ in the base B

Triangular scheme of a basis B for $C_2^{(1)}$

$$
\begin{array}{llll}
11 & & & \\
12 & 22 & & \\
1\underline{2} & 2\underline{2} & \underline{22} & \\
1\underline{1} & 2\underline{1} & \underline{21} & \underline{11}
\end{array}
\qquad
\begin{array}{llll}
a_1 & & & \\
a_2 & a_3 & & \\
a_4 & a_5 & a_6 & \\
a_7 & a_8 & a_9 & a_{10}
\end{array}
$$

1	2	4	7
3	2	4	7
3	5	4	7
3	5	8	7
6	5	4	7
6	5	8	7
6	9	8	7
10	9	8	7

Cascade $\mathcal{C}$ with multiplicities (in the base B)

- Definition
  We say that $\mathcal{C}$ is a *cascade with multiplicities* if for each $X_{a_i b_i}$ in $\mathcal{C}$ a multiplicity $m_{a_i b_i} \in \mathbb{Z}_{\geq 0}$ is given. By abuse of language, we shall say that in the cascade $\mathcal{C}$ with multiplicities $X_{a_i b_i}$ is the *place* $a_i b_i \in \mathcal{C} \subset B$ with $m_{a_i b_i}$ *points*. We shall also write a cascade with multiplicities $\mathcal{C}$ in the basis B as a monomial

$$\prod_{\alpha \in \mathcal{C}} X_\alpha^{m_\alpha}.$$

Admissible pair of cascades $\mathcal{C}$ (in the base B)

- Definition

  We say that two cascades are an admissible pair $(\mathcal{B}, \mathcal{A})$ if

  $$\mathcal{B} \subset \triangle_r, \quad \text{and} \quad \mathcal{A} \subset {}^r\triangle$$

  for some r. We shall also consider the case when $\mathcal{B}$ is empty and $\mathcal{A} \subset {}^1\triangle (= B)$.

  For general rank we may visualize admissible pair of cascades as figure below

Visualization of admissible pair of cascades

For general rank we may visualize admissible pair of cascades as figure below

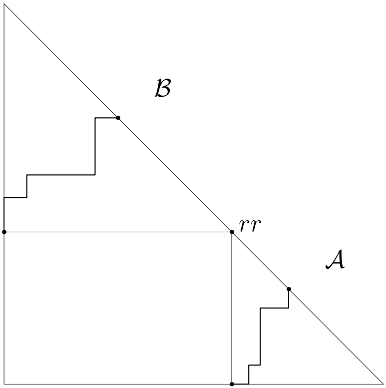


Figure 1.

Leading terms theorem related to $\mathfrak{g}$ of the type C_n

Theorem

Let

$$(-j-1)^b(-j)^a, \quad j \in \mathbb{Z}, \quad a+b=k+1, \quad b \geq 0, \tag{3}$$

be a fixed shape and let $\mathcal{B}$ *and* $\mathcal{A}$ *be two cascades in degree* $-j-1$ *and* $-j$*, with multiplicities* $(m_{\beta,j+1}, \beta \in \mathcal{B})$ *and* $(m_{\alpha,j}, \alpha \in \mathcal{A})$*, such that*

$$\sum_{\beta\in\mathcal{B}} m_{\beta,j+1} = b, \quad \sum_{\alpha\in\mathcal{A}} m_{\alpha,j} = a. \tag{4}$$

Let $r \in \{1, \cdots, n, \underline{n}, \cdots, \underline{1}\}$*. If the points of cascade* $\mathcal{B}$ *lie in the upper triangle* $\triangle_r$ *and the points of cascade* $\mathcal{A}$ *lie in the lower triangle* ${}^r\triangle$*, than*

$$\prod_{\beta\in\mathcal{B}} X_\beta(-j-1)^{m_{\beta,j+1}} \prod_{\alpha\in\mathcal{A}} X_\alpha(-j)^{m_{\beta,j}} \tag{5}$$

Proof:

