Combinatorial bases of principal subspaces for affine Lie algebras of type $B_l^{(1)}$ and $C_l^{(1)}$

Marijana Butorac
University of Rijeka
supported by CSF, grant. 2634.

Representation Theory XIV
Dubrovnik, Croatia,
June 21 - 27, 2015

- Quasi-particle bases of principal subspaces for the affine Lie algebras of types $B_l^{(1)}$ and $C_l^{(1)}$, arXiv:1505.00450 [math.QA].

Motivation

- Lepowsky and Primc: character formulas for all standard modules of affine Lie algebra of type $A_1^{(1)}$
- B. L. Feigin and A. V. Stoyanovsky: introducing principal subspaces of standard $A_1^{(1)}$-modules; the principal subspace of level one standard module of affine Lie algebra of type $A_1^{(1)}$ has a combinatorial basis that satisfies the difference-two condition;
- Rogers-Ramanujan identities:

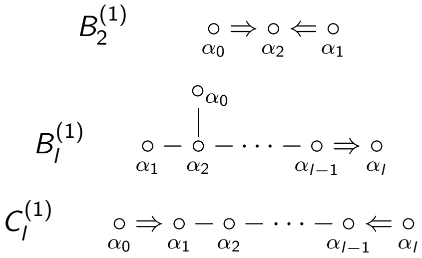
$$\sum_{r\geq 0}\frac{q^{r^2}}{(q)_r}=\prod_{i\geq 0}\frac{1}{(1-q^{5i+1})(1-q^{5i+4})}$$

$$\sum_{r\geq 0}\frac{q^{r^2+r}}{(q)_r}=\prod_{i\geq 0}\frac{1}{(1-q^{5i+2})(1-q^{5i+3})}$$

  where $(q)_r=(1-q)(1-q^2)\cdots(1-q^r)$

- G. Georgiev: combinatorial bases of principal subspaces of standard $A_n^{(1)}$-modules ($n\geq 1$); character formulas for affine Lie algebras of $A-D-E$ types

Combinatorial bases of principal subspaces



1. Principal subspace
   - Modules of affine Lie algebra
   - Principal subspaces
   - Quasi-particle

2. Combinatorial bases
   - Combinatorial basis in the case of $B_l^{(1)}$, $l \geq 2$
   - Combinatorial basis in the case of $C_l^{(1)}$, $l \geq 2$
   - Combinatorial basis for the pricipal subspace W_N

3. Character of the principal subspaces
   - Characters of the principal subspace W_L
   - Characters of the principal subspace W_N

Simple Lie algebra of type X_l

$\mathfrak{g}$ simple Lie algebra of type X_l

- $\mathfrak{h}$ the Cartan subalgebra
- $\langle \cdot, \cdot \rangle$ the normalized symmetric invariant nondegenerate bilinear form
- $\mathfrak{h} \cong \mathfrak{h}^*$
- $\Pi = \{\alpha_1, ..., \alpha_l\}$ simple roots
- R the set of roots of $\mathfrak{g}$
- $\langle \alpha, \alpha \rangle = 2$, $\alpha \in R$ long root
- $R = R_+ \cup R_-$
- triangular decomposition:
  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$
- x_α fixed root vectors
- $\omega_1, \ldots, \omega_l$ fundamental weights

Affine Lie algebra of type $X_l^{(1)}$

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$- associated affine Lie algebra

- c canonical central element
- $x(j) = x \otimes t^j, \quad x \in \mathfrak{g}, \quad j \in \mathbb{Z}$
- $[x(j_1), y(j_2)] = [x, y]\,(j_1 + j_2) + \langle x, y \rangle\, j_1 \delta_{j_1+j_2,0}\, c, \quad [c, x(j)] = 0$
- d degree operator $\quad [c, d] = 0, \quad [d, x(j)] = jx(j)$
- $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d$
- $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ the Cartan subalgebra
- $\tilde{\Pi} = \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_l^\vee\}$ simple roots
- $\Lambda_0, \Lambda_1, \ldots, \Lambda_l$ fundamental weights
- $L(\Lambda_0), L(\Lambda_1), \ldots, L(\Lambda_l)$ standard $\tilde{\mathfrak{g}}$-modules of level 1 with highest weight vectors $v_{\Lambda_0}, v_{\Lambda_1}, \ldots, v_{\Lambda_l}$

Modules of affine Lie algebra

$k \in \mathbb{N}$

$N(k\Lambda_0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} \mathbb{C}v_N$ - generalized Verma module

- $\hat{\mathfrak{g}}_{\geq 0} = \coprod_{n \geq 0} \mathfrak{g} \otimes t^n \oplus \mathbb{C}c$ - subalgebra of $\hat{\mathfrak{g}}$
- $1 \otimes v_N = v_N$ - highest weight vector
- $dv_N = 0$

$L(k\Lambda_0)$ - standard (integrable highest weight) $\tilde{\mathfrak{g}}$-module of level k

- v_L - a highest weight vector of $L(k\Lambda_0)$

Principal subspaces

- $k \in \mathbb{N}$- fixed
- $\mathfrak{n}_+ = \oplus_{\alpha \in R_+} \mathfrak{n}_\alpha, \quad \mathfrak{n}_\alpha = \mathbb{C} x_\alpha$
- $\mathcal{L}(\mathfrak{n}_+) = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}]$

Principal subspace W_L of $L(k\Lambda_0)$

$$W_L := U(\mathcal{L}(\mathfrak{n}_+))v_L$$

Principal subspace W_N of $N(k\Lambda_0)$

$$W_N := U(\mathcal{L}(\mathfrak{n}_+))v_N$$

Principal subspaces

- Feigin and Stoyanovsky: introduced quasi-particles of color 1, charge 1 and energy -m, coefficients of formal Laurent series

$$x_\alpha(z)^r = x_\alpha(z_1)x_\alpha(z_2)\cdots x_\alpha(z_r)|_{z=z_1=z_2=\cdots=z_r};$$

  energies satisfy the difference-two condition

- Georgiev: combinatorial bases of principal subspaces of standard $A_l^{(1)}$-modules $(n \geq 1)$ are given in terms of quasi-particles of color i, $1 \leq i \leq l$, a charge r, $1 \leq r \leq k$ and energy $-m$ such that the energies of the same color and charge satisfy certain difference conditions;

$$\underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_l}{\circ}$$

Vertex operator algebra structure

$N(k\Lambda_0)$ - a vertex operator algebra

- v_N - a vacuum vector
- $Y(x(-1)v_N, z) = x(z) = \sum_{j \in \mathbb{Z}} x(j) z^{-j-1}$ - vertex operator associated with the vector $x(-1)v_N \in N(k\Lambda_0)$, $x \in \mathfrak{g} \cong \mathfrak{g}(-1)v_N \subset N(k\Lambda_0)$

