On a free piston problem for irrotational ideal fluid flow

Boris Muha[*] Zvonimir Tutek[†]

Abstract

Our goal is to model and analyze a stationary and evolutionary ideal fluid flow through the junction of two pipes in the gravity field. Inside the 'vertical' pipe there is a heavy piston which can freely move along the pipe. In the stationary case we are interested in the equilibrium position of the piston in dependence on the geometry of junction and in the evolutionary case we study motion of the piston also in dependence on geometry. We formulate corresponding initial and boundary value problems and prove an existence results. The problem is nonlinear because the domain is unknown. Furthermore we study some qualitative properties of the solutions and compare them to the qualitative properties of a free piston problem for Newtonian fluid flow. All theoretical results are illustrated with numerical experiments.

1 Introduction

In this paper we consider irrotational, incompressible ideal fluid flow in a system of two pipes with a piston inserted inside the vertical one. This is a fluid-rigid body interaction problem, but also a free boundary problem. Motivation for considering this kind of problem is multiple, for example blood clots in blood vessels or valves in the oil-pipes. This can also be understood as a control problem with respect to geometry or mass of the piston. Of course, the study of incompressible ideal fluid flow is just a first step in analysis of this kind of problems. It is also interesting for comparison with more realistic models and better understanding of some noticed effects. For example we would like to understand which effects are consequences of incompressibility of the fluid and which effects come from the other properties (viscosity, compressibility, etc) of the fluid.

Problems of motion of the rigid body (and solids in general) in a fluid were intensively studied in the last decade (see for example [2], [11], [4], [5], [8] and references within). However in these papers the rigid body is fully immersed in a viscous fluid and there is no contact between the body and the domain boundary. Recently rigid body-ideal fluid interaction models were also studied in [20], [19] and [21]; rigid body is immersed into

---

[*] Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia, borism@math.hr

[†] Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia, tutek@math.hr

incompressible, ideal fluid filling the whole space. Non-potential fluid flow was considered and existence as well as uniqueness of classical solutions has been proved. Houot and Munnier in [13] proved the local existence result for the rigid bodies-ideal fluid interaction system in either bounded or unbounded domain. They also showed that, unlike in the viscous fluid case, collision with non zero relative velocity can occur; more recently, Houot, San Martin and Tucsnak in [14] proved existence and uniqueness of local in time strong solution in $\mathbb{R}^3$. In our case rigid body is part of the domain boundary and therefore different techniques should be applied in mathematical analysis.

Evolution free piston problem for gas dynamics in $1D$ case has been studied by Takeno ([22]) and D'Acunto and Rionero ([3]).

Stationary free piston problem for incompressible viscous fluid flow and its numerical analysis has been considered by authors ([16], [17]).

Goal of this paper is to give precise mathematical formulation of stationary and evolutionary problem and prove corresponding existence results. Furthermore we will analyse qualitative properties of the solutions and illustrate them with numerical examples. We will also analyse these properties in dependence on geometry of the problem.

In the remaining part of introduction we will introduce some notations and precisely describe considered geometry. In section two first we formulate the stationary problem. Then by using similar techniques as in [16] we prove the existence theorem. We also prove non-uniqueness of the solution and give a few numerical examples. The main results of this paper are contained in section three. There we give formulation of the evolution problem and then we prove the existence results. Existence of a solution local in time for arbitrary data is proved by using Schauder's fixed point theorem. Then the solution for arbitrary time interval $(0, T)$ is established for small data. Existence of global the solution depends on the geometry of the problem. Finally, the stability of equilibrium of the piston is analyzed. Theoretical results are illustrated with numerical examples.

1.1 Notation and geometry of the problem

First we will precisely describe geometry of the problem. We consider system of two pipes of constant cross-section in the gravity field. The system is constituted by infinite horizontal pipe $\mathcal{F}_1$ and semi-infinite "vertical" pipe $\mathcal{F}_2$. The angle between the horizontal and the vertical pipe is denoted by α; by definition, α is the angle between direction opposite to the gravity and vertical pipe. We consider only the control volume of pipe $\mathcal{F}_1$ of length $2L$ with two artificial exits, Σ_p and Σ_k. Diameter of the cross-section of the horizontal pipe $\mathcal{F}_1$ is d_1 and diameter of cross-section of the "vertical" pipe $\mathcal{F}_2$ is d_2 (see Figure 1).

Inside the "vertical" pipe we have a heavy piston of constant cross section of diameter d. The upper and lower face of the piston are horizontal, while its lateral surface aligns to the pipe. Therefore $d_2 = d \cos \alpha$.

Friction is neglected. Fluid is modeled as an ideal fluid and enters the "vertical" pipe only up to a height of the lower face of the piston. The piston is modeled as a rigid body, so its motion is given by Newton's second law.

Let us now introduce some notations and precise assumptions on the geometry of the

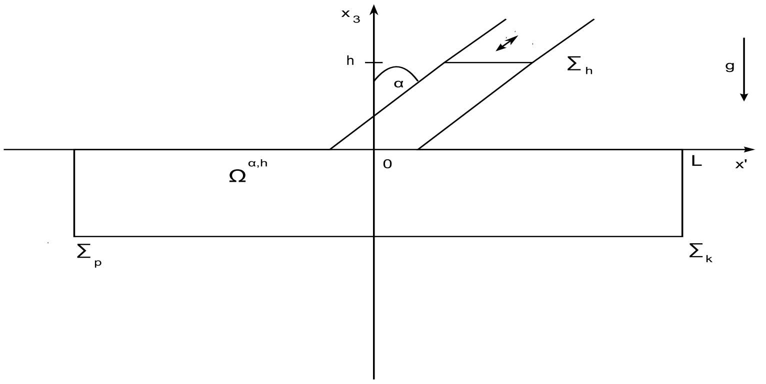

Figure 1: Ω_h^α

control volume. Coordinate frame is chosen in such a way that the lower end of the "vertical" pipe Σ_0 is a subset of $x_3 = 0$ plane. The coordinate x_1 is along the horizontal pipe and x_3 is in the opposite direction of the acceleration of gravity (x_2 in the $2D$ case). Let h be height of the piston in selected the coordinate frame. By $\Omega_h^\alpha \subset \mathbb{R}^n,\ n = 2, 3$ we denote the domain occupied by the fluid. More precisely, let S_1 be the cross-section of $\mathcal{F}_1$ and let

$$\Omega_1 = \{(x_1, x_2, x_3)\,;\, -L \leq x_1 \leq L,\ (x_2, x_3) \in S_1\}, \quad S_1 \subset \mathbb{R}^2.$$

Next, let $\mathbf{s} = \sin\alpha \mathbf{e}_1 + \cos\alpha \mathbf{e}_3$ be direction of the "vertical" pipe $\mathcal{F}_2$ and $\Sigma \subset \mathbb{R}^2$ the lower face of the piston. Then $\Omega_2^{h,\alpha}$ is a set having in non-orthogonal coordinate frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{s})$ the form:

$$\Omega_2^{h,\alpha} = \{(z_1, z_2, z_3)\,;\, 0 \leq z_3 \leq h/\cos\alpha,\ (z_1, z_2) \in \Sigma\}, \quad \Sigma \subset \mathbb{R}^2.$$

Now

$$\Omega_h^\alpha = \Omega_1 \cup \Omega_0 \cup \Omega_2^{h,\alpha}.$$

Note that only the "vertical" pipe $\Omega_2^{h,\alpha}$ depends on h and α. The lower face of the piston Σ_h is a subset of the $x_3 = \mathit{const}$ plane. We assume that Σ_0 is symmetric w.r.t. plane perpendicular to the central axis of the horizontal pipe. Furthermore, we suppose that $\Omega_0 \cup \Omega_1$ (domain without the "vertical" pipe) is also symmetric w.r.t. plane perpendicular to the central axis of the horizontal pipe; this is a technical assumption and not a restriction. Note that Ω_0 is an extension of the vertical pipe up to the boundary of Ω_1; its shape is complicated in general in $3D$ case, in $2D$ case it is an empty set. Inflow and outflow regions are denoted by Σ_p and Σ_k respectively; they are artificial boundaries of fluid domain Ω_h^α. $\Gamma = \partial\Omega_h^\alpha \setminus (\Sigma_p \cup \Sigma_k \cup \Sigma_h)$ is rigid boundary. We will consider only the $3D$ case but formulation of the problem in the $2D$ case is straightforward and all proven results are also valid in the $2D$ case with analogous proofs.

In Section 2 we suppose that the domain is locally Lipschitzian and Σ_0 is smooth (at least C^2). We also suppose that contacts between two pipes are smoothen. This is a technical assumption because we use the regularity result that requires that all angles are less than π. However, for technical reasons, in Section 3 we suppose that the whole domain is smooth, i.e. all angles are smoothen. We describe this modification of domain more precisely in that Section.

All numerical experiments are done in the $2D$ case for simplicity. Numerical experiments are done using FreeFem++ 3.4. and the visualization is made by Mathematica 7. Boundary value problems are solved using finite elements method with piecewise quadratic elements.