... by precisely defined application of arrows $[rs] = \operatorname{ad} X_{rs}$ on the colored partition

$$Z_0 = X_{11}(-j-1)^b X_{11}(-j)^a.$$

Using smart strategy to combine arrows (eight technical lemmas) we succeeded in

- **Preparation of upper barrier**
- **Construction of upper cascade**
- **Preparation of lower barrier**
- **Construction of lower cascade**

Proof:

$$arrow = X_{\varepsilon_3-\varepsilon_2}$$

$$
\begin{array}{lllllllllll}
11 & & & & & & & & & & \\
12 & 22 & & & & & & & & & \\
\downarrow & \downarrow & & & & & & & & & \\
13 & 23 & \rightarrow & 33 & & & & & & & \\
14 & 24 & \rightarrow & 34 & 44 & & & & & & \\
15 & 25 & \rightarrow & 35 & 45 & 55 & & & & & \\
1\underline{5} & 2\underline{5} & \rightarrow & 3\underline{5} & 4\underline{5} & 5\underline{5} & \underline{55} & & & & \\
1\underline{4} & 2\underline{4} & \rightarrow & 3\underline{4} & 4\underline{4} & 5\underline{4} & \underline{54} & \underline{44} & & & \\
1\underline{3} & 2\underline{3} & & 3\underline{3} & 4\underline{3} & 5\underline{3} & \underline{53} & \underline{43} & \underline{33} & & \\
\downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
1\underline{2} & 2\underline{2} & \rightarrow & 3\underline{2} & 4\underline{2} & 5\underline{2} & \underline{52} & \underline{42} & \underline{32} & \underline{22} & \\
1\underline{1} & 2\underline{1} & \rightarrow & 3\underline{1} & 4\underline{1} & 5\underline{1} & \underline{51} & \underline{41} & \underline{31} & \underline{21} & \underline{11}
\end{array}
$$

Proof:

$arrow = X_{-2\varepsilon_5}$

$$
\begin{array}{cccccccccccc}
11 & & & & & & & & & & \\
12 & 22 & & & & & & & & & \\
13 & 23 & 33 & & & & & & & & \\
14 & 24 & 34 & 44 & & & & & & & \\
15 & 25 & 35 & 45 & 55 & & & & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & & & & \\
1\underline{5} & 2\underline{5} & 3\underline{5} & 4\underline{5} & 5\underline{5} & \rightarrow & \underline{55} & & & & \\
1\underline{4} & 2\underline{4} & 3\underline{4} & 4\underline{4} & 5\underline{4} & \rightarrow & \underline{54} & \underline{44} & & & \\
1\underline{3} & 2\underline{3} & 3\underline{3} & 4\underline{3} & 5\underline{3} & \rightarrow & \underline{53} & \underline{43} & \underline{33} & & \\
1\underline{2} & 2\underline{2} & 3\underline{2} & 4\underline{2} & 5\underline{2} & \rightarrow & \underline{52} & \underline{42} & \underline{32} & \underline{22} & \\
1\underline{1} & 2\underline{1} & 3\underline{1} & 4\underline{1} & 5\underline{1} & \rightarrow & \underline{51} & \underline{41} & \underline{31} & \underline{21} & \underline{11}
\end{array}
$$

Remarks

As we have already mentioned, the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ may be regarded as of type C_n for $n = 1$, with the standard basis B

$$x = x_{11} \succ h = x_{1\underline{1}} \succ y = x_{\underline{1}\underline{1}}.$$

The standard basis B can be written as the triangle

$$\begin{array}{ll} 11 & \\ 1\underline{1} & \underline{1}\underline{1} \end{array}$$

Remarks

Theorem (monomials as in 5) applies: for the shape $(-j-1)^b(-j)^a$, $j \in \mathbb{Z}$, $a+b=k+1$, all leading terms of relations for level k standard $\tilde{\mathfrak{g}}$-modules are monomials

$$
\begin{aligned}
& x(-j-1)^b h(-j)^{a_2} x(-j)^{a_1}, \quad a_1+a_2=a, \\
& x(-j-1)^b y(-j)^{a_2} h(-j)^{a_1}, \quad a_1+a_2=a, \\
& h(-j-1)^{b_1} x(-j-1)^{b_2} y(-j)^a, \quad b_1+b_2=b, \\
& y(-j-1)^{b_1} h(-j-1)^{b_2} y(-j)^a, \quad b_1+b_2=b
\end{aligned} \tag{6}
$$