- an induced vertex operator algebra structure on $L(k\Lambda_0)$
- every irreducible module for the vertex operator algebra $L(k\Lambda_0)$ is a standard $\tilde{\mathfrak{g}}$-module

Quasi-particle

- $\alpha_i \in \Pi$,
- $r \in \mathbb{N}$

$$x_{r\alpha_i}(z) = \underbrace{x_{\alpha_i}(z) \cdots x_{\alpha_i}(z)}_{r-\text{ factors}} = Y((x_{\alpha_i}(-1))^r v_N, z)$$

- $m \in \mathbb{Z}$

Quasi-particle of color i, charge r and energy -m

$$x_{r\alpha_i}(m) = \mathrm{Res}_z \left\{ z^{m+r-1} x_{r\alpha_i}(z) \right\}$$

$$x_{r\alpha_i}(m) = \sum_{\substack{m_1,\dots,m_r \in \mathbb{Z} \\ m_1+\cdots+m_r=m}} x_{\alpha_i}(m_r) \cdots x_{\alpha_i}(m_1)$$

- $x_{r\alpha_i}(z)$ - represents the generating function for quasi-particles of color i and charge r

Quasi-particle monomials

Quasi-particle monomial $b(\alpha_i)$ is colored with color-type r_i, if the sum of all quasi-particle charges in monomial $b(\alpha_i)$ is r_i.
Partiton of a positive integer r_i

$$r_i = (r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(s)}),$$

$$r_i^{(1)} \geq r_i^{(2)} \geq \ldots \geq r_i^{(s)} \geq 0, \ \sum_{t=1}^{s} r_i^{(t)} = r_i$$

Conjugate of $(r_i^{(1)}, r_i^{(2)}, \ldots, r_i^{(s)})$

$$\left(n_{r_i^{(1)},i}, \ldots, n_{1,i}\right),$$

$$0 \leq n_{r_i^{(1)},i} \leq \cdots \leq n_{1,i}, \ \sum_{t=1}^{r_i^{(1)}} n_{t,i} = r_i$$

Quasi-particle monomials

Quasi-particle monomial

$$b(\alpha_l)\cdots b(\alpha_1)=$$

$$x_{n_{r_l^{(1)},l}\alpha_l}(m_{r_l^{(1)},l})\cdots x_{n_{1,l}\alpha_l}(m_{1,l})\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})$$

- color-type $(r_l,\ldots,r_1)$
- dual-charge-type $\left(r_l^{(1)},\ldots,r_l^{(s_l)};\ldots;r_1^{(1)},\ldots,r_1^{(s_1)}\right)$

$$r_i^{(1)}\geq r_i^{(2)}\geq\ldots\geq r_i^{(s_i)}\geq 0,\ \sum_{t=1}^{s_i} r_i^{(t)}=r_i,\ \ s_i\in\mathbb{N}$$

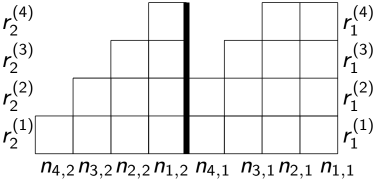
- charge-type $\left(n_{r_l^{(1)},l},\ldots,n_{1,l};\ldots;n_{r_l^{(1)},l},\ldots,n_{1,1}\right)$

$$0<n_{r_i^{(1)},i}\leq\ldots\leq n_{1,i},\ \sum_{t=1}^{r_i^{(1)}} n_{t,i}=r_i$$

Quasi-particle monomials

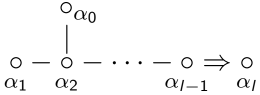
- Example:

$$x_{\alpha_2}(m_{4,2})x_{2\alpha_2}(m_{3,2})x_{3\alpha_2}(m_{2,2})x_{4\alpha_2}(m_{1,2})x_{2\alpha_1}(m_{4,1})$$
$$x_{3\alpha_1}(m_{3,1})x_{4\alpha_1}(m_{2,1})x_{4\alpha_1}(m_{1,1})$$



Construction of quasi-particle bases

the case of $B_l^{(1)}$, $l \geq 2$



- $\mathfrak{n}_\alpha = \mathbb{C}x_\alpha, \quad \alpha \in \Pi$
- $\mathcal{L}(\mathfrak{n}_\alpha) = \mathfrak{n}_\alpha \otimes \mathbb{C}[t, t^{-1}]$

Principal subspace W_L of $L(k\Lambda_0)$

$$W_L = U(\mathcal{L}(\mathfrak{n}_{\alpha_l})) \cdots U(\mathcal{L}(\mathfrak{n}_{\alpha_1}))v_L$$

Construction of quasi-particle bases

In the construction of quasi-particle basis:

- we use relations

$$x_{(k+1)\alpha_i}(z) = 0, \quad 1 \leq i \leq l-1, \quad x_{(2k+1)\alpha_l}(z) = 0$$

  on $L(k\Lambda_0)$;
- we obtain relations among quasi-particles which are true on every $\tilde{\mathfrak{g}}$-module:
  - relations among quasi-particles of the same colors, that is expressions for the products of the form

$$x_{n\alpha}(m)x_{n'\alpha}(m'),$$

    where $\alpha = \alpha_i$, $n, n' \in \mathbb{N}$, $n \leq n'$ and $m, m' \in \mathbb{Z}$ (the same as at Georgiev)
  - relations among quasi-particles of different colors, that is expressions for the products of the form

$$x_{n_i\alpha_i}(m_i)x_{n'_j\alpha_j}(m'_j),$$

    where $i, j = 1, 2, \ldots, l$, $i \neq j$ $n_i, n'_j \in \mathbb{N}$ and $m_i, m'_j \in \mathbb{Z}$