2 The stationary problem

2.1 Formulation of the problem

We model the fluid as an ideal fluid i.e. incompressible fluid with constant density ϱ and constitutive equation $\mathbf{T} = -p\mathbf{I}$; here, $\mathbf{T}$ is the stress tensor and p the pressure. Moreover, we assume that fluid the flow is irrotational. Hence the total fluid force on the piston in direction $\mathbf{s}$ of the "vertical" pipe at the height h is given by:

$$-\int_{\Sigma_h} \mathbf{T}\mathbf{s} \cdot \mathbf{n} = \cos\alpha \int_{\Sigma_h} p,$$

where $\mathbf{n}$ in general denotes the unit outer normal; here $\mathbf{n} = \mathbf{e}_3$. Since domain Ω_h^α is simply connected and flow is irrotational, the flow is potential. Let us denote potential of velocity $\mathbf{v}$ by Φ, i. e. $\mathbf{v} = \nabla\Phi$. Now we can give differential formulation of our problem:
find $(\Phi, h) \in H^2(\Omega_h^\alpha) \times \mathbb{R}_+$ such that

$$\begin{gathered} \triangle\Phi = 0 \quad \text{in } \Omega_h^\alpha, \\ \tfrac{\partial}{\partial \mathbf{n}}\Phi = 0 \quad \text{on } \Gamma, \\ \tfrac{\partial}{\partial \mathbf{n}}\Phi = \tfrac{Fl}{|\Sigma_p|} \quad \text{on } \Sigma_p, \\ \tfrac{\partial}{\partial \mathbf{n}}\Phi = -\tfrac{Fl}{|\Sigma_k|} \quad \text{on } \Sigma_k, \\ \tfrac{\partial}{\partial \mathbf{n}}\Phi = 0 \quad \text{on } \Sigma_h, \\ \textstyle\int_{\Sigma_h} p = P_0. \end{gathered} \tag{1}$$

The equation $(1)_1$ is the equation for potential of velocity of irrotational flow of an ideal fluid (see [10]). Conditions $(1)_2$ and $(1)_5$ are due to the fact that the rigid boundary Γ and the piston are material boundaries and therefore velocity is tangential. Furthermore, conditions $(1)_3$ and $(1)_4$ are inflow and outflow boundary conditions with prescribed flux Fl; it is assumed that the longitudinal component of velocity is constant at both artificial boundaries. The equation $(1)_6$ is the balance of forces on the piston, where constant P_0 is supposed to be given and takes into account the weight of the piston and outer forces.

Let us examine $(1)_6$ more closely. There are three types of forces that act upon the piston: gravity, contact force from the fluid and force that comes from outer pressure. Hence the total force density on the piston is given by

$$-\mathbf{T}\mathbf{e}_3 - (\varrho_k g + p_V)\mathbf{e}_3, \tag{2}$$

where ϱ_k is constant area mass density of the piston, p_V constant outer pressure, and g the gravity constant. Since the piston can not rotate, relevant quantity is the total force on the piston in the direction $\mathbf{s}$:

$$\cos\alpha(\int_{\Sigma_h} p - mg - p_V|\Sigma_h|), \tag{3}$$

where m is the mass of the piston. Since the fluid is incompressible, pressure is determined only up to a constant. However in (3) we have difference of pressures p and p_V so the total force is uniquely determined.

Since problem (1) is given in terms of potential Φ, we also need to express the function of total force on piston F in terms of potential Φ. Let us now introduce the Bernoulli function

$$B = \frac{v^2}{2} + \frac{p}{\varrho} + gx_3.$$

It is well-known that B is constant function if flow is irrotational and stationary ([10]). Since we have not yet fixed pressure, we can take this constant to be 0. By fixing pressure inside the pipe, we have also redefined constant p_V. Therefore, we have

$$p = -\varrho(\frac{v^2}{2} + gx_3) = p_H - \frac{\varrho}{2}|\nabla\Phi|^2,$$

where $p_H(x_3) = -\varrho g x_3$ denotes hydrostatic pressure. Finally we have the formula for total contact force from fluid on the piston on height h:

$$F^\alpha(h) := \int_{\Sigma_h} p = -\varrho g h|\Sigma_0| - \frac{\varrho}{2}\int_{\Sigma_h} |\nabla\Phi|^2. \tag{4}$$

Now if we set $P_0 = mg + p_V|\Sigma_h|$, balance of forces $(1)_6$ can be written as $F^\alpha(h) = P_0$. Notice that $F^\alpha(h) < 0$, $h > 0$ and P_0 is greater for heavier piston and greater outer pressure. At first sight $F^\alpha(h) < 0$ may seem non-intuitive, but this comes from the fact that we have fixed pressure in a such way that $B = 0$. Note that formula (4) makes sense because we impose that $\Phi \in H^2(\Omega_h^\alpha)$ in problem (1).

At the end of this subsection let us note two symmetry properties of the considered problem (1).

Remark 1.

- *Let $\alpha = 0$, i. e. pipes are perpendicular and let $h > 0$ be fixed. Then by direct computation we can verify that potential Φ_h, which is solution of problem $(1)_{1-5}$, satisfies symmetry property $\Phi_h(x_1, x_2, x_3) = -\Phi_h(-x_1, x_2, x_3)$. Hence, we have following symmetry properties for corresponding velocity:*

$$v_1(x_1, x_2, x_3) = v_1(-x_1, x_2, x_3), \ v_i(x_1, x_2, x_3) = -v_i(-x_1, x_2, x_3), \ i = 2, 3.$$

- *Let us now examine transformation $(x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3)$ for general α in more details. It maps Ω_h^α on $\Omega_h^{-\alpha}$. Furthermore, if Φ_h^α is solution of problem $(1)_{1-5}$ in Ω_h^α, then it is easy to verify that $\tilde{\Phi}(x_1, x_2, x_3) = -\Phi(-x_1, x_2, x_3)$ is solution of problem $(1)_{1-5}$ in $\Omega_h^{-\alpha}$. From formula (4) we have equality $F^\alpha(h) = F^{-\alpha}(h)$, $h \in \mathbb{R}_+$, where*

$F^\alpha(h)$ is contact force from fluid on the piston in domain Ω_h^α. Hence in case of stationary flow contact force from fluid on the piston is the same for angles α and $-\alpha$. On the contrary, in Newtonian fluid model we have significant difference between opposite angles (see [16], [17]).

2.2 The existence result

We will now prove existence of the solution of the problem (1). Since proof does not depend on the angle α, in this section we will omit superscript α for simplicity of notation. Recall that F denotes contact force from the fluid on the piston. Essential part of the proof of existence is to estimate part of F that does not come from hydrostatic pressure. Since we know hydrostatic pressure explicitly, then we know asymptotic behavior of F. We will get necessary estimates by comparison of solution of problem $(1)_{1-5}$ for h fixed with solution of auxiliary problem in infinite domain $\Omega_\infty = \cup_{h>0}\Omega_h$:
find $\Phi \in H^2_{\mathrm{loc}}(\Omega_\infty)$ such that:

$$\begin{aligned} \triangle\Phi = 0 \quad & \text{in } \Omega_\infty, \\ \tfrac{\partial}{\partial \mathbf{n}}\Phi = 0 \quad & \text{on } \Gamma, \\ \tfrac{\partial}{\partial \mathbf{n}}\Phi = \tfrac{Fl}{|\Sigma_p|} \quad & \text{on } \Sigma_p, \\ \tfrac{\partial}{\partial \mathbf{n}}\Phi = -\tfrac{Fl}{|\Sigma_k|} \quad & \text{on } \Sigma_k, \\ \nabla\Phi \in L^2(\Omega_\infty) & \end{aligned} \tag{5}$$

Condition $(5)_5$, i.e. boundedness of the Dirichlet integral $\int_{\Omega_\infty} |\nabla\Phi|^2$ is standard. Let us now prove existence of the solution of the problem (5) and give precise description of its behaviour at infinity. Problem (5) can be viewed as analogue of Leray's problem for the Navier-Stokes equations (see [7], [6]) and analogous results hold.

Lemma 1. *Problem (5) has a solution Φ_∞ which is unique up to a constant. Constant can be chosen is such way that there exist $\beta > 0$ such that $\Phi_\infty \exp(\beta x_3) \in H^2(\Omega_\infty)$.*

Proof.
First we will prove existence of a H^1 solution. Let us define function space

$$\mathcal{V} = \{\varphi \in L^2(\Omega_\infty)\,;\, \nabla\varphi \in L^2(\Omega_\infty)\}.$$

It is immediate that $\mathcal{V}_{/\mathbb{R}}$ is a Hilbert space with scalar product

$$(\varphi, \psi) = \int_{\Omega_\infty} \nabla\varphi \cdot \nabla\psi.$$

Now the existence of a weak solution $\Phi_\infty \in \mathcal{V}$ of problem (5) follows directly from Riesz theorem on representation of linear functionals on Hilbert spaces. Since our domain is regular we have $\Phi_\infty \in H^2_{\mathrm{loc}}(\Omega_\infty)$. Furthermore, we know that there exists $\beta_1 > 0$ such that

$\Phi_\infty \exp(\beta_1 x_3) \in L^2(\Omega_\infty)$, see appendix in [15]. Using techniques from [8] and [7] we can get analogous behavior at infinity of derivatives. We will present main steps of the proof. Let $R > 0$ and $R_1 > R_2 > 0$ and let

$$\Omega(R) = \{\mathbf{x} \in \Omega_\infty, x_3 \geq R\},$$

$$\Omega_{R_1,R_2} = \Omega(R_2) \setminus \Omega(R_1), \quad \Omega_R = \Omega_\infty \setminus \Omega(R).$$

By multiplying equation $(5)_1$ with Φ_∞ and integrating over Ω_{R_1,R_2} we get

$$\int_{\Omega_{R_1,R_2}} |\nabla\Phi_\infty|^2 = \int_{\Sigma_{R_1}} \Phi_\infty \frac{\partial}{\partial x_3}\Phi_\infty - \int_{\Sigma_{R_2}} \Phi_\infty \frac{\partial}{\partial x_3}\Phi_\infty.$$