Remarks

We believe that all leading terms of level k relations $\bar{R}$ are given by (5). In the case $k = 1$ and 2 we can check this by direct calculation. On one side, by using Weyl's character formula for simple Lie algebra C_n, we have

$$\begin{aligned} \dim L(2\theta) &= \binom{2n+3}{4}, \\ \dim L(3\theta) &= \binom{2n+5}{6}. \end{aligned}$$

Remarks

On the other side, in the case $k = 1$ for the shape $(-j)^2$ the number of leading terms (5) is

$$\sum_{i_1=1}^{2n} \sum_{j_1=1}^{i_1} \sum_{i_2=i_1}^{2n} \sum_{j_2=1}^{j_1} 1 = \binom{2n+3}{4},$$

and for the shape $(-j-1)(-j)$

$$\sum_{i_1=1}^{2n} \sum_{j_1=1}^{i_1} \sum_{i_2=i_1}^{2n} \sum_{j_2=i_1}^{i_2} 1 = \binom{2n+3}{4}.$$

Remarks

In the case $k = 2$ and the shape $(-j)^3$ the number of leading terms (5) is

$$\sum_{i_1=1}^{2n}\sum_{j_1=1}^{i_1}\sum_{i_2=i_1}^{2n}\sum_{j_2=1}^{j_1}\sum_{i_3=i_2}^{2n}\sum_{j_3=1}^{j_2} 1 = \binom{2n+5}{6},$$

for the shape $(-j-1)^2(-j)$

$$\sum_{i_1=1}^{2n}\sum_{j_1=1}^{i_1}\sum_{i_2=i_1}^{2n}\sum_{j_2=1}^{j_1}\sum_{i_3=i_2}^{2n}\sum_{j_3=i_2}^{i_3} 1 = \binom{2n+5}{6},$$

and for the shape $(-j-1)(-j)^2$

$$\sum_{i_1=1}^{2n}\sum_{j_1=1}^{i_1}\sum_{i_2=i_1}^{2n}\sum_{j_2=i_1}^{i_2}\sum_{i_3=i_2}^{2n}\sum_{j_3=i_1}^{j_2} 1 = \binom{2n+5}{6}.$$

Remarks

- Unfortunately, we have not completed the the job!
- We did not prove (but we a quite sure) that the set of $\mathcal{LT}$ parametrized a basis of $L(k\Lambda_0)$.
- All of the above remarks suggested us that we are on the right way.
- Moreover, we have a proof for basic modules (i.e. level k=1) for arbitrary n (i.e. affine Lie algebra $C_n^{(1)}$)

Conjectured colored Rogers-Ramanujan type identities

Let $n \geq 2$ *and* $k \geq 2$. *We consider the standard module* $L(k\Lambda_0)$ *for the affine Lie algebra of type* $C_n^{(1)}$ *with the basis*

$$\{X_{ab}(j) \mid ab \in B,\ j \in \mathbb{Z}\} \cup \{c, d\},$$

where $B = \{ab \mid b \in \{1, 2, \cdots, n, \underline{n}, \cdots, \underline{2}, \underline{1}\},\ a \in \{1, \cdots, b\}\}$. *We conjecture that the set of monomial vectors*

$$\prod_{ab \in B,\, j>0} X_{ab}(-j)^{m_{ab;j}}\, v_0, \tag{7}$$

satisfying difference conditions

$$\sum_{ab \in \mathcal{B}} m_{ab;j+1} + \sum_{ab \in \mathcal{A}} m_{ab;j} \leq k$$

for any admissible pair of cascades $(\mathcal{B}, \mathcal{A})$, *is a basis of* $L(k\Lambda_0)$.