Relation among quasi-particles of different colors

Lemma

Let $n_i \in \mathbb{N}$, $1 \leq i \leq l$ be fixed. One has

- $(z_l - z_{l-1})^{min\{2n_{l-1},n_l\}} x_{n_{l-1}\alpha_{l-1}}(z_{l-1}) x_{n_l\alpha_l}(z_l) =$
  $(z_l - z_{l-1})^{min\{2n_{l-1},n_l\}} x_{n_l\alpha_l}(z_l) x_{n_{l-1}\alpha_{l-1}}(z_{l-1})$
- $(z_{i+1} - z_i)^{min\{n_{i+1},n_i\}} x_{n_i\alpha_i}(z_i) x_{n_{i+1}\alpha_{i+1}}(z_{i+1}) =$
  $(z_{i+1} - z_i)^{min\{n_{i+1},n_i\}} x_{n_{i+1}\alpha_{i+1}}(z_{i+1}) x_{n_i\alpha_i}(z_i)$; $1 \leq i \leq l-2$

The spanning set of pricipal subspace W_L of $B_l^{(1)}$

Basis for the principal subspace W_L of $B_l^{(1)}$

$$\mathfrak{B}_{W_L} = \{b = b(\alpha_l) \cdots b(\alpha_1) v_L$$

$$= x_{n_{r_l^{(1)},l}\alpha_l}(m_{r_l^{(1)},l}) \cdots x_{n_{1,l}\alpha_l}(m_{1,l}) \cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1}) \cdots x_{n_{1,1}\alpha_1}(m_{1,1}) v_L :$$

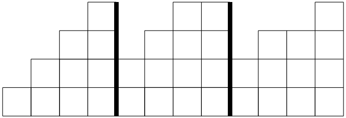
$$\left| \begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i-1}^{(1)}} min\{n_{q,i-1}, n_{p,i}\} - \sum_{p>p'>0} 2\ min\{n_{p,i}, n_{p',i}\}, \\ \qquad\qquad 1 \leq p \leq r_i^{(1)},\ 1 \leq i \leq l-1; \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i}\ if\ n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1,\ 1 \leq i \leq l-1; \\ m_{p,l} \leq -n_{p,l} + \sum_{q=1}^{r_{l-1}^{(1)}} min\{2n_{q,l-1}, n_{p,l}\} - \sum_{p>p'>0} 2\ min\{n_{p,l}, n_{p',l}\},\ 1 \leq p \leq r_l^{(1)}; \\ m_{p+1,l} \leq m_{p,l} - 2n_{p,l}\ if\ n_{p,l} = n_{p+1,l},\ 1 \leq p \leq r_l^{(1)} - 1 \end{array} \right\}$$

- charge-type: $\left(n_{r_1^{(1)},1}, \ldots, n_{1,1}; \ldots; n_{r_l^{(1)},l}, \ldots, n_{1,l}\right)$
  $n_{r_1^{(1)},1} \leq \ldots \leq n_{1,1} \leq k, \quad \ldots, \quad n_{r_{l-1}^{(1)},l-1} \leq \ldots \leq n_{1,l-1} \leq k, \quad n_{r_l^{(1)},l} \leq \ldots \leq n_{1,l} \leq 2k$

- dual-charge-type: $\left(r_1^{(1)}, \ldots, r_1^{(k)}; \ldots; r_l^{(1)}, \ldots, r_l^{(2k)}\right)$
  $r_1^{(1)} \geq \cdots \geq r_1^{(k)} \geq 0, \quad \ldots, r_{l-1}^{(1)} \geq \cdots \geq r_{l-1}^{(k)} \geq 0, \quad r_l^{(1)} \geq \cdots \geq r_l^{(2k)} \geq 0$

Combinatorial bases of principal subspaces in the case of $B_l^{(1)}$

- Example:



$$x_{\alpha_l}(m_{4,l})x_{2\alpha_l}(m_{3,l})x_{3\alpha_l}(m_{2,l})x_{4\alpha_2}(m_{1,l})$$

$$x_{2\alpha_{l-1}}(m_{4,l-1})x_{3\alpha_{l-1}}(m_{3,l-1})x_{4\alpha_{l-1}}(m_{2,l-1})x_{4\alpha_{l-1}}(m_{1,l-1})$$

$$x_{2\alpha_{l-2}}(m_{4,l-2})x_{3\alpha_{l-2}}(m_{3,l-2})x_{4\alpha_{l-2}}(m_{2,l-2})x_{4\alpha_{l-2}}(m_{1,l-2})v_L$$

$$\begin{array}{lclclclclcl}
m_{1,l} & \leq & 12; & \quad & m_{1,l-1} & \leq & 8; & \quad & m_{1,l-2} & \leq & -4; \\
m_{2,l} & \leq & 3; & & m_{2,l-1} & \leq & 0; & & m_{2,l-2} & \leq & -9; \\
m_{3,l} & \leq & -2; & & m_{3,l-1} & \leq & -4; & & m_{3,l-2} & \leq & -15; \\
m_{4,l} & \leq & -3 & & m_{3,l-1} & \leq & -6; & & m_{4,l-2} & \leq & 3
\end{array}$$

Construction of quasi-particle bases

the case of $C_l^{(1)}$, $l \geq 2$

$$\underset{\alpha_0}{\circ} \Rightarrow \underset{\alpha_1}{\circ} - \underset{\alpha_2}{\circ} - \cdots - \underset{\alpha_{l-1}}{\circ} \Leftarrow \underset{\alpha_l}{\circ}$$

- $\mathfrak{n}_\alpha = \mathbb{C} x_\alpha, \quad \alpha \in \Pi$
- $\mathcal{L}(\mathfrak{n}_\alpha) = \mathfrak{n}_\alpha \otimes \mathbb{C}[t, t^{-1}]$

Principal subspace W_L of $L(k\Lambda_0)$

$$W_L = U(\mathcal{L}(\mathfrak{n}_{\alpha_1})) \cdots U(\mathcal{L}(\mathfrak{n}_{\alpha_l})) v_L$$