Then we can pass to limit as $R_1 \to \infty$; using already known asymptotic properties of Φ_∞ we have

$$H(R) := \int_{\Omega(R)} |\nabla\Phi_\infty|^2 = -\int_{\Sigma_R} \Phi_\infty \frac{\partial}{\partial x_3}\Phi_\infty.$$

Now by using standard inequalities, for every $\lambda > 0$ we get

$$H(R) = \|\nabla\Phi_\infty\|^2_{L^2(\Omega(R))} \leq \|\Phi_\infty\|_{L^2(\Sigma_R)} \|\nabla\Phi_\infty\|_{L^2(\Sigma_R)} \leq \frac{1}{\lambda^2}\|\Phi_\infty\|^2_{L^2(\Sigma_R)} + \lambda^2\|\nabla\Phi_\infty\|^2_{L^2(\Sigma_R)}$$

$$\leq \frac{C}{\lambda^2}(\|\nabla\Phi_\infty\|^2_{L^2(\Omega(R))} + \|\Phi_\infty\|^2_{L^2(\Omega_R)}) + \lambda^2\|\nabla\Phi_\infty\|^2_{L^2(\Sigma_R)}.$$

By taking λ large enough so that $C/\lambda^2 \leq 1/2$ and using the expression for derivative of H, $H'(R) = -\int_{\Sigma_R} \|\nabla\Phi_\infty|^2$, we have proved that for every $R > 0$ the following inequality holds

$$H(R) \leq C_1(\exp(-\beta_1 R) - H'(R)),$$

where C_1 is positive constants. Hence, we have

$$H'(R) \leq C_2(\exp(-\beta R) - H(R)).$$

From the Gronwall inequality follows that first derivatives of Φ_∞ decay to 0 exponentially fast. Assertion of lemma now follows from analogue of lemma VI.1.2. from [6] for Poisson's equation. ■

Now we can prove the existence theorem:

Theorem 1. *There exists $P < 0$ such that for all $P_0 \leq P$ problem (1) has at least one solution.*

Proof.

First we notice that F is a continuous function. Really, for every $h \in \mathbb{R}_+$ problem $(1)_{1-5}$ has a unique solution Φ_h up to the constant. However, the definition of function F involves only $\nabla\Phi_h$ which is uniquely determined and therefore F is well defined. Now, continuity of F follows from continuity of the trace operator and continuous dependence of the solution of the boundary value problem on change of a domain (see [18]). If we prove that $\int_{\Sigma_h} |\nabla \Phi_h|^2$ is bounded in h, then from (4) we have

$$\lim_{h\to\infty} F(h) = -\infty.$$

This equality and continuity of F gives assertion of the theorem. Hence, we just have to prove the following lemma:

Lemma 2. *Let Φ_h be solution of problem $(1)_{1-5}$ in Ω_h. Then*

$$\lim_{h\to\infty} \int_{\Sigma_h} |\nabla \Phi_h|^2 = 0.$$

Proof.
Let us define $\varphi_h = \Phi_h - \Phi_\infty$. Then φ_h satisfies:

$$\begin{aligned} \triangle \varphi_h &= 0 \quad \text{in } \Omega_h, \\ \tfrac{\partial}{\partial \mathbf{n}} \varphi_h &= 0 \quad \text{on } \partial\Omega_h \setminus \Sigma_h, \\ \tfrac{\partial}{\partial \mathbf{n}} \varphi_h &= -\tfrac{\partial}{\partial \mathbf{n}} \Phi_\infty \quad \text{on } \Sigma_h. \end{aligned}$$

Now we have

$$\|\nabla \varphi_h\|_{L^2(\Sigma_h)} \le C(h) \|\varphi_h\|_{H^2(\Omega_h)} \le C(h) \|\frac{\partial}{\partial \mathbf{n}} \Phi_\infty\|_{H^{\frac{1}{2}}(\Sigma_h)}.$$

By change of variables (explicit formula is given in section 3.2, see also [16]) one can easily show that $C(h)$ depends on h at most polynomially. Therefore from exponential decay of Φ_∞ we get that $\|\nabla \varphi_h\|_{L^2(\Sigma_h)} \to 0$ as $h \to \infty$, thus $\|\nabla \Phi_h\|_{L^2(\Sigma_h)} \to 0$. ■

Remark 2. *1. Restriction on P_0 is physically expected since the fluid can not support the piston if it is too heavy.*

2. Solution of the problem (1) is not unique in general because F is not necessarily monotone. From (4) and lemma 1. follow that F is asymptotically decreasing. Furthermore, from lemma 1. follows that there exist $0 < h_1 < h_2$ such that $-\int_{\Sigma_{h_1}} |\nabla \Phi_{h_1}|^2 < -\int_{\Sigma_{h_2}} |\nabla \Phi_{h_2}|^2$. Now we notice that Φ depends linearly on given flux Fl. Therefore from previous observation and (4) we see that for flux Fl large enough function F is not monotone and in that case the original problem (1) does not have a unique solution.

Using techniques from [16] it can be proved that non-uniqueness has a form of saddle-point bifurcation or a fold. Furthermore the following result holds:

Proposition 1. $F \in C^\infty(0, \infty)$.

Example 1. *In this example we will numerically illustrate proved results and compare function F^α for various α. Geometric parameters are $l = 5$, $d_1 = d = 1.6$ and fixed data are $Fl = \varrho = 1$ and $g = 0$. Condition $g = 0$ is not a restriction because we can compute hydrostatic pressure p_H by formula. First we fix α. Then we numerically evaluate function $F^\alpha(h)$ for various h. Notice that for evaluation $F^\alpha(h)$ we need to solve the boundary value problem (1)$_{1-5}$ in Ω_h^α. We want to emphasize that we do not solve full problem (1), but only determine values of function F^α at various heights.*

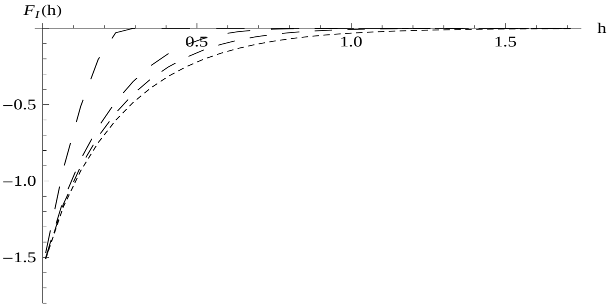

Figure 2: Graphs of functions F^0, $F^{\pi/4}$, $F^{\pi/3}$ and $F^{1.4}$

Figure 2 shows graphs of functions F^0 (shortest dashes), $F^{\pi/4}$, $F^{\pi/3}$ and $F^{1.4}$ (longest dashes). One can see that all functions have the same qualitative behavior.

3 The evolutionary problem

3.1 Formulation of the problem

In the evolutionary case, like in the stationary, we consider irrotational flow of an ideal fluid, so a scalar function Φ exists (velocity potential) such that $\mathbf{v}(t, \mathbf{x}) = \nabla_{\mathbf{x}}\Phi(t, \mathbf{x})$. In the sequel we write only ∇ instead of $\nabla_{\mathbf{x}}$. Motion of the piston is given by Newton's second law. Let $\mathbf{v}_P = h'\mathbf{s}/\cos\alpha$ denote the velocity of the piston.

In this section we will suppose that Ω_h^α is smooth domain for every $h > 0$. We get smooth domain by smoothing angles in the domain that is described in the introduction. Changed parts of the domain are subsets of Σ_h, Σ_p and Σ_k. Now whole lower face of the piston Σ_h is not flat anymore (i.e. is not subset of $x_3 = const$ plane), so term height of the piston h is now referring to the height of the flat part of the lower face of the piston. Furthermore, we suppose that smoothing of the domain is done in the same way for all α and h. More precisely, Σ_{h_1} is translation of Σ_{h_2} in direction $\mathbf{s}$, h_1, $h_2 > 0$ and transformation $(x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3)$ maps Ω_h^α onto $\Omega_h^{-\alpha}$.

Hence, formal differential formulation of considered problem is:

find (Φ, h) such that

$$\begin{gathered}
\triangle\Phi(t,.) = 0 \quad \text{in } \Omega^\alpha_{h(t)}, \\
\tfrac{\partial}{\partial \mathbf{n}}\Phi = 0 \quad \text{on } \Gamma, \\
\tfrac{\partial}{\partial \mathbf{n}}\Phi = \mathbf{e}_1 \cdot \mathbf{n}(Fl(t)/Fl_b) \quad \text{on } \Sigma_p, \\
\tfrac{\partial}{\partial \mathbf{n}}\Phi = \mathbf{e}_1 \cdot \mathbf{n}(Fl(t)/Fl_b + h'(t)Fl_p/(\cos\alpha Fl_b)) \quad \text{on } \Sigma_k, \\
\tfrac{\partial}{\partial \mathbf{n}}\Phi = (h'(t)/\cos\alpha)\mathbf{s}\cdot\mathbf{n} \quad \text{on } \Sigma_{h(t)}, \\
\textstyle\int_{\Sigma_p} \Phi(t,.)d\mathbf{x} = 0, \\
mh''(t)/\cos\alpha = F^\alpha_V(t), \\
h(0) = h_0,\ h'(0) = h'_0,
\end{gathered} \tag{6}$$