Case $C_2^{(1)}$ and $k = 2$

If our conjecture is true, then we have a combinatorial Rogers-Ramanujan type identities by using Lepowsky's product formula for principaly specialized characters of standard modules. In the case of $n = 2$ and $k \geq 1$ we have product formulas for principally specialized characters of standard $C_2^{(1)}$-modules $L(k\Lambda_0)$

$$\prod_{\substack{j\geq 1\\ j\not\equiv 0 \bmod 2}} \frac{1}{1-q^j} \times \prod_{\substack{j\geq 1\\ j\not\equiv 0,\pm1,\pm2,\pm3 \bmod 2k+6}} \frac{1}{1-q^j} \times \tag{8}$$

$$\times \prod_{\substack{j\geq 1\\ j\not\equiv 0,\pm1,\pm(k+1),\pm(k+2),k+3 \bmod 2k+6}} \frac{1}{1-q^j}.$$

Case $C_2^{(1)}$ and $k=2$

This product can be interpreted combinatorially in the following way: For fixed k let $\mathcal{C}_k$ be a disjoint union of integers in three colors, say j_1, j_2, j_3 is the integer j in colors $1, 2, 3$, satisfying the following congruence conditions

$$\begin{aligned}
&\{j_1 \mid j \geq 1, j \not\equiv 0 \bmod 2\}, \\
&\{j_2 \mid j \geq 1, j \not\equiv 0, \pm 1, \pm 2, \pm 3 \bmod 2k+6\}, \\
&\{j_3 \mid j \geq 1, j \not\equiv 0, \pm 1, \pm(k+1), \pm(k+2), k+3 \bmod 2k+6\}.
\end{aligned} \tag{9}$$

Case $C_2^{(1)}$ and $k=2$

For $k=2$ we have

$$\mathcal{C}_2 = \{1_1, 3_1, 5_1, 7_1, \dots\} \sqcup \{4_2, 5_2, 6_2, 14_2, \dots\} \sqcup \{2_3, 8_3, 12_3, 18_3 \dots\};$$

and all colored partitions of 5 with colored parts in $\mathcal{C}_2$ are

5_1

5_2

$4_2 + 1_1$

$3_1 + 2_3$

$3_1 + 1_1 + 1_1$

$2_3 + 2_3 + 1_1$

$2_3 + 1_1 + 1_1 + 1_1$

$1_1 + 1_1 + 1_1 + 1_1 + 1_1$

Case $C_2^{(1)}$ and $k = 2$

Let $n = k = 2$. Then the first nine terms of Taylor series (8) are

$$1 + q + 2q^2 + 3q^3 + 5q^4 + 8q^5 + 12q^6 + 17q^7 + 25q^8 + \cdots \quad (10)$$

Case $C_2^{(1)}$ and $k=2$

On the other hand, in the principal specialization $e^{-\alpha_i} \mapsto q^1$, $i=0,1,2$, the sequence of root subspaces in $C_2^{(1)}$

$$X_{ab}(-1), ab \in B, \quad X_{ab}(-2), ab \in B, \quad X_{ab}(-3), ab \in B, \quad \dots \tag{11}$$

obtains degrees

$$\begin{matrix} 1 & & & & 5 & & & & 9 & & & & \\ 2 & 3 & & & 6 & 7 & & & 10 & 11 & & & \\ 3 & 4 & 5 & & 7 & 8 & 9 & & 11 & 12 & 13 & & \dots \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \end{matrix} \tag{12}$$

Case $C_2^{(1)}$ and $k = 2$

In order to make numbers distinct, we consider four colors $1, 2, 3, 4$, say

$$
\begin{array}{cccccccccccccc}
1_1 & & & & 5_1 & & & & 9_1 & & & & & \\
2_2 & 3_2 & & & 6_2 & 7_2 & & & 10_2 & 11_2 & & & & \\
3_3 & 4_3 & 5_3 & & 7_3 & 8_3 & 9_3 & & 11_3 & 12_3 & 13_3 & & & \dots, \\
4_4 & 5_4 & 6_4 & 7_4 & 8_4 & 9_4 & 10_4 & 11_4 & 12_4 & 13_4 & 14_4 & 15_4 & &
\end{array} \tag{13}
$$

so that numbers in the first row have color 1, numbers in the second row have color 2, and so on.