Relation among quasi-particles of different colors

Lemma

Let $n_i \in \mathbb{N}$, $1 \leq i \leq l$ be fixed. One has

- $(z_l - z_{l-1})^{min\{n_{l-1},2n_l\}} x_{n_{l-1}\alpha_{l-1}}(z_{l-1}) x_{n_l\alpha_l}(z_l) =$
  $(z_l - z_{l-1})^{min\{n_{l-1},2n_l\}} x_{n_l\alpha_l}(z_l) x_{n_{l-1}\alpha_{l-1}}(z_{l-1})$
- $(z_{i+1} - z_i)^{min\{n_{i+1},n_i\}} x_{n_i\alpha_i}(z_i) x_{n_{i+1}\alpha_{i+1}}(z_{i+1}) =$
  $(z_{i+1} - z_i)^{min\{n_{i+1},n_i\}} x_{n_{i+1}\alpha_{i+1}}(z_{i+1}) x_{n_i\alpha_i}(z_i)$; $1 \leq i \leq l-2$

Basis for the pricipal subspace W_L of $C_l^{(1)}$

Basis for the principal subspace W_L of $C_l^{(1)}$

$$\mathfrak{B}_{W_L} = \{b = b(\alpha_1)\cdots b(\alpha_l)v_L$$

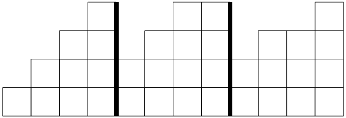
$$= x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})\cdots x_{n_{r_l^{(1)},l}\alpha_l}(m_{r_l^{(1)},l})\cdots x_{n_{1,l}\alpha_l}(m_{1,l})v_L :$$

$$\left| \begin{array}{l} m_{p,l} \leq -n_{p,l} - \sum_{p>p'>0} 2\ min\{n_{p,l}, n_{p',l}\},\ 1 \leq p \leq r_l^{(1)}; \\ m_{p+1,l} \leq m_{p,l} - 2n_{p,l}\ if\ n_{p,l} = n_{p+1,l},\ 1 \leq p \leq r_l^{(1)} - 1; \\ m_{p,l-1} \leq -n_{p,l-1} + \sum_{q=1}^{r_l^{(1)}} min\{2n_{q,l}, n_{p,l-1}\} - \sum_{p>p'>0} 2\ min\{n_{p,l}, n_{p',l}\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \leq p \leq r_l^{(1)}; \\ m_{p+1,l-1} \leq m_{p,l-1} - 2n_{p,l-1}\ if\ n_{p+1,l-1} = n_{p,l-1},\ 1 \leq p \leq r_{l-1}^{(1)} - 1; \\ m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i+1}^{(1)}} min\{n_{q,i+1}, n_{p,i}\} - \sum_{p>p'>0} 2\ min\{n_{p,i}, n_{p',i}\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \leq p \leq r_i^{(1)},\ 1 \leq i \leq l-2; \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i}\ if\ n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1,\ 1 \leq i \leq l-2 \end{array} \right\}$$

- charge-type: $\left(n_{r_l^{(1)},l}, \ldots, n_{1,l}; \ldots; n_{r_1^{(1)},1}, \ldots, n_{1,1}\right)$
  $n_{r_1^{(1)},l} \leq \ldots \leq n_{1,l} \leq 2k, \quad n_{r_{l-1}^{(1)},l-1} \leq \ldots \leq n_{1,l-1} \leq 2k, \quad \ldots, \quad n_{r_l^{(1)},1} \leq \ldots \leq n_{1,1} \leq k$
- dual-charge-type: $\left(r_l^{(1)}, \ldots, r_l^{(k)}; \ldots; r_1^{(1)}, \ldots, r_1^{(2k)}\right)$
  $r_l^{(1)} \geq \cdots \geq r_l^{(2k)} \geq 0, \quad r_{l-1}^{(1)} \geq \cdots \geq r_{l-1}^{(2k)} \geq 0, \quad \ldots, \quad r_1^{(1)} \geq \cdots \geq r_1^{(2k)} \geq 0$

Combinatorial bases of principal subspaces in the case of $C_l^{(1)}$

- Example:



$$x_{2\alpha_{l-2}}(m_{4,l-2})x_{3\alpha_{l-2}}(m_{3,l-2})x_{4\alpha_{l-2}}(m_{2,l-2})x_{4\alpha_{l-2}}(m_{1,l-2})$$

$$x_{2\alpha_{l-1}}(m_{4,l-1})x_{3\alpha_{l-1}}(m_{3,l-1})x_{4\alpha_{l-1}}(m_{2,l-1})x_{4\alpha_{l-1}}(m_{1,l-1})$$

$$x_{\alpha_l}(m_{4,l})x_{2\alpha_l}(m_{3,l})x_{3\alpha_l}(m_{2,l})x_{4\alpha_2}(m_{1,l})v_L$$

$$\begin{array}{lclclcl}
m_{1,l-2} & \leq & 9; & m_{1,l-1} & \leq & 12; & m_{1,l} \leq -4; \\
m_{2,l-2} & \leq & 2; & m_{2,l-1} & \leq & 4; & m_{2,l} \leq -9; \\
m_{3,l-2} & \leq & -2; & m_{3,l-1} & \leq & -3; & m_{3,l} \leq -15; \\
m_{4,l-2} & \leq & -3 & m_{3,l-1} & \leq & 0; & m_{4,l} \leq 3
\end{array}$$

Proof of linear independence

The idea of the proof of linear independence of the set $\mathfrak{B}_{W_L}$:

- we prove by an induction on the linear lexicographic ordering "$<$" on quasi-particle monomials
- assume that we have

$$\sum_{a \in A} c_a b_a x_{n_{1,1}}(m_{1,1}) v_L = 0,$$

  with $b_a x_{n_{1,1}}(m_{1,1}) v_L \in \mathfrak{B}_{W_L}$ having the same color-type
- Let b be the smallest monomial in the linear order of charge-type $\mathcal{N}$ and dual-charge type $\mathcal{R}$ such that $c_a \neq 0$ and $m_{1,1} \geq -j$, where j is energy of quasi-particle $x_{n_{1,1}\alpha}(j)$ of monomial b
- we use a projection $\pi_{\mathfrak{R}}$ to distribute factors of b_a along $v_{\Lambda_0}^{\otimes k}$:

$$\sum_{a \in A} c_a \pi_{\mathfrak{R}} b_a x_{n_{1,1}}(m_{1,1}) v_L = 0$$

Proof of linear independance

- we act repeatedly by a coefficient of a certain intertwining operator and a simple current operator e_{ω_1}