where $Fl(t)$ is a given function of flux on inflow part of boundary Σ_p and $Fl_b = \int_{\Sigma_p} \mathbf{e}_1 \cdot \mathbf{n} = -\int_{\Sigma_k} \mathbf{e}_1 \cdot \mathbf{n}$. Furthermore, h_0 and h'_0 are initial height and x_3 component of velocity of the piston, respectively. Condition $(6)_5$ is equality of the normal components of fluid and piston velocities, i.e. $\mathbf{v}\cdot\mathbf{n} = \mathbf{v}_P \cdot \mathbf{n}$. Unlike the stationary case, in the evolution case movement of the piston generates non-trivial flux in the vertical pipe. Since fluid is incompressible, flux that "enters" on Σ_h must "exit" somewhere and therefore we have an additional term $h'(t)Fl_p/(\cos\alpha Fl_b)$ in boundary conditions $(6)_4$, where $Fl_p = \int_{\Sigma_0} \mathbf{s}\cdot\mathbf{n}$ is unit flux generated by the piston. Notice that problem $(6)_{1-5}$ for Φ is the Neumann problem and does not have time derivative of Φ. Therefore we need an additional condition $(6)_6$ to have unique Φ. Of course we can fix potential in different ways, but it will be important that potential is fixed on part of boundary that does not depend on $h(t)$ and where all data are given. Finally, $(6)_7$ is second Newton's law which determines motion of the piston, where $F^\alpha_V(t)$ is total force on the piston in direction $\mathbf{s}$ at time t and $h''/\cos\alpha$ acceleration of the piston in direction $\mathbf{s}$. Analogously as in stationary case, we have

$$F^\alpha_V(t) = \int_{\Sigma_h} (p - p_V(t))\mathbf{n}\cdot\mathbf{s} - mg\cos\alpha, \tag{7}$$

where $p_V(t)$ is outer pressure, which is now a function of time.

Like in the stationary case we express total fluid force on piston F in terms of potential Φ. We define Bernoulli's function

$$B(t,\mathbf{x}) = \frac{1}{2}|\mathbf{v}(t,\mathbf{x})|^2 + \frac{p(t,\mathbf{x})}{\varrho} + gx_3.$$

It is well-known that (see [10])

$$\frac{\partial}{\partial t}\Phi(t,.) + B(t,.) = \alpha(t).$$

Now we can redefine pressure in such a way that $\alpha \equiv 0$ (for example fix $\int_{\Sigma_p} p$). Of course that means that we have also redefined outer pressure $p_V(t)$, which we will still denote by $p_V(t)$. Since this redefinition is done on the part of the boundary where data are given,

p_V does not depend on h. Notice that $p - p_V$ and therefore $F_V^\alpha(t)$ remains the same after these redefinitions.

We denote total outer force on the piston in direction $-\mathbf{s}$ at time t by $P_0^\alpha(t)$ and we have:

$$P_0^\alpha(t) = \cos\alpha\, mg + p_V(t)\int_{\Sigma_{h(t)}} \mathbf{n}\cdot\mathbf{s}.$$

Hence on piston $\Sigma_{h(t)}$ we have

$$p = -\varrho\Big(\frac{1}{2}(|\mathbf{v}_\tau|^2 + v_n^2) + gx_3 + \frac{\partial}{\partial t}\Phi(t,.)\Big) = -\varrho\Big(\frac{h'(t)^2(\mathbf{n}\cdot\mathbf{s})^2}{2\cos^2\alpha} + gx_3 + \frac{|v_\tau|^2}{2} + \frac{\partial}{\partial t}\Phi(t,.)\Big), \quad (8)$$

where v_n and $\mathbf{v}_\tau$ are normal and tangential component of velocity, respectively. We shorten the notations by introducing an auxiliary function $B(\alpha) = \int_{\Sigma_h}(\mathbf{s}\cdot\mathbf{n})^3/\cos^2\alpha$. The initial value problem $(6)_{7,8}$ can be written as

$$\begin{gathered} mh''(t)/\cos\alpha + \varrho\Big(\frac{h'(t)^2}{2}B(\alpha) + g\int_{\Sigma_{h(t)}} x_3\mathbf{s}\cdot\mathbf{n}\Big) + P_0^\alpha(t) = -\varrho\int_{\Sigma_{h(t)}}\Big(\frac{v_\tau^2}{2} + \frac{\partial}{\partial t}\Phi\Big)\mathbf{n}\cdot\mathbf{s}\,dx'. \\ h(0) = h_0,\ h'(0) = h'_0, \end{gathered} \quad (9)$$

3.2 Existence of the solution local in time

Let us now assume that motion of the piston is given by some fixed function $h \in H^2(\mathbb{R}_+)$ such that $0 < h_{\min} < h(t) < h_{\max}$, $t \in \mathbb{R}_+$. In this section we omit superscript α for simplicity of notation, since proofs and results do not depend on α. All boundary value problems considered in this section have an additional condition $\int_{\Sigma_p} r = 0$ to ensure uniqueness of the solution, where r is unknown of the given problem.

Our first goal is to obtain estimates of total fluid force on the piston. We begin with analysis of $\int_{\Sigma_h}\partial_t\Phi\,\mathbf{n}\cdot\mathbf{s}$ term. From lemma 5. and (6) we get that $\Theta = \partial_t\Phi$ is solution of the following boundary value problem

$$\begin{gathered} \triangle\Theta(t,.) = 0 \quad \text{in } \Omega_{h(t)}, \\ \tfrac{\partial}{\partial\mathbf{n}}\Theta = 0 \quad \text{on } \Gamma, \\ \tfrac{\partial}{\partial\mathbf{n}}\Theta = \mathbf{e}_1\cdot\mathbf{n}Fl'(t)/Fl_b \quad \text{on } \Sigma_p, \\ \tfrac{\partial}{\partial\mathbf{n}}\Theta = \mathbf{e}_1\cdot\mathbf{n}(Fl'(t)/Fl_b + h''(t)Fl_p/(\cos\alpha Fl_b)) \quad \text{on } \Sigma_k, \\ \tfrac{\partial}{\partial\mathbf{n}}\Theta = h''(t)\mathbf{s}\cdot\mathbf{n}/\cos\alpha - (\nabla\mathbf{v})\mathbf{v}_P\cdot\mathbf{n}, \quad \text{on } \Sigma_{h(t)}. \end{gathered} \quad (10)$$

We see that Θ depends on $h''(t)$. Now let us analyze this dependence more closely. Following idea from [14] we define auxiliary function χ^h which satisfies:

$$\begin{gathered} \triangle\chi^h(t,.) = 0 \quad \text{in } \Omega_{h(t)}, \\ \tfrac{\partial}{\partial\mathbf{n}}\chi^h = 0 \quad \text{on } \Gamma, \\ \tfrac{\partial}{\partial\mathbf{n}}\chi^h = 0 \quad \text{on } \Sigma_p, \\ \tfrac{\partial}{\partial\mathbf{n}}\chi^h = \mathbf{e}_1\cdot\mathbf{n}(Fl_p/Fl_b) \quad \text{on } \Sigma_k, \\ \tfrac{\partial}{\partial\mathbf{n}}\chi^h = \mathbf{s}\cdot\mathbf{n} \quad \text{on } \Sigma_{h(t)}, \end{gathered} \quad (11)$$

We set $\Theta_1 = h''(t)\chi^h/\cos\alpha$. Now we have

$$\int_{\Sigma_h}\Theta_1(\mathbf{s}\cdot\mathbf{n}) = \frac{h''(t)}{\cos\alpha}\int_{\Sigma_h}\chi^h\frac{\partial}{\partial\mathbf{n}}\chi^h = \frac{h''(t)}{\cos\alpha}\Big(\int_{\Omega(t)}|\nabla\chi^h|^2 + \frac{Fl_p}{|\Sigma_p|}\int_{\Sigma_k}\chi^h\Big).$$

We set $D(h)(t) = \int_{\Omega(t)}|\nabla\chi^h|^2 + \frac{Fl_p}{|\Sigma_p|}\int_{\Sigma_k}\chi^h$. Notice that D is continuous function.

Now let us estimate the remaining part of $\int_{\Sigma_h}\partial_t\Phi\mathbf{n}\cdot\mathbf{s}$ term. Let Θ_2^h and χ_1^h be solutions of the following boundary value problems:

$$\begin{cases} \triangle\Theta_2^h(t,.) = 0 \quad \text{in } \Omega_{h(t)}, \\ \frac{\partial}{\partial\mathbf{n}}\Theta_2^h = -(\nabla\mathbf{v})\mathbf{s}\cdot\mathbf{n}, \quad \text{on } \Sigma_{h(t)}, \\ \frac{\partial}{\partial\mathbf{n}}\Theta_2^h = 0, \quad \text{on } \Sigma_{k/p}, \\ \frac{\partial}{\partial\mathbf{n}}\Theta_2^h = 0, \quad \text{on } \Gamma, \end{cases} \tag{12}$$

$$\begin{cases} \triangle\chi_1^h(t,.) = 0 \quad \text{in } \Omega_{h(t)}, \\ \frac{\partial}{\partial\mathbf{n}}\chi_1^h = 0 \quad \text{on } \Gamma, \\ \frac{\partial}{\partial\mathbf{n}}\chi_1^h = \mathbf{e}_1\cdot\mathbf{n} \quad \text{on } \Sigma_{p/k}, \\ \frac{\partial}{\partial\mathbf{n}}\chi_1^h = 0 \quad \text{on } \Sigma_{h(t)}. \end{cases} \tag{13}$$