Case $C_2^{(1)}$ and $k = 2$

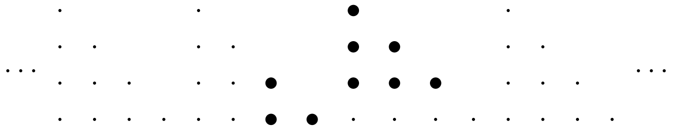
In other words, for fixed $n = 2$ we consider a disjoint union $\mathcal{D}_2$ of integers in four colors, say j_1, j_2, j_3, j_4 is the integer j in colors $1, 2, 3, 4$.satisfying the congruence conditions

$$\begin{aligned} &\{j_1 \mid j \geq 1, j \equiv 1 \bmod 4\}, \\ &\{j_2 \mid j \geq 2, j \equiv 2, 3 \bmod 4\}, \\ &\{j_3 \mid j \geq 3, j \equiv 0, 1, 3 \bmod 4\}, \\ &\{j_4 \mid j \geq 4, j \equiv 0, 1, 2, 3 \bmod 4\} \end{aligned} \tag{14}$$

and arranged in a sequence of triangles (13).

Case $C_2^{(1)}$ and $k = 2$

For example, for the third row we have $r = \underline{2}$ and two triangles denoted by bullets



(15)

are ${}^{\underline{2}}\triangle$ on the left and $\triangle_{\underline{2}}$ on the right. We say that two cascades

$$\mathcal{A} \subset {}^{r}\triangle \quad \text{and} \quad \mathcal{B} \subset \triangle_{r}$$

form an admissible pair of cascades in the sequence (13).

Case $C_2^{(1)}$ and $k = 2$

By enumerating all admissible cascades for the basis B of simple Lie algebra C_2 we made a list of $4 \times 8 = 32$ difference conditions. From the list of difference conditions and the list of ordinary partitions, direct calculation gives all colored partitions of $m = 1, 2, \cdots, 8$ with colored parts in $\mathcal{D}_2$:

$$
\begin{aligned}
8 &= 8_3 = 8_4 = 7_2 + 1_1 = 7_3 + 1_1 = 7_4 + 1_1 = 6_2 + 2_2 = 6_4 + 2_2 \\
&= 6_2 + 1_1 + 1_1 = 6_4 + 1_1 + 1_1 = 5_1 + 3_2 = 5_1 + 3_3 = 5_2 + 3_2 \\
&= 5_2 + 3_3 = 5_3 + 3_2 = 5_3 + 3_3 = 5_3 + 2_2 + 1_1 = 5_4 + 2_2 + 1_1 \\
&= 4_3 + 4_3 = 4_3 + 4_4 = 4_4 + 4_4 = 4_3 + 3_2 + 1_1 = 4_3 + 3_3 + 1_1 \\
&= 4_4 + 3_2 + 1_1 = 4_3 + 2_2 + 2_2 = 3_2 + 3_2 + 1_1 + 1_1 \; .
\end{aligned}
$$

Hence the number of partitions satisfying difference conditions coincides with the coefficients of above Taylor series for $m = 1, 2, \cdots, 8$.

Case $C_2^{(1)}$ and $k=2$

$$
\begin{aligned}
1 &= 1_1 \\
2 &= 2_2 = 1_1 + 1_1 \\
3 &= 3_2 = 3_3 = 2_2 + 1_1 \\
4 &= 4_3 = 4_4 = 3_2 + 1_1 = 3_3 + 1_1 = 2_2 + 2_2 \\
5 &= 5_1 = 5_3 = 5_4 = 4_3 + 1_1 = 4_4 + 1_1 = 3_2 + 2_1 = 3_3 + 2_1 \\
&= 3_2 + 1_1 + 1_1 \\
6 &= 6_2 = 6_4 = 5_1 + 1_1 = 5_3 + 1_1 = 5_4 + 1_1 = 4_3 + 2_2 = 4_4 + 2_2 \\
&= 4_3 + 1_1 + 1_1 \\
7 &= 7_2 = 7_3 = 7_4 = 6_2 + 1_1 = 6_4 + 1_1 = 5_1 + 2_2 = 5_3 + 2_2 \\
&= 5_4 + 2_2 = 5_3 + 1_1 + 1_1 = 5_4 + 1_1 + 1_1 = 4_3 + 3_2 \\
&= 4_3 + 3_3 = 4_4 + 3_2 = 4_4 + 3_3 = 4_3 + 2_2 + 1_1 \\
&= 3_2 + 3_2 + 1_1 = 3_2 + 3_3 + 1_1
\end{aligned}
$$