$$e_{\omega_1} : L(\Lambda_0) \to L(\Lambda_1)$$

  to raise the energies of all quasi-particles of charge $n_{1,1}$ to maximal value

$$\sum_{a \in A} c_a \pi_{\mathfrak{R}} b'_a x_{n_{1,1}}(m'_{1,1}) v_L = 0,$$

  with $b'_a x_{n_{1,1}}(m'_{1,1}) v_L \in \mathfrak{B}_{W_L}$

Proof of linear independance

- we use the "Weyl group translation" operator e_{α_1}

$$e_{\alpha_1} : L(\Lambda_0) \to L(\Lambda_0)$$

  to get to

$$\sum_{a \in A} c_a \pi_{\mathfrak{R}} b''_a v_L = 0,$$

  with $b''_a x_{n_{1,1}}(m'_{1,1}) v_L \in \mathfrak{B}_{W_L}$
- $c_a = 0$

Combinatorial basis for the principal subspace W_N

Principal subspace W_N of $N(k\Lambda_0)$

$$W_N = U(\mathcal{L}(\mathfrak{n}_{\alpha_l})) \cdots U(\mathcal{L}(\mathfrak{n}_{\alpha_1}))v_N$$

- we use set of relations which hold among quasi-particles of the same and different colors;
- the proof of linear independence of $\mathfrak{B}_{W_N}$:

$$W_{N(k\Lambda_0)} \longrightarrow W_{N(K\Lambda_0)} \longrightarrow W_{L(K\Lambda_0)}$$

$$K \neq k$$

$$K \gg 0$$

$$W_{N(K\Lambda_0)} = U(\mathcal{L}(\mathfrak{n}_+))v_{N(K\Lambda_0)} \subset N(K\Lambda_0)$$
$$W_{L(K\Lambda_0)} = U(\mathcal{L}(\mathfrak{n}_+))v_{L(K\Lambda_0)} \subset L(K\Lambda_0)$$

Basis for the pricipal subspace W_N of $B_l^{(1)}$

Basis for the principal subspace W_N of $B_l^{(1)}$

$$\mathfrak{B}_{W_N} = \{b = b(\alpha_l)\cdots b(\alpha_1)v_N$$

$$= x_{n_{r_l^{(1)},l}\alpha_l}(m_{r_l^{(1)},l})\cdots x_{n_{1,l}\alpha_l}(m_{1,l})\cdots x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})v_N :$$

$$\left| \begin{array}{l} m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i-1}^{(1)}} min\{n_{q,i-1}, n_{p,i}\} - \sum_{p>p'>0} 2\, min\{n_{p,i}, n_{p',i}\}, \\ \qquad\qquad 1 \leq p \leq r_i^{(1)},\ 1 \leq i \leq l-1; \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i} \text{ if } n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1,\ 1 \leq i \leq l-1; \\ m_{p,l} \leq -n_{p,l} + \sum_{q=1}^{r_{l-1}^{(1)}} min\{2n_{q,l-1}, n_{p,l}\} - \sum_{p>p'>0} 2\, min\{n_{p,l}, n_{p',l}\},\ 1 \leq p \leq r_l^{(1)}; \\ m_{p+1,l} \leq m_{p,l} - 2n_{p,l} \text{ if } n_{p,l} = n_{p+1,l},\ 1 \leq p \leq r_l^{(1)} - 1 \end{array} \right\}$$

- charge-type: $\left(n_{r_1^{(1)},1},\ldots,n_{1,1};\ldots;n_{r_l^{(1)},l},\ldots,n_{1,l}\right)$

$$n_{r_1^{(1)},i} \leq \ldots \leq n_{1,i} \quad \sum_{p=1}^{r_i^{(1)}} n_{p,i} = r_i$$

- dual-charge-type: $\left(r_1^{(1)},\ldots,r_1^{(s_1)};\ldots;r_l^{(1)},\ldots,r_l^{(s_l)}\right)$

$$r_i^{(1)} \geq \cdots \geq r_i^{(s_i)} \geq 0, \quad \sum_{t=1}^{s_i} r_i^{(t)} = r_i, \quad s_i \in \mathbb{N}$$

Basis for the pricipal subspace W_N of $C_l^{(1)}$

Basis for the principal subspace W_N of $C_l^{(1)}$

$$\mathfrak{B}_{W_N} = \{b = b(\alpha_1)\cdots b(\alpha_l) v_N$$

$$= x_{n_{r_1^{(1)},1}\alpha_1}(m_{r_1^{(1)},1})\cdots x_{n_{1,1}\alpha_1}(m_{1,1})\cdots x_{n_{r_l^{(1)},l}\alpha_l}(m_{r_l^{(1)},l})\cdots x_{n_{1,l}\alpha_l}(m_{1,l}) v_N :$$

$$\left| \begin{array}{l} m_{p,l} \leq -n_{p,l} - \sum_{p>p'>0} 2\ min\{n_{p,l}, n_{p',l}\},\ 1 \leq p \leq r_l^{(1)}; \\ m_{p+1,l} \leq m_{p,l} - 2n_{p,l}\ if\ n_{p,l} = n_{p+1,l},\ 1 \leq p \leq r_l^{(1)} - 1; \\ m_{p,l-1} \leq -n_{p,l-1} + \sum_{q=1}^{r_l^{(1)}} min\{2n_{q,l}, n_{p,l-1}\} - \sum_{p>p'>0} 2\ min\{n_{p,l}, n_{p',l}\}, \\ \hfill 1 \leq p \leq r_l^{(1)}; \\ m_{p+1,l-1} \leq m_{p,l-1} - 2n_{p,l-1}\ if\ n_{p+1,l-1} = n_{p,l-1},\ 1 \leq p \leq r_{l-1}^{(1)} - 1; \\ m_{p,i} \leq -n_{p,i} + \sum_{q=1}^{r_{i+1}^{(1)}} min\{n_{q,i+1}, n_{p,i}\} - \sum_{p>p'>0} 2\ min\{n_{p,i}, n_{p',i}\}, \\ \hfill 1 \leq p \leq r_i^{(1)},\ 1 \leq i \leq l-2; \\ m_{p+1,i} \leq m_{p,i} - 2n_{p,i}\ if\ n_{p+1,i} = n_{p,i},\ 1 \leq p \leq r_i^{(1)} - 1,\ 1 \leq i \leq l-2 \end{array} \right\}$$