Straightforward calculation yields $\Theta = \Theta_1 + h'(t)\Theta_2/\cos\alpha + \chi_1^h Fl'(t)/Fl_b$. Now we use lemma 5. and classical regularity result for elliptic equations (see [9]) to obtain the following estimate

$$D_1(h(t)) = \int_{\Sigma_{h(t)}}(\Theta - \Theta_1^h)\mathbf{n}\cdot\mathbf{s} = \int_{\Sigma_{h(t)}}(h'(t)\Theta_2/\cos\alpha + \chi_1^h Fl'(t)/Fl_b)\mathbf{n}\cdot\mathbf{s}$$

$$\leq C(h_{\min},h_{\max})(\|\nabla\mathbf{v}\|_{L^2(\Sigma_h)}|h'(t)| + |Fl'(t)|) \leq C(h_{\min},h_{\max})(\|\mathbf{v}\|_{H^1(\Sigma_h)}|h'(t)| + |Fl'(t)|)$$

$$\leq C(h_{\min},h_{\max})(\|\Phi\|_{H^3(\Omega_{h(t)})}|h'(t)| + |Fl'(t)|) \leq C(h_{\min},h_{\max})(|h'(t)|^2 + |Fl(t)|^2 + |Fl'(t)|).$$

Analogously we estimate tangential velocity term

$$G(h(t)) := \int_{\Sigma_{h(t)}}\frac{v_\tau^2}{2}\mathbf{n}\cdot\mathbf{s} \leq C(h_{\min},h_{\max})\|\Phi\|^2_{H^2(\Omega_{h(t)})} \leq C(h_{\min},h_{\max})(|h'(t)|^2 + |Fl(t)|^2).$$

Finally, let $B_1(h)(t) = \int_{\Sigma_{h(t)}}x_3\mathbf{s}\cdot\mathbf{n}$. It is immediate that $B_1(h(t)) = B_1(\alpha)h(t)$. With this notation, Newton's equation (9) can be rewritten as:

$$(m + \varrho D(h))h''/\cos\alpha + \varrho\Big(\frac{h'^2}{2}B(\alpha) + gB_1(\alpha)h\Big) + P_0^\alpha = -\varrho(D_1(h,h') + G(h,h')). \tag{14}$$

Now we have all necessary estimates needed to prove the following theorem:

Theorem 2. *Let m, $h_0 > 0$ be such that $\varrho D(h_0) + m \neq 0$, and let $\alpha \in (-\frac{\pi}{2},\frac{\pi}{2})$, $Fl \in H^2_{\mathrm{loc}}(\mathbb{R})$, $P_0 \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $h_0' \in \mathbb{R}$. Then there exists $T > 0$ such that problem (6) has a solution (Φ, h), where $h \in H^3(0,T)$ and $\Phi(t,.) \in C^\infty(\Omega^\alpha_{h(t)})$. Furthermore, if Fl, $P_0 \in H^m(0,T)$, then $h \in H^{m+2}(0,T)$. Specially, if Fl, $P_0 \in C^\infty[0,T]$, then $h \in C^\infty[0,T]$.*

Proof.

Since D is a continuous function and $\varrho D(h_0) + m \neq 0$, we take $0 < h_{\min} < h_0 < h_{\max}$ such that $\varrho D(h_0) + m \neq 0$ on $[h_{\min}, h_{\max}]$. Because $h_{\min}$ and $h_{\max}$ are now fixed, for brevity of notation we will denote by C a generic positive constant that does not depend on h (instead of $C(h_{\min}, h_{\max})$).

We will carry out the proof of existence of a local solution by decoupling the original problem (6) into two problems; boundary value problem $(6)_{1-6}$ with given function of motion of the piston h and problem $(6)_{7,8}$ with given total fluid force on the piston $\int_{\Sigma_{h(t)}} p$. Then we will complete the proof by using Schauder's fixed point theorem.

First we define function $H_0(t) = h_0 + h_0' t$ which homogenizes initial conditions $(6)_7$ and introduce the function space

$$\mathcal{A}(0,T) = \{f \in H^2(0,T);\ f(0) = f'(0) = 0\}, \quad T > 0.$$

Let $K(0,R) \subset \mathcal{A}(0,T)$ be a closed ball such that for function h in $K(0,R)$ it holds $0 < h_{\min} < H_0(t) + h(t) < h_{\max}$, $t \in [0,T)$, i. e. $H_0 + h$ is a function that gives allowed motion of the piston. We can find such R and T because of the inequalities:

$$\|f\|_{L^\infty(0,T)} \leq \sqrt{T}\|f'\|_{L^2(0,T)}, \ \|f\|_{L^2(0,T)} \leq T\|f'\|_{L^2(0,T)}. \tag{15}$$

Hence, $\mathcal{A}(0,T)$ is the Hilbert space with scalar product $< f, g >= \int_0^T f'' g''$.

Let us now take $h \in K(0,R)$ and consider the problem $(6)_{1-6}$ with motion of the piston given by $h + H_0$, denote the corresponding pressure with p^{h+H_0}. We define the operator S on $K(0,R)$ by

$$S(h)(t) = -\varrho\Big(D_1(h(t) + H_0(t), h'(t) + h_0') + G(h(t) + H_0(t), h'(t) + h_0') +$$
$$+ \frac{(h'(t) + h_0')^2}{2} B + B_1 g(h(t) + H_0(t)) - P_0(t).$$

Let $N(h)$ be solution of the Cauchy problem:

$$(m + \varrho D(h + H_0))((N(h))'' = \cos\alpha S(h), \ N(h)(0) = 0, \ (N(h))'(0) = 0. \tag{16}$$

Notice that $S(h)(t)$ is exactly the total force on the piston without term that depends on $h''(t)$ $(-\varrho D(h+H_0)h''/\cos\alpha)$. Therefore we reduce problem (6) to a problem of finding a fixed point of operator N. Really, let us suppose that operator N has a fixed point h_f. Then it is immediate that $(\Phi^{h_f + H_0}, h_f + H_0)$ is the solution of the problem (6).

From lemma 5. follows that the regularity of function $S(h)$ is fully determined by the regularity of h. Since definition of S involves only h, h' and because $h \in H^2(0,T)$, we have $S(h) \in H^1(0,T)$. Therefore it is immediate that $N(h) \in H^3(0,T)$. We have proved that $\mathrm{Im}N \subset H^3(0,T)$, so because of the compactness of embedding $H^3(0,T) \hookrightarrow H^2(0,T)$ the set $\mathrm{Im}N$ is relatively compact set in $H^2(0,T)$. Let us now prove that for T small enough, $N(K(0,R)) \subset K(0,R)$. From estimates for $G(h,h')$, lemma 5. and the trace theorem we get

$$|S(h)(t)| \leq C\Big(|h'(t)|^2 + |Fl'(t)| + |Fl(t)|^2 + |h_0'|^2 + |h_0'|t + |h_0| + |h(t)| + |P_0(t)|\Big), \quad h \in K(0,R). \tag{17}$$

Therefore by using (15) we get

$$\|S(h)\|_{L^2(0,T)}$$

$$\leq C(T^{3/2}R^2 + \|Fl\|_{H^2(0,T)}(T + T^{5/2}) + T\|P_0\|_{H^1(0,T)} + RT^2 + T + T^{3/2}), \quad h \in K(0,R). \tag{18}$$

Now, for T small enough the following inequality holds:

$$\|N(h)\|_{\mathcal{A}(0,T)} \leq \frac{1}{\min_{t\in[0,T]} |\varrho D(h(t) + H_0(t)) + m|} \|S(h)\|_{L^2(0,T)} \leq R.$$

Hence, we have proved $N(K_{\mathcal{A}(0,T)}(0,R)) \subset K_{\mathcal{A}(0,T)}(0,R)$. It remains to prove the continuity of operator N. Let us take a convergent sequence $(h_n)_n \subset \mathcal{A}(0,T)$ and denote its limit by h. Furthermore, let us denote corresponding solutions of the problem $(6)_{1-6}$ by Φ^{h_n}, Φ^h. Convergence of sequence h_n in $\mathcal{A}(0,T)$ gives also convergence of sequences h_n and h_n' in $L^\infty(0,T)$ and therefore from lemma 3. we have $S(h_n) \to S(h)$ strongly in $L^\infty(0,T)$.

Now from the definition of operator N it is immediate that $N(h_n) \to N(h)$ in $\mathcal{A}(0,T)$ and therefore the proof of continuity of operator N is proved.

From the Schauder fixed point theorem (see [23]) we conclude that N has a fixed point h_f. Since $h_f \in \mathrm{Im}N$ we have $h_f \in H^3(0,T)$. For smooth Fl we can iterate this procedure and conclude that $h_f \in C^\infty[0,T]$.

Now it only remains to prove the following lemma

Lemma 3. *Let $h_n \to h$ in $C^1(0,T)$. Then $S(h_n) \to S(h)$ strongly in $L^\infty(0,T)$.*

Proof.
This is variant of results on smooth dependence of solution of boundary value problem on smooth change of the domain and boundary data. Therefore we only present the main idea of the proof (for similar results see for example [13], [18]).