Case $C_2^{(1)}$ and $k = 2$

$$\begin{aligned} 8 &= 8_3 = 8_4 = 7_2 + 1_1 = 7_3 + 1_1 = 7_4 + 1_1 = 6_2 + 2_2 = 6_4 + 2_2 \\ &= 6_2 + 1_1 + 1_1 = 6_4 + 1_1 + 1_1 = 5_1 + 3_2 = 5_1 + 3_3 = 5_2 + 3_2 \\ &= 5_2 + 3_3 = 5_3 + 3_2 = 5_3 + 3_3 = 5_3 + 2_2 + 1_1 = 5_4 + 2_2 + 1_1 \\ &= 4_3 + 4_3 = 4_3 + 4_4 = 4_4 + 4_4 = 4_3 + 3_2 + 1_1 = 4_3 + 3_3 + 1_1 \\ &= 4_4 + 3_2 + 1_1 = 4_3 + 2_2 + 2_2 = 3_2 + 3_2 + 1_1 + 1_1 \ . \end{aligned}$$

Case $C_2^{(1)}$ and $k = 2$

Difference conditions (1×8) of a basis B for $C_2^{(1)}$

1			
2	3		
4	5	6	
7	8	9	10

$$\prod_{\beta \in \mathcal{C}_{j+1}} X_\beta(-j-1)^{n_{\beta,j+1}} \prod_{\alpha \in \mathcal{C}_j} X_\alpha(-j)^{n_{\alpha,j}}$$

$$\begin{aligned}
b_1 + a_1 + a_2 + a_4 + a_7 &\leq 2 \\
b_1 + a_2 + a_3 + a_5 + a_7 &\leq 2 \\
b_1 + a_3 + a_4 + a_5 + a_7 &\leq 2 \\
b_1 + a_3 + a_5 + a_7 + a_8 &\leq 2 \\
b_1 + a_4 + a_5 + a_6 + a_7 &\leq 2 \\
b_1 + a_5 + a_6 + a_7 + a_8 &\leq 2 \\
b_1 + a_6 + a_7 + a_8 + a_9 &\leq 2 \\
b_1 + a_7 + a_8 + a_9 + a_{10} &\leq 2
\end{aligned}$$

Case $C_2^{(1)}$ and $k = 2$

Difference conditions (2×8) of a basis B for $C_2^{(1)}$

1		
2	3	
	5	6
	8	9	10

$$\prod_{\beta \in \mathcal{C}_{j+1}} X_\beta(-j-1)^{n_{\beta,j+1}} \prod_{\alpha \in \mathcal{C}_j} X_\alpha(-j)^{n_{\alpha,j}}$$

$$\begin{aligned}
b_1 + b_2 + a_3 + a_5 + a_8 &\leq 2 \\
b_1 + b_2 + a_5 + a_6 + a_8 &\leq 2 \\
b_1 + b_2 + a_6 + a_8 + a_9 &\leq 2 \\
b_1 + b_2 + a_8 + a_9 + a_{10} &\leq 2 \\
b_2 + b_3 + a_3 + a_5 + a_8 &\leq 2 \\
b_2 + b_3 + a_5 + a_6 + a_8 &\leq 2 \\
b_2 + b_3 + a_6 + a_8 + a_9 &\leq 2 \\
b_2 + b_3 + a_8 + a_9 + a_{10} &\leq 2
\end{aligned}$$