- charge-type: $\left(n_{r_l^{(1)},l}, \ldots, n_{1,l}; \ldots; n_{r_1^{(1)},1}, \ldots, n_{1,1}\right)$

$$n_{r_1^{(1)},i} \leq \ldots \leq n_{1,i} \quad \sum_{p=1}^{r_i^{(1)}} n_{p,i} = r_i$$

- dual-charge-type: $\left(r_l^{(1)}, \ldots, r_1^{(s_1)}; \ldots; r_l^{(1)}, \ldots, r_1^{(s_l)}\right)$

$$r_i^{(1)} \geq \cdots \geq r_i^{(s_i)} \geq 0, \quad \sum_{t=1}^{s_i} r_i^{(t)} = r_i, \quad s_i \in \mathbb{N}$$

Characters of the principal subspaces

The space $N(k\Lambda_0)$ has certain gradings:

- the action of the operator $-d$: weight grading
- the action of the Cartan subalgebra $\mathfrak{h}$: color-type grading

We restrict these gradings to the principal subspaces W_L and W_N.

Characters of the principal subspace W_L

Definition of the character of the principal subspace W_L

$$ch\ W_L = \sum_{m,r_1,\ldots,r_l \geq 0} dim\ W_{L(m,r_1,\ldots,r_l)} q^m y_1^{r_1} \cdots y_l^{r_l},$$

- $q,\ y_1, \ldots y_l$ are formal variables
- $W_{L(m,r_1,\ldots,r_l)} = W_{L-m\delta+r_1\alpha_1+\ldots+r_l\alpha_l}$ is the $\widetilde{\mathfrak{h}}$-weight subspace of weight $-m\delta + r_1\alpha_1 + \ldots + r_l\alpha_l$.

Characters of the principal subspace W_L in the case of $B_l^{(1)}$

- $(q)_r := (1-q)(1-q^2)\cdots(1-q^r)$

Characters of the principal subspace W_L of $B_l^{(1)}$

$$\mathrm{ch}\ W_L = \sum_{r_1^{(1)}\geq\ldots\geq r_1^{(k)}\geq 0} \frac{q^{r_1^{(1)^2}+\cdots+r_1^{(k)^2}}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(k)}}} y_1^{r_1}$$

$$\sum_{r_2^{(1)}\geq\ldots\geq r_2^{(k)}\geq 0} \frac{q^{r_2^{(1)^2}+\cdots+r_2^{(k)^2}-r_1^{(1)}r_2^{(1)}-\cdots-r_1^{(k)}r_2^{(k)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(k)}}} y_2^{r_2}$$

$$\cdots$$

$$\sum_{r_{l-1}^{(1)}\geq\ldots\geq r_{l-1}^{(k)}\geq 0} \frac{q^{r_{l-1}^{(1)^2}+\cdots+r_{l-1}^{(k)^2}-r_{l-2}^{(1)}r_{l-1}^{(1)}-\cdots-r_{l-2}^{(k)}r_{l-1}^{(k)}}}{(q)_{r_{l-1}^{(1)}-r_{l-1}^{(2)}}\cdots(q)_{r_{l-1}^{(k)}}} y_{l-1}^{r_{l-1}}$$

$$\sum_{r_l^{(1)}\geq\ldots\geq r_l^{(2k)}\geq 0} \frac{q^{r_l^{(1)^2}+\cdots+r_l^{(2k)^2}-r_{l-1}^{(1)}(r_l^{(1)}+r_l^{(2)})-\cdots-r_{l-1}^{(k)}(r_l^{(2k-1)}+r_l^{(2k)})}}{(q)_{r_l^{(1)}-r_l^{(2)}}\cdots(q)_{r_l^{(2k)}}} y_l^{r_l}.$$

Characters of the principal subspace W_L in the case of $C_l^{(1)}$

Characters of the principal subspace W_L of $C_l^{(1)}$

$$\mathrm{ch}\ W_L = \sum_{r_1^{(1)}\geq\ldots\geq r_1^{(2k)}\geq 0} \frac{q^{r_1^{(1)^2}+\cdots+r_1^{(2k)^2}}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(2k)}}} y_1^{r_1}$$

$$\sum_{r_2^{(1)}\geq\ldots\geq r_2^{(2k)}\geq 0} \frac{q^{r_2^{(1)^2}+\cdots+r_2^{(2k)^2}-r_1^{(1)}r_2^{(1)}-\cdots-r_1^{(2k)}r_2^{(2k)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(2k)}}} y_2^{r_2}$$

$$\cdots$$

$$\sum_{r_{l-1}^{(1)}\geq\ldots\geq r_{l-1}^{(2k)}\geq 0} \frac{q^{r_{l-1}^{(1)^2}+\cdots+r_{l-1}^{(2k)^2}-r_{l-2}^{(1)}r_{l-1}^{(1)}-\cdots-r_{l-2}^{(2k)}r_{l-1}^{(2k)}}}{(q)_{r_{l-1}^{(1)}-r_{l-1}^{(2)}}\cdots(q)_{r_{l-1}^{(2k)}}} y_{l-1}^{r_{l-1}}$$

$$\sum_{r_l^{(1)}\geq\ldots\geq r_l^{(k)}\geq 0} \frac{q^{r_l^{(1)^2}+\cdots+r_l^{(k)^2}-r_l^{(1)}(r_{l-1}^{(1)}+r_{l-1}^{(2)})-\cdots-r_l^{(k)}(r_{l-1}^{(2k-1)}+r_{l-1}^{(2k)})}}{(q)_{r_l^{(1)}-r_l^{(2)}}\cdots(q)_{r_l^{(k)}}} y_l^{r_l}.$$

Characters of the principal subspace W_L

- Example: for $k = 1$ and one color $i = 1$
  $x_{\alpha_1}(m_r)\cdots x_{\alpha_1}(m_1)v_{\Lambda_0} \in W_{L(m,0,\dots,0,r)}$,
  color-type $(0,\dots,0,r)$,
  weight $m = -m_r - \cdots - m_1$

$$m_1 \leq -1,\ m_2 \leq m_1 - 2,\ \dots\ , m_r \leq m_{r-1} - 2$$

$$\sum_{m,r\geq 0} \dim\ W_{L(\Lambda_0)_{(m,r,0)}} q^m y^r = \sum_{r\geq 0} \frac{q^{r^2}}{(q)_r} y^r$$

