Itôv integral - prvi dio

Neka je $B = \{B_t : t \geq 0\}$ Brownovo gibanje u odnosu na filtraciju $\{\mathcal{F}_t : t \geq 0\}$.

1. Neka je $\Pi_n = \{0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T\}$. Izračunajte

$$L^2 - \lim_{\|\Pi_n\| \to 0+} \sum_{j=1}^{n} B_{\frac{t_{j-1}+t_j}{2}} (B_{t_j} - B_{t_{j-1}}) \, .$$

2. Neka je $X = \int_0^1 f(t)(\sin B_t + \cos B_t)\, dB_t$, gdje je $f \in L^2([0,1])$. Dokažite da je $\mathrm{Var} X = \int_0^1 f(t)^2 \, dt$.

3. Izračunajte varijancu sljedećih slučajnih varijabli:

$$\text{(a)} \int_0^t |B_s| dB_s \qquad \text{(b)} \int_0^t \operatorname{sh} B_s dB_s \qquad \text{(c)} \int_0^t s\, e^{B_s} dB_s \, .$$

4. Dokažite da vrijedi

$$\text{(a)} \int_0^t \frac{dB_s}{1+B_s^2} = \operatorname{arctg} B_t + \int_0^t \frac{B_s}{(1+B_s^2)^2}\, ds$$

$$\text{(b)} \int_0^t \frac{B_s}{1+B_s^2}\, dB_s = \frac{1}{2} \ln(1+B_t^2) - \frac{1}{2} \int_0^t \frac{1-B_s^2}{(1+B_s^2)^2}\, ds \, .$$

5. Dokažite da sljedeći procesi rješavaju dane stohastičke diferencijalne jednadžbe:

   (a) $X_t = e^{B_t}$ rješava

$$dX_t = \frac{1}{2} X_t dt + X_t dB_t \, ;$$

   (b) $X_t = (1+t)^{-1} B_t$ rješava

$$dX_t = -\frac{1}{1+t} X_t \, dt + \frac{1}{1+t} dB_t; \quad X_0 = 0 \, ,$$

   (c) $X_t = \left(x^{1/3} + \frac{1}{3} B_t\right)^3$ rješava

$$dX_t = \frac{1}{3} X_t^{1/3} \, dt + X_t^{2/3} dB_t; \quad X_0 = x \, .$$

6. Dokažite da su sljedeći procesi martingali:

(a) $e^{\frac{t}{2}} \cos B_t$ (b) $e^{\frac{t}{2}} \sin B_t$ (c) $e^{-B_t - \frac{t}{2}}(B_t + t)$.

7. Označimo $\beta_k(t) = \mathbb{E}[B_t^k]$ za $k \geq 0$ i $t \geq 0$.

(a) Pomoću Itôve formule dokažite da je

$$\beta_k(t) = \frac{k(k-1)}{2} \int_0^t \beta_{k-2}(s)\, ds \quad \text{za} \quad k \geq 2\,.$$

(b) Izračunajte $\mathbb{E}[B_t^4]$.

(c) Izračunajte $\mathbb{E}[B_t^n]$, $n \geq 1$.

Upute i rješenja: **1.** $\frac{B_T^2}{2}$ **3.** (a) $t^2/2$ (b) $\frac{e^{2t}-1-t}{2}$ (c) $\frac{e^{2t}(2t^2-2t+1)-1}{4}$ **7.** (a) $3t^2$ (b) $\mathbb{E}[B_t^{2k+1}] = 0$ i $\mathbb{E}[B_t^{2k}] = \frac{(2k)!t^k}{2^k k!}$ za $k \geq 1$.