The main idea is to rewrite boundary value problems for Φ^{h_n} on some fixed domain and then use the classical result for elliptic equations. Therefore we define a family of mappings $\beta_h(h_n, .) : \Omega_h \to \Omega_{h_n}$:

$$\beta_h(h_n, x_1, x_2, x_3) = \left\{ \begin{array}{c} (x_1, x_2, x_3), \\ (x_1 - x_3 \tan\alpha(1 - \frac{h_n}{h}), x_2, x_3 \frac{h_n}{h}). \end{array} \right.$$

Now for every $t \in [0,T]$, we define $\Phi(h_n)(t,.) := \Phi^{h_n} \circ \beta_{h(t)}(h_n(t), .)$ which satisfies the following variational equality

$$\begin{array}{c} \int_{\Omega_h(t)} \nabla^{h(t),h_n(t)} \nabla\Phi(h_n)(t,.) \cdot \nabla^{h(t),h_n(t)}\varphi = \int_{\Sigma_p} \frac{Fl(t)}{Fl_b}\mathbf{e}_1 \cdot \mathbf{n}\varphi + \\ + \int_{\Sigma_k} \mathbf{e}_1 \cdot \mathbf{n}(\frac{Fl(t)}{Fl_b} + h_n'(t)\frac{Fl_p}{\cos\alpha Fl_b})\varphi + \int_{\Sigma_h(t)} \frac{h_n'(t)}{\cos\alpha}\mathbf{s} \cdot \mathbf{n}\varphi, \quad \varphi \in \mathcal{W}_h, \end{array} \tag{19}$$

where $\mathcal{W} = \{\varphi \in H^1(\Omega_h); \int_{\Sigma_p} \varphi = 0\}$ and

$$\nabla^{h,h_n} = \left\{ \begin{array}{cc} \nabla & x_3 < 0, \\ \frac{h_n}{h} \left(\begin{array}{c} \nabla_{x'} \\ (\frac{h}{h_n} - 1)\tan\alpha\partial_{x_1} + \frac{h}{h_n}\partial_{x_3} \end{array} \right) & x_3 > 0. \end{array} \right.$$

We know that $h_n(t) \to h(t)$ and $h_n'(t) \to h'(t)$ uniformly on $(0,T)$. By taking $\Phi(h_n(t,.))$ for the test function in (19) we conclude that set $\{\Phi(h_n)(t,.)), n \in \mathbb{N}\}$ is bounded in $\mathcal{W}_{h(t)}$. Now we can subtract (19) for $\Phi(h_n)$ and $\Phi(h)$, and by using previous two observations get $\Phi(h_n)(t,.) \to \Phi(h)(t,.)$ in $\mathcal{W}_{h(t)}$ uniformly on $(0,T)$.

Now let us take $0 < h_c < h_{min}$ and $\theta \in C^\infty(\Omega_{h_{max}})$ such that $\text{supp}(\theta) \in \Omega_{h_{max}} \setminus \Omega_{h_c}$ and $\theta \equiv 1$ on $\Omega_{h_{max}} \setminus \Omega_{h_{min}}$. Now we can use the standard local regularity result for the Laplace equations on $\Omega_{h(t)}$ to get $\theta\Phi(h_n)(t,.) \to \theta\Phi(h)(t,.)$ in $H^m(\Omega_{h(t)})$ uniformly on $(0,T)$, $m \in \mathbb{N}$. Assertion of the lemma follows from definition of S and θ and continuity of trace operator. ■

Remark 3. *Convergence from previous lemma is not local, i.e. one could also prove* $\Phi(h_n)(t,.) \to \Phi(h)(t,.)$ *in* $H^m(\Omega_{h(t)})$. *Proof is analogous but one needs to construct smooth* β. *Since we need only local convergence we decided to prove lemma 3. with simplest possible* β *which is because of its simplicity most convenient for numerical computations.*

Remark 4. *Condition* $\varrho D(h_0)+m \neq 0$ *is necessary, i.e. is not always satisfied. Numerical experiments show that this condition depends on geometry, more precisely on the ratio between diameters of vertical and horizontal pipe.*

Example 2. *In this example we numerically solve problem (6) for* $\alpha = \pi/4$, $l = 5$, $d = d_1 = 1.6$, $Fl(t) = 1.5\cos(t)$, $g = 9.81$, $m = \varrho = 1$, $h_0 = 1.1$, $h_0' = 0.3$ *and* $P_0 = -15 + 1.5\cos(t)$. *We use the same decoupling method as in proof of theorem 2. For solving boundary value problem* $(6)_{1-5}$ *in fixed domain* $\Omega^\alpha_{h(t)}$ *we use finite elements method and for solving Cauchy problem* $(6)_{6,7}$ *we use explicit Euler method with time step* $dt = 0.01$.

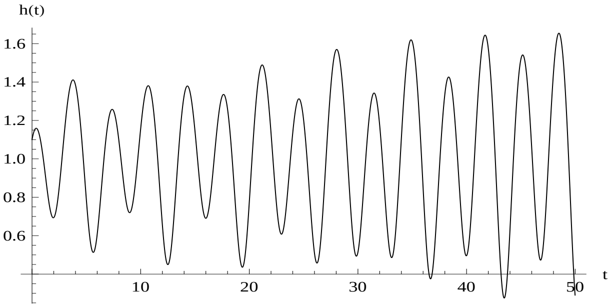

Figure 3: Evolution of the piston height $h(t)$

Figure 3 shows evolution of the piston height $h(t)$. Using obtained estimates we can prove the existence of a solution on $(0,T)$ for an arbitrary T if the data are small.

Corollary 1. *Let* $T > 0$, $Fl \in H^2(0,T)$, $h_0' \in \mathbb{R}$, $h_0 > 0$ *and* $P_0 \in H^1(0,T)$ *be such that quantities* $\|Fl\|_{H^2(0,T)}$, $|h_0'|$ *and* $\|-\varrho g B_1 h_0 - P_0\|_{H^1(0,T)}$ *are small enough. If the mass of the piston* m *is great enough, then there exists solution* (Φ, h) *of the problem (6) such that* $\Phi(t,.) \in C^\infty(\Omega^\alpha_{h(t)})$, $t \in [0,T]$ *and* $h \in H^3(0,T)$.

Proof.
Directly from the estimates (17), (18) and definitions of operator N from proof of Theorem 2. we see that we can choose R such that $N(K_{\mathcal{A}(0,T)}(0,R)) \subset K_{\mathcal{A}(0,T)}(0,R)$. Then we can proceed exactly as in proof of Theorem 2. to get assertion of Corollary. ■

3.3 Global solutions and stability of the equilibrium point of the piston

In this section we consider autonomous case, i. e. the case when inflow flux is constant, $Fl(t) = Fl$ and for simplicity we suppose that $Fl_b = 1$. We can do that without loss of generality because we can just redefine Fl as Fl/Fl_b. Furthermore we suppose that outer pressure p_V is also constant. Because of these assumptions we have $P_0^\alpha(t) = P_0^\alpha$ is also a constant. For subsequent analysis we will have to express all unknowns and parameters in (9) as explicitly as possible.

Recall that χ^h, χ_1^h and Θ_2^h are solutions of auxiliary boundary value problems (11), (13) and (12) respectively. It follows that $\Phi(t,.) = Fl(t)\chi_1^h + h'(t)\chi^h/\cos\alpha$. Let

$$\begin{aligned} &G_1(h)(t) = \textstyle\int_{\Sigma_{h(t)}} |\nabla\chi_1^h|^2\mathbf{n}\cdot\mathbf{s}, \quad G_2(h)(t) = \int_{\Sigma_{h(t)}} |\nabla\chi^h|^2\mathbf{n}\cdot\mathbf{s}, \\ &G_3(h)(t) = 2\textstyle\int_{\Sigma_{h(t)}} \nabla\chi_1^h\cdot\nabla\chi^h\mathbf{n}\cdot\mathbf{s}. \end{aligned} \tag{20}$$

Then we have

$$G_0(h)(t) = \int_{\Sigma_{h(t)}} |\nabla\Phi|^2\mathbf{n}\cdot\mathbf{s} = \int_{\Sigma_{h(t)}} (Fl^2(t)|\nabla\chi_1^h|^2 + h'(t)^2|\nabla\chi^h|^2 +$$

$$+Fl(t)h'(t)\nabla\chi_1^h\cdot\nabla\chi^h) = Fl(t)^2G_1(h(t)) + h'(t)^2G_2(h(t)) + Fl(t)h'(t)G_3(h(t)),$$

Furthermore, we know asymptotic behavior of G_1, G_2 and G_3,

$$\lim_{s\to\infty} G_2(s) = Fl_p, \ \lim_{s\to\infty} G_1(s) = \lim_{s\to\infty} G_3(s) = 0.$$

Both assertions follow now from lemma 1.. First one directly, while for the second one we apply lemma 1. to function $\Phi_2(.,x_1,x_2,x_3) + \cos\alpha x_3 + \sin\alpha x_1$. Moreover, we have

$$D_1(h(t),h'(t)) = Fl'(t)\int_{\Sigma_{h(t)}} \chi_1^h + \frac{h'(t)}{\cos\alpha}\int_{\Sigma_{h(t)}} \Theta_2^h = h'(t)D_2(h(t),h'(t)).$$

Now Newton's equation (9) can be written as a system of first order ODEs:

$$\begin{pmatrix} h \\ v \end{pmatrix}' = \begin{pmatrix} v \\ \frac{\cos\alpha}{m+\varrho D(h)}\Big(- P_0 - \varrho g B_1 h - \varrho(Fl^2G_1(h) + v^2G_2(h) + v(FlG_3(h) + D_2(h,v)))\Big) \end{pmatrix},$$
$$\begin{pmatrix} h(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} h_0 \\ h_0' \end{pmatrix}. \tag{21}$$

We will consider linearization around a stationary point. Right-hand side of (21) is denoted by $W(h,v)$. We look for a stationary point of the system (21), i. e. (h_c, v_c) such that

$W(h_c, v_c) = \mathbf{0}$. It is immediate that $v_c = 0$. Therefore the determination of the stationary point is reduced to solving the equation:

$$-B_1 \varrho g h - |Fl|^2 G_1(h) = P_0. \tag{22}$$

Equation (22) is just a balance of forces in the stationary problem $(1)_6$. From Theorem 1. we know that there exists a solution which we denote by h_c. Stability of the stationary point and asymptotic behavior of the solution depends on the derivative $DW(h_c, 0)$ (see [12]). Let us now calculate $DW(h_c, 0)$:

$$DW(h_c, 0) = \begin{pmatrix} 0 & 1 \\ \frac{\varrho \cos \alpha}{m + \varrho D(h_c)}(-B_1 g - Fl^2 G_1'(h_c)) & \frac{\varrho \cos \alpha}{m + \varrho D(h_c)}(-FlC_3(h_c) - D_2(h_c, 0)) \end{pmatrix}.$$