- Rogers-Ramanujan identities

$$\sum_{r\geq 0} \frac{q^{r^2}}{(q)_r} = \prod_{i\geq 0} \frac{1}{(1-q^{5i+1})(1-q^{5i+4})}$$

$$\sum_{r\geq 0} \frac{q^{r^2+r}}{(q)_r} = \prod_{i\geq 0} \frac{1}{(1-q^{5i+2})(1-q^{5i+3})}$$

Character of principal subspace W_N

Definition of the character of the principal subspace W_N

$$ch\ W_N = \sum_{m,r_1,\dots,r_l \geq 0} dim\ W_{N(m,r_1,\dots,r_l)} q^m y_1^{r_1} \cdots y_l^{r_l},$$

- $q,\ y_1, \dots y_l$ are formal variables
- $W_{(m,r_1,\dots,r_l)} = W_{N_{-m\delta + r_1\alpha_1 + \dots + r_l\alpha_l}}$ is the $\widetilde{\mathfrak{h}}$-weight subspace of weight $-m\delta + r_1\alpha_1 + \dots + r_l\alpha_l$.

Characters of the principal subspace W_N in the case of $B_l^{(1)}$

Characters of the principal subspace W_N

$$\mathrm{ch}\ W_N$$

$$= \sum_{\substack{r_1^{(1)}\geq \ldots \geq r_1^{(u_1)}\geq 0\\ u_1\geq 0}} \frac{q^{r_1^{(1)^2}+\cdots+r_1^{(u_1)^2}}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(u_1)}}} y_1^{r_1}$$

$$\sum_{\substack{r_2^{(1)}\geq \ldots \geq r_2^{(u_2)}\geq 0\\ u_2\geq 0}} \frac{q^{r_2^{(1)^2}+\cdots+r_2^{(u_2)^2}-r_1^{(1)}r_2^{(1)}-\cdots-r_1^{(u_2)}r_2^{(u_2)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(u_2)}}} y_2^{r_2}$$

$$\cdots$$

$$\sum_{\substack{r_{l-1}^{(1)}\geq \ldots \geq r_{l-1}^{(u_{l-1})}\geq 0\\ u_{l-1}\geq 0}} \frac{q^{r_{l-1}^{(1)^2}+\cdots+r_{l-1}^{(u_{l-1})^2}-r_{l-2}^{(1)}r_{l-1}^{(1)}-\cdots-r_{l-2}^{(u_{l-1})}r_{l-1}^{(u_{l-1})}}}{(q)_{r_{l-1}^{(1)}-r_{l-1}^{(2)}}\cdots(q)_{r_{l-1}^{(u_{l-1})}}} y_{l-1}^{r_{l-1}}$$

$$\sum_{\substack{r_l^{(1)}\geq \ldots \geq r_l^{(2u_l)}\geq 0\\ u_l\geq 0}} \frac{q^{r_l^{(1)^2}+\cdots+r_l^{(2u_l)^2}-r_{l-1}^{(1)}(r_l^{(1)}+r_l^{(2)})-\cdots-r_{l-1}^{(u_l)}(r_l^{(2u_l-1)}+r_l^{(2u_l)})}}{(q)_{r_l^{(1)}-r_l^{(2)}}\cdots(q)_{r_l^{(2u_l)}}} y_l^{r_l}.$$

Characters of the principal subspace W_N in the case of $C_l^{(1)}$

Characters of the principal subspace W_N

$$\mathrm{ch}\ W_N = \sum_{\substack{r_1^{(1)}\geq\ldots\geq r_1^{(2u_1)}\geq 0\\ u_1\geq 0}} \frac{q^{r_1^{(1)^2}+\cdots+r_1^{(2u_1)^2}}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(2u_1)}}} y_1^{r_1}$$

$$\sum_{\substack{r_2^{(1)}\geq\ldots\geq r_2^{(2u_2)}\geq 0\\ u_2\geq 0}} \frac{q^{r_2^{(1)^2}+\cdots+r_2^{(2u_2)^2}-r_1^{(1)}r_2^{(1)}-\cdots-r_1^{(2u_2)}r_2^{(2u_2)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(2u_2)}}} y_2^{r_2}$$

$$\cdots$$

$$\sum_{\substack{r_{l-1}^{(1)}\geq\ldots\geq r_{l-1}^{(2u_{l-1})}\geq 0\\ u_{l-1}\geq 0}} \frac{q^{r_{l-1}^{(1)^2}+\cdots+r_{l-1}^{(2u_{l-1})^2}-r_{l-2}^{(1)}r_{l-1}^{(1)}-\cdots-r_{l-2}^{(2u_{l-1})}r_{l-1}^{(2u_{l-1})}}}{(q)_{r_{l-1}^{(1)}-r_{l-1}^{(2)}}\cdots(q)_{r_{l-1}^{(2u_{l-1})}}} y_{l-1}^{r_{l-1}}$$

$$\sum_{\substack{r_l^{(1)}\geq\ldots\geq r_l^{(u_l)}\geq 0\\ u_l\geq 0}} \frac{q^{r_l^{(1)^2}+\cdots+r_l^{(u_l)^2}-r_l^{(1)}(r_{l-1}^{(1)}+r_{l-1}^{(2)})-\cdots-r_l^{(u_l)}(r_{l-1}^{(2u_l-1)}+r_{l-1}^{(2u_l)})}}{(q)_{r_l^{(1)}-r_l^{(2)}}\cdots(q)_{r_l^{(u_l)}}} y_l^{r_l}.$$