Case $C_2^{(1)}$ and $k = 2$

Difference conditions (3×8) of a basis B for $C_2^{(1)}$

1				
2	3			
4	5	6		
		9	10	

$$\prod_{\beta \in \mathcal{C}_{j+1}} X_\beta(-j-1)^{n_{\beta,j+1}} \prod_{\alpha \in \mathcal{C}_j} X_\alpha(-j)^{n_{\alpha,j}}$$

$$\begin{aligned}
b_1 + b_2 + b_4 + a_6 + a_9 &\leq 2 \\
b_2 + b_3 + b_4 + a_6 + a_9 &\leq 2 \\
b_3 + b_4 + b_5 + a_6 + a_9 &\leq 2 \\
b_4 + b_5 + b_6 + a_6 + a_9 &\leq 2 \\
b_1 + b_2 + b_4 + a_9 + a_{10} &\leq 2 \\
b_2 + b_3 + b_4 + a_9 + a_{10} &\leq 2 \\
b_3 + b_4 + b_5 + a_9 + a_{10} &\leq 2 \\
b_4 + b_5 + b_6 + a_9 + a_{10} &\leq 2
\end{aligned}$$

Case $C_2^{(1)}$ and $k = 2$

Difference conditions (4×8) of a basis B for $C_2^{(1)}$

1			
2	3		
4	5	6	
7	8	9	10

$$\prod_{\beta\in\mathcal{C}_{j+1}} X_\beta(-j-1)^{n_{\beta,j+1}} \prod_{\alpha\in\mathcal{C}_j} X_\alpha(-j)^{n_{\alpha,j}}$$

$$\begin{aligned}
b_1 + b_2 + b_4 + b_7 + a_{10} &\leq 2 \\
b_2 + b_3 + b_5 + b_7 + a_{10} &\leq 2 \\
b_3 + b_4 + b_5 + b_7 + a_{10} &\leq 2 \\
b_3 + b_5 + b_7 + b_8 + a_{10} &\leq 2 \\
b_4 + b_5 + b_6 + b_7 + a_{10} &\leq 2 \\
b_5 + b_6 + b_7 + b_8 + a_{10} &\leq 2 \\
b_6 + b_7 + b_8 + b_9 + a_{10} &\leq 2 \\
b_7 + b_8 + b_9 + b_{10} + a_{10} &\leq 2
\end{aligned}$$

Case $C_2^{(1)}$ and $k=2$

How difference conditions eliminated the colored partition $5_1+2_2+1_1$ in the case $m=8$?
First of all, notice that 5_1 belongs to the triangle $X_{ab}(-2)$, and 2_2 and 1_1 belong to the triangle $X_{ab}(-1)$.
Now we chose $r=1$ and consider the triangles ${}^1\triangle$ and $\triangle_1$ and the pair of admissible cascades is

$$\begin{array}{llllllll c llllllll}
m_{11;1} & & & & m_{11;2} & & & & & 1_1 & & & & 5_1 & & & \\
m_{12;1} & \cdot & & & \cdot & \cdot & & & \curvearrowright & 2_2 & \cdot & & & \cdot & \cdot & & \\
m_{1\underline{2};1} & \cdot & \cdot & & \cdot & \cdot & \cdot & & & 0 & \cdot & \cdot & & \cdot & \cdot & \cdot & \\
m_{1\underline{1};1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Case $C_2^{(1)}$ and $k = 2$

The corresponded difference condition—one of 32 conditions—is given by

$$m_{11;2} + m_{11;1} + m_{12;1} + m_{1\underline{2};1} + m_{1\underline{1};1} \leq 2\ .$$

$$(b_1 + a_1 + a_2 + a_4 + a_7 \leq 2 \ \rightsquigarrow 1^{st} \text{ one})$$

Since
$m_{11;2} + m_{11;1} + m_{12;1} + m_{1\underline{2};1} + m_{1\underline{1};1} = 1 + 1 + 1 + 0 + 0 = 3 > 2$, the observed colored partition is eliminated from the list.

Combinatorial version of Conjecture

Let $n = 2$ and $k \geq 2$. We conjecture that for every $m \in \mathbb{N}$ the number of colored partitions

$$m = \sum_{j_a \in \mathcal{C}_k} j_a f_{j_a}$$

in three colors satisfying congruence conditions (9) equals the number of colored partitions

$$m = \sum_{j_a \in \mathcal{D}_2} j_a f_{j_a}$$

in four colors satisfying congruence conditions (14) and difference conditions for every admissible pair of cascades in the sequence (13).