Eigenvalues $\lambda_{1,2}$ of $WG(h_c, 0)$ are

$$\lambda_{1,2} = (-FlG_3(h_c) - D_2(h_c, 0)) \frac{\varrho \cos \alpha}{m + \varrho D(h_c)} \pm$$

$$\pm \sqrt{\frac{\varrho \cos \alpha}{m + \varrho D(h)} \Big(\frac{\varrho \cos \alpha}{m + \varrho D(h)} (D_2(h_c, 0) + FlG_3(h)) - 4(B_1 g + Fl^2 G_1(h)) \Big)}$$

If stationary state h_c is far from the junction we can use asymptotic properties of Φ to describe conditions for stability more precisely. We know that $\chi^h \approx C + \sin \alpha x_1 + \cos \alpha x_3$ for large x_3. Therefore $D(h_c) > 0$ for h_c large enough. Furthermore, since all terms under the square root, besides $-4B_1 g$, are asymptotically small, the expression under the square root is negative. Hence,

$$\mathrm{Re}(\lambda_{1,2}) = (-FlG_3(h_c) - D_2(h_c, 0)) \frac{\varrho \cos \alpha}{m + \varrho D(h)}$$

and stability depends on sign of $\mathrm{Re}(\lambda_{1,2})$.

Lemma 4. *$D_2(h_c, 0)$ depends linearly on Fl, i.e. $D_2(h_c, 0) = FlD_3(h_c)$.*

Proof.
Since $h'(t) = 0$, $\mathbf{v} = \nabla \Phi = Fl \nabla \chi_1$. From (12) we see that Θ_2 depends linearly on $\mathbf{v}$ and therefore also linearly on Fl. ∎

Therefore we have proved the following theorem

Theorem 3. *Let h_0' be small enough, $Fl(t) = Fl$ and let P_0 be such that there exist corresponding stationary state h_c which is far enough from the junction and such that $m + \varrho D(h_c) \neq 0$ and $G_3(h_c) + D_3(h_c) \neq 0$.*

If

$$Fl \frac{G_3(h_c) + D_3(h_c)}{m + \varrho D(h_c)} > 0,$$

then h_c is asymptotically stable stationary state. If Fl has opposite sign, then h_c is unstable stationary state.

Remark 5. *Dependence of stability of stationary states on α*

Furthermore, by using transformation from remark 1. straightforward calculation yields $G_1^\alpha = G_1^{-\alpha}$, thus $h_c^\alpha = h_c^{-\alpha}$. By using the same transformation and the fact that asymptotically $\nabla\chi^h \approx \mathbf{s}$, we get $G_3^\alpha(h_c) \approx -G_3^{-\alpha}(h_c)$ for large h_c, i.e. $G_3^\alpha(h_c)$ and $G_3^{-\alpha}(h_c)$ have different signs. Is is also easy to see that $(\nabla\mathbf{v}^\alpha)\mathbf{n}^\alpha \cdot \mathbf{s} = -(\nabla\mathbf{v}^{-\alpha})\mathbf{n}^{-\alpha} \cdot \mathbf{s}$. These observations together with the numerical experiments strongly suggest the dependence of stability of stationary states on the angle α.

Example 3. *In this example we will illustrate Theorem 3.. We take $l = 5$, $d = d_1 = 1.6$, $Fl = 3$, $g = 9.81$, $m = \varrho = 1$, $h_0 = 0.94$, $h_0' = 0.02$ and $P_0 = -15$. Then we numerically solve system (6) for $\alpha_1 = \pi/4$ and $\alpha_2 = -\pi/4$ using method described in Example 2..*

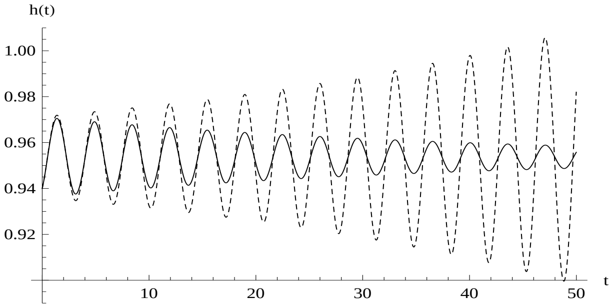

Figure 4: Evolution of the piston height $h(t)$ for $\alpha_1 = \pi/4$ and $\alpha_2 = -\pi/4$

Figure 4 shows evolution of the piston height $h(t)$ for $\alpha_1 = \pi/4$ (full line) and $\alpha_2 = -\pi/4$ (dashed). We see that for exactly the same data, there exists a global solution for $\alpha = \pi/4$ and for $\alpha = -\pi/4$ the solution exists only on some finite time-interval.

Remark 6. *In remark 1. we saw that this model in stationary case does not "see" the difference between angles α and $-\alpha$. However, Example 4 shows us that even the equilibrium height is the same in both cases, in one case we have a stable equilibrium and in another we have not. Therefore in the evolutionary case this model can "see" differences between angles α and $-\alpha$.*

3.3.1 Non-autonomous case

In this subsection we will prove the global existence theorem for problem (6). As we have seen, problem (6) does not have a global solution in general, but if we take data close to the stationary solution and consider a stable case, then a global solution exists. We state slight modification of theorem 2.77 from [1] and therefore we only present the main steps of the proof.

Theorem 4. *Let $\alpha \neq 0$ and let h_c be stable equilibrium height of the piston for flux Fl_s and pressure P_s on Σ_p. Furthermore, let $Fl(t)$ and $P_0(t)$ be small perturbations of Fl_s and P_s, respectively, i.e.*

$$Fl(t) = Fl_s + f(t), \quad P_0(t) = P_s + g(t) \quad \textit{with} \quad \lim_{t\to\infty} f(t) = 0 \ \textit{and} \ \lim_{t\to\infty} g(t) = 0.$$

Moreover, let (h_0, h_0') be close enough to $(h_c, 0)$. Then if $\|f\|_{L^\infty}$ and $\|g\|_{L^\infty}$ are small enough, then a global solution (Φ, h) of the problem (6) exists.

Proof.
Let us introduce notations

$$\mathbf{h}(t) = \begin{pmatrix} h(t) - h_c \\ h'(t) \end{pmatrix}, \quad \mathbf{h}_0 = \begin{pmatrix} h_0 - h_c \\ h_0' \end{pmatrix} \text{ and } \tilde{W}(h, v) = W(h + h_c, v).$$

Now the Cauchy problem (9) can be written as

$$\mathbf{h}'(t) = \tilde{G}(\mathbf{h}(t), t), \quad \mathbf{h}(0) = \mathbf{h}_0;$$

here $\tilde{G}(\mathbf{h}(t), t) = A\mathbf{h}(t) + B(t)\mathbf{h}(t) + g(\mathbf{h}(t), t)$, where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{\varrho\cos\alpha}{m+\varrho D(h_c)}(-B_1 g - Fl_s^2 G_1'(h_c)) & \frac{\varrho\cos\alpha}{m+\varrho D(h_c)}(-Fl_s C_3(h_c) - D_2(h_c, 0)) \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 & 1 \\ \frac{-\varrho\cos\alpha}{m+\varrho D(h_c)}(f(t)^2 G_1'(h_c)) & \frac{-\varrho\cos\alpha}{m+\varrho D(h_c)}(f(t) C_3(h_c) + f(t) D_3(h_c)) \end{pmatrix}.$$

It is immediate that $\|B(t)\| \to 0$ as $t \to \infty$. Since $D_h\tilde{G}(h, t) = A + B(t)$, we have that $g(h, t) = \tilde{G}(\mathbf{0}, t) + o(|h|^2)$. Note that $\|\tilde{G}(\mathbf{0}, t)\| \to 0$ as $t \to \infty$ because h_c is the equilibrium height of the corresponding stationary problem. Let us now take $0 < \delta < h_c$ and maximal $T > 0$ such that the solution of problem (6) exists on $(0, T)$ and $\|\mathbf{h}(t)\| \leq \delta,\ t \in (0, T)$. The solution is given by variation of parameters formula, (see [1]),:

$$\mathbf{h}(t) = \mathbf{h}_0 e^{tA} + e^{tA} \int_0^t e^{-sA}(B(s)\mathbf{h}(s) + g(\mathbf{h}(s), s) + \mathbf{P}_0(s))ds.$$

Since all eigenvalues of A have negative real parts, there exists $\lambda > 0$ and $C > 1$ such that $\|e^{tA}\| \leq Ce^{-\lambda t}$. Now we can get the following estimate:

$$\|\mathbf{h}(t)\| \leq \|\mathbf{h}_0\| e^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda s}(\|B(t)\|\delta + \|\tilde{G}(\mathbf{0}, s)\| + C\delta^2 + \|\mathbf{P}_0(t)\|).$$

Assertion of the theorem follows from the fact that

$$\lim_{t\to\infty} e^{-\lambda t} \int_0^t e^{\lambda s}|f(s)|ds = \frac{-1}{\lambda} \lim_{t\to\infty} |f(t)| = 0$$

and smallness of f and g.

■

The following example illustrates theorem 4.