Identity in the case of $B_l^{(1)}$

$$\prod_{m>0}\frac{1}{(1-q^m y_1)}\frac{1}{(1-q^m y_1 y_2)}\cdots\frac{1}{(1-q^m y_1\cdots y_l)}\frac{1}{(1-q^m y_1 y_2^2\cdots y_l^2)}\cdots\frac{1}{(1-q^m y_1 y_2\cdots y_l^2)}$$

$$\frac{1}{(1-q^m y_2)}\frac{1}{(1-q^m y_2 y_3)}\cdots\frac{1}{(1-q^m y_2\cdots y_l)}\frac{1}{(1-q^m y_2 y_3^2\cdots y_l^2)}\cdots\frac{1}{(1-q^m y_2 y_3\cdots y_l^2)}$$

$$\cdots$$

$$\frac{1}{(1-q^{l-1})}\frac{1}{(1-q^m y_{l-1} y_l)}\frac{1}{(1-q^m y_{l-1} y_l^2)}\frac{1}{(1-q^m y_l)}$$

$$=\sum_{\substack{r_1^{(1)}\geq\ldots\geq r_1^{(u_1)}\geq 0\\ u_1\geq 0}}\frac{q^{r_1^{(1)^2}+\cdots+r_1^{(u_1)^2}}}{(q)_{r_1^{(1)}-r_1^{(2)}}\cdots(q)_{r_1^{(u_1)}}}y_1^{r_1}$$

$$\sum_{\substack{r_2^{(1)}\geq\ldots\geq r_2^{(u_2)}\geq 0\\ u_2\geq 0}}\frac{q^{r_2^{(1)^2}+\cdots+r_2^{(u_2)^2}-r_1^{(1)}r_2^{(1)}-\cdots-r_1^{(u_2)}r_2^{(u_2)}}}{(q)_{r_2^{(1)}-r_2^{(2)}}\cdots(q)_{r_2^{(u_2)}}}y_2^{r_2}$$

$$\cdots$$

$$\sum_{\substack{r_{l-1}^{(1)}\geq\ldots\geq r_{l-1}^{(u_{l-1})}\geq 0\\ u_{l-1}\geq 0}}\frac{q^{r_{l-1}^{(1)^2}+\cdots+r_{l-1}^{(u_{l-1})^2}-r_{l-2}^{(1)}r_{l-1}^{(1)}-\cdots-r_{l-2}^{(u_{l-1})}r_{l-1}^{(u_{l-1})}}}{(q)_{r_{l-1}^{(1)}-r_{l-1}^{(2)}}\cdots(q)_{r_{l-1}^{(u_{l-1})}}}y_{l-1}^{r_{l-1}}$$

$$\sum_{\substack{r_l^{(1)}\geq\ldots\geq r_l^{(2u_l)}\geq 0\\ u_l\geq 0}}\frac{q^{r_l^{(1)^2}+\cdots+r_l^{(2u_l)^2}-r_{l-1}^{(1)}(r_l^{(1)}+r_l^{(2)})-\cdots-r_{l-1}^{(u_l)}(r_l^{(2u_l-1)}+r_l^{(2u_l)})}}{(q)_{r_l^{(1)}-r_l^{(2)}}\cdots(q)_{r_l^{(2u_l)}}}y_l^{r_l}.$$

Identity in the case of $C_l^{(1)}$

$$\prod_{m>0} \frac{1}{(1-q^m y_1)} \frac{1}{(1-q^m y_1 y_2)} \cdots \frac{1}{(1-q^m y_1 \cdots y_{l-1})} \frac{1}{(1-q^m y_1 y_2^2 \cdots y_l^2)} \cdots \frac{1}{(1-q^m y_1 y_2 \cdots y_l^2)} \frac{1}{(1-q^m y_1^2 y_2^2 \cdots y_l)}$$

$$\frac{1}{(1-q^m y_2)} \frac{1}{(1-q^m y_2 y_3)} \cdots \frac{1}{(1-q^m y_2 \cdots y_{l-1})} \frac{1}{(1-q^m y_2 y_3^2 \cdots y_l^2)} \cdots \frac{1}{(1-q^m y_2 y_3 \cdots y_l^2)} \frac{1}{(1-q^m y_2^2 y_3^2 \cdots y_l)}$$

$$\cdots$$

$$\frac{1}{(1-q^{l-1})} \frac{1}{(1-q^m y_{l-1} y_l^2)} \frac{1}{(1-q^m y_{l-1}^2 y_l)} \frac{1}{(1-q^m y_l)}$$

$$= \sum_{\substack{r_1^{(1)} \geq \ldots \geq r_1^{(2u_1)} \geq 0 \\ u_1 \geq 0}} \frac{q^{r_1^{(1)^2} + \cdots + r_1^{(2u_1)^2}}}{(q)_{r_1^{(1)} - r_1^{(2)}} \cdots (q)_{r_1^{(2u_1)}}} y_1^{r_1}$$

$$\sum_{\substack{r_2^{(1)} \geq \ldots \geq r_2^{(2u_2)} \geq 0 \\ u_2 \geq 0}} \frac{q^{r_2^{(1)^2} + \cdots + r_2^{(2u_2)^2} - r_1^{(1)} r_2^{(1)} - \cdots - r_1^{(2u_2)} r_2^{(2u_2)}}}{(q)_{r_2^{(1)} - r_2^{(2)}} \cdots (q)_{r_2^{(2u_2)}}} y_2^{r_2}$$

$$\cdots$$

$$\sum_{\substack{r_{l-1}^{(1)} \geq \ldots \geq r_{l-1}^{(2u_{l-1})} \geq 0 \\ u_{l-1} \geq 0}} \frac{q^{r_{l-1}^{(1)^2} + \cdots + r_{l-1}^{(2u_{l-1})^2} - r_{l-2}^{(1)} r_{l-1}^{(1)} - \cdots - r_{l-2}^{(2u_{l-1})} r_{l-1}^{(2u_{l-1})}}}{(q)_{r_{l-1}^{(1)} - r_{l-1}^{(2)}} \cdots (q)_{r_{l-1}^{(2u_{l-1})}}} y_{l-1}^{r_{l-1}}$$

$$\sum_{\substack{r_l^{(1)} \geq \ldots \geq r_l^{(u_l)} \geq 0 \\ u_l \geq 0}} \frac{q^{r_l^{(1)^2} + \cdots + r_l^{(u_l)^2} - r_l^{(1)}(r_{l-1}^{(1)} + r_{l-1}^{(2)}) - \cdots - r_l^{(u_l)}(r_{l-1}^{(2u_l-1)} + r_{l-1}^{(2u_l)})}}{(q)_{r_l^{(1)} - r_l^{(2)}} \cdots (q)_{r_l^{(u_l)}}} y_l^{r_l}.$$