Example 4. *We take* $l = 5$, $d = d_1 = 1.6$, $Fl(t) = 3 - \frac{1}{10(1+t)}$, $g = 9.81$, $m = \varrho = 1$, $h_0 = 0.94$, $h_0' = 0.02$ *and* $P_0(t) = -15 + \frac{1}{10(1+t)}$. *Then we numerically solve system (6) for* $\alpha = \frac{\pi}{4}$ *using the method described in Example 2. Figure 5 shows evolution of the piston height* $h(t)$.

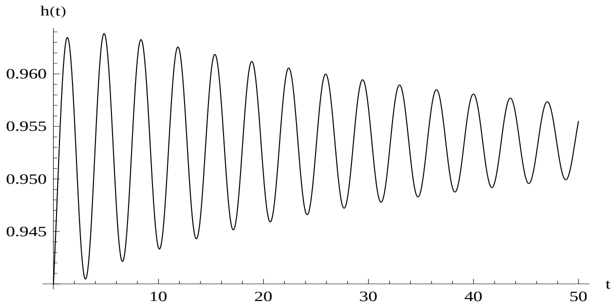

Figure 5: Evolution of the piston height $h(t)$.

4 Appendix

Throughout Section 3 we used results on Neumann's problem for Laplace's equations. Since our domain $\Omega_{h(t)}$ is variable, we need some nonstandard versions of the classical existence and regularity results. However, these results follow from the classical ones directly with obvious modifications. Therefore we just state the following lemma with a sketch of its proof.

Lemma 5. *Let* $h \in H^2(0,T)$ *be such that* $0 < h_{\min} < h(t) < h_{\max}$ *for* $t \in [0,T]$. *Moreover, let* $g_i \in H^1((0,T); H^{3/2}(\partial\Omega_{h(t)}))$, $i = 1,2,3$ *be such that* $\int_{\Sigma_p} g_1 + \int_{\Sigma_k} g_2 + \int_{\Sigma_{h(t)}} g_3 = 0$ *and* $\mathbf{v}_P$ *is defined in Section 3. Then the problem*

$$\begin{aligned} \triangle u(t,.) = 0 \quad & \text{in } \Omega_{h(t)}, \\ \frac{\partial}{\partial \mathbf{n}} u = 0 \quad & \text{on } \Gamma, \\ \frac{\partial}{\partial \mathbf{n}} u = g_1 \quad & \text{on } \Sigma_p, \\ \frac{\partial}{\partial \mathbf{n}} u = g_2 \quad & \text{on } \Sigma_k, \\ \frac{\partial}{\partial \mathbf{n}} u = g_3 \quad & \text{on } \Sigma_{h(t)}, \\ \textstyle\int_{\Sigma_p} u(t,.) d\mathbf{x} = 0, & \end{aligned} \tag{23}$$

has a unique solution $u \in L^\infty((0,T); H^3(\Omega_{h(t)}))$ *such that for every* $t \in [0,T]$,

$$\|u(t.)\|_{H^3(\Omega_{h(t)})} \leq C(h_{\min}, h_{\max}) \sum_{i=1}^{3} \|g(t,.)\|_{H^{3/2}}(\partial\Omega_{h(t)}) \tag{24}$$

Furthermore, $z = \partial_t u \in L^2((0,T); H^2(\Omega_{h(t)}))$ *is a unique solution of the boundary value problem*

$$\begin{aligned} \triangle z(t,.) &= 0 \quad \text{in } \Omega_{h(t)}, \\ \tfrac{\partial}{\partial \mathbf{n}} z &= 0 \quad \text{on } \Gamma, \\ \tfrac{\partial}{\partial \mathbf{n}} z &= g_1' \quad \text{on } \Sigma_p, \\ \tfrac{\partial}{\partial \mathbf{n}} z &= g_2' \quad \text{on } \Sigma_k, \\ \tfrac{\partial}{\partial \mathbf{n}} z &= -\nabla(\nabla u)\mathbf{n} \cdot \mathbf{v}_P + g_3' + \nabla g_3 \cdot \mathbf{v}_P \quad \text{on } \Sigma_{h(t)}, \\ \textstyle\int_{\Sigma_p} z(t,.) d\mathbf{x} &= 0. \end{aligned} \tag{25}$$

Moreover, we have the estimate

$$\int_0^T \|\partial_t u(t,.)\|^2_{H^2(\Omega_{h(t)})} \le C(h_{\min}, h_{\max})\Big(\sum_{i=1}^3 \|g_i'(t,.)\|^2_{H^{1/2}}(\partial\Omega_{h(t)}) + \|h'\|^2_{L^\infty(0,T)} \sum_{i=1}^3 \|g_i(t,.)\|^2_{H^{3/2}}(\partial\Omega_{h(t)})\Big).$$

Proof.
Existence of the solution of problem (23) is standard (see for example [9]). Classical results also give estimate (24), but with constant $C(\Omega_{h(t)})$ depending on $h(t)$. Nevertheless, by transforming domain $\Omega_{h(t)}$ onto some fixed domain and calculating the corresponding norms, after long but straightforward calculation we get that $C(\Omega_{h(t)}) \le C(h_{\min}, h_{\max})$.

The only non-trivial part of the second part of lemma is to get condition $(25)_5$. We obtain it by formally taking the derivative of condition $(23)_5$ w.r.t. t and using the fact that normal $\mathbf{n}$ does not depend on $h(t)$. More precisely, let $\beta(h.,.) : \Omega_1 \to \Omega_h$ be a family of diffeomorphisms.

Notice that $\mathbf{n}(t, \beta(h(t), \mathbf{x})) = \mathbf{n}(\mathbf{x})$. Therefore we have

$$\partial_t(\nabla u(t, \beta(h(t), \mathbf{x})) \cdot \mathbf{n}(\mathbf{x})) = \Big(\partial_t \nabla u(t, \beta(h(t), \mathbf{x})) + ((\nabla(\nabla u))\mathbf{v}_P)(t, \beta(h(t), \mathbf{x}))\Big) \cdot \mathbf{n}(\mathbf{x})$$

Now the assertion of the lemma follows from results of the first part of the lemma. ∎

References

[1] C. Chicone. *Ordinary Differential Equations with Applications.* Springer, 2006.

[2] C. Conca, J. San Martín H., and M. Tucsnak. Motion of a rigid body in a viscous fluid. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(6):473–478, 1999.

[3] B. D'Acunto and S. Rionero. A note on the existence and uniqueness of solutions to a free piston problem. *Rend. Accad. Sci. Fis. Mat. Napoli (4)*, 66:75–84, 1999.

[4] B. Desjardins and M. J. Esteban. On weak solutions for fluid-rigid structure interaction: compressible and incompressible models. *Comm. Partial Differential Equations*, 25(7-8):1399–1413, 2000.

[5] B. Desjardins, M. J. Esteban, C. Grandmont, and P. Le Tallec. Weak solutions for a fluid-elastic structure interaction model. *Rev. Mat. Complut.*, 14(2):523–538, 2001.

[6] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady problems.

[7] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II*, volume 39 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Nonlinear steady problems.

[8] G. P. Galdi. Mathematical problems in classical and non-Newtonian fluid mechanics. In *Hemodynamical flows*, volume 37 of *Oberwolfach Semin.*, pages 121–273. Birkhäuser, Basel, 2008.

[9] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[10] M. E. Gurtin. *An introduction to continuum mechanics*, volume 158 of *Mathematics in Science and Engineering*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.

[11] M. Hillairet and D. Serre. Chute stationnaire d'un solide dans un fluide visqueux incompressible le long d'un plan incliné. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(5):779–803, 2003.

[12] M. W. Hirsch and S. Smale. *Differential equations, dynamical systems, and linear algebra*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. Pure and Applied Mathematics, Vol. 60.

[13] J. Houot and A. Munnier. On the motion and collisions of rigid bodies in an ideal fluid. *Asymptot. Anal.*, 56(3-4):125–158, 2008.

[14] J. G. Houot, J. San Martin, and M. Tucsnak. Existence of solutions for the equations modeling the motion of rigid bodies in an ideal fluid. *J. Funct. Anal.*, 259(11):2856–2885, 2010.

[15] E. Marušić-Paloka and I. Pažanin. Non-isothermal fluid flow through a thin pipe with cooling. *Appl. Anal.*, 88(4):495–515, 2009.

[16] B. Muha and Z. Tutek. On a free piston problem for Stokes and Navier-Stokes equations. Preprint, http://web.math.hr/~borism/preprints/free_piston.pdf, 2009.

[17] B. Muha and Z. Tutek. Numerical analysis of a free piston problem. *Math. Commun.*, 15(2):573–585, 2010.

[18] J. Nečas. *Les méthodes directes en théorie des équations elliptiques.* Masson et Cie, Éditeurs, Paris, 1967.

[19] J. Ortega, L. Rosier, and T. Takahashi. On the motion of a rigid body immersed in a bidimensional incompressible perfect fluid. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(1):139–165, 2007.

[20] J. H. Ortega, L. Rosier, and T. Takahashi. Classical solutions for the equations modelling the motion of a ball in a bidimensional incompressible perfect fluid. *M2AN Math. Model. Numer. Anal.*, 39(1):79–108, 2005.

[21] C. Rosier and L. Rosier. Smooth solutions for the motion of a ball in an incompressible perfect fluid. *J. Funct. Anal.*, 256(5):1618–1641, 2009.

[22] S. Takeno. Free piston problem for isentropic gas dynamics. *Japan J. Indust. Appl. Math.*, 12(2):163–194, 1995.

[23] E. Zeidler. *Nonlinear functional analysis and its applications. I.* Springer-Verlag, New York, 1986. Fixed-point theorems, Translated from the German by Peter R. Wadsack.