Itôva formula

Neka je $X = \{X_t : t \geq 0\}$ Itôv proces definiran s

$$X_t = X_0 + \int_0^t H_s \, dB_s + \int_0^t V_s \, ds, t \geq 0 \,,$$

gdje je $B = \{B_t : t \geq 0\}$ Brownovo gibanje u odnosu na filtraciju $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$, a $H = \{H_t : t \geq 0\}$ i $V = \{V_t : t \geq 0\}$ su $\mathbb{F}$-adaptirani procesi koji zadovoljavaju sljedeće uvjete

$$\mathbb{E}\left[\int_0^t H_s^2 \, ds\right] < \infty \quad \text{i} \quad \int_0^t |V_s| < \infty \ \ \mathbb{P}-\text{g.s.} \quad \text{za sve} \quad t \geq 0 \,. \tag{1}$$

Kvadratna varijacija Itôvog procesa je definirana s

$$\langle X \rangle_t = \int_0^t H_s^2 \, ds, \quad t \geq 0 \,.$$

Teorem (Itôva formula). Neka je X Itôv proces i neka je $f \in C^2([0,\infty) \times \mathbb{R})$. Tada

za sve $t \geq 0$ vrijedi

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) \, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) \, dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) d\langle X \rangle_s$$

$$= f(0, X_0) + \int_0^t \frac{\partial f}{\partial x}(s, X_s) H_s \, dB_s + \int_0^t \left(\frac{\partial f}{\partial t}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s) V_s + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, X_s) H_s^2 \right) ds$$

Skica dokaza.

Dokaz provodimo samo za funkcije $f \in C^2([0,\infty) \times \mathbb{R})$ takve da su funkcije $f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ omeđene, recimo konstantom $M > 0$.

Neka je $t > 0$ i neka su $(H^{(n)})_n$ i $(V^{(n)})_n$ nizovi jednostavnih procesa takvi da

$$\lim_{n \to \infty} \mathbb{E}\left[\int_0^t |H_u^{(n)} - H_u|^2 \, du\right] = 0 \quad \text{i} \quad \lim_{n \to \infty} \int_0^t |V_u^{(n)} - V_u| \, du = 0 \quad \mathbb{P}-\text{g.s.}$$

i označimo pripadnu particiju intervala s

$$\Pi_n = \{0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t\} \,.$$

Pretpostavimo da smo dokazali Itôvu formulu za jednostavne procese

$$f(t, X_t^{(n_k)}) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial x}(s, X_s^{(n_k)}) H_s^{(n_k)} \, dB_s \tag{2}$$

$$+ \int_0^t \left(\frac{\partial f}{\partial t}(s, X_s^{(n_k)}) + \frac{\partial f}{\partial x}(s, X_s^{(n_k)}) V_s^{(n_k)} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, X_s^{(n_k)}) (H_s^{(n_k)})^2 \right) ds \,. \tag{3}$$

Zaista, zbog neprekidnosti procesa X, omeđenosti funkcije $\frac{\partial f}{\partial x}$ i teorema o dominiranoj konvergenciji je

$$\begin{aligned}
\mathbb{E}\int_0^t \left|\frac{\partial f}{\partial x}(s, X_s^{(n_k)})H_s^{(n_k)} - \frac{\partial f}{\partial x}(s, X_s)H_s\right|^2 ds = \\
= \mathbb{E}\int_0^t \Big(\frac{\partial f}{\partial x}(s, X_s^{(n_k)})^2 (H_s^{(n_k)})^2 - 2\frac{\partial f}{\partial x}(s, X_s^{(n_k)})\frac{\partial f}{\partial x}(s, X_s)H_s^{(n_k)}H_s + \\
+ \frac{\partial f}{\partial x}(s, X_s)^2 H_s^2\Big)\, ds \underset{\|\Pi\|\to 0}{\longrightarrow} 0\,.
\end{aligned}$$

Dakle, Itôv integral u (2) konvergira u L^2 (pa onda i $\mathbb{P}-$g.s. na podnizu) po definiciji, tj.

$$L^2 - \lim_{k\to\infty}\int_0^t \frac{\partial f}{\partial x}(s, X_s^{(n_k)})H_s^{(n_k)}\, dB_s = \int_0^t \frac{\partial f}{\partial x}(s, X_s)H_s\, dB_s\,.$$

Na sličan način (koristeći teorem o dominiranoj konvergenciji) se pokaže da integral prva dva člana u (3) konvergira g.s. prema

$$\int_0^t \left(\frac{\partial f}{\partial t}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s)V_s\right) ds\,.$$

Zbog omeđenosti i neprekidnosti $\frac{\partial^2 f}{\partial x^2}$, iz teorema o dominiranoj konvergenciji slijedi

$$\begin{aligned}
\lim_{k\to\infty}\mathbb{E}\left[\left|\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s^{(n_k)})(H_s^{(n_k)})^2\, ds - \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)H_s^2\, ds\right|\right] \leq \\
\leq 2M \lim_{k\to\infty}\mathbb{E}\left[\int_0^t ((H_s^{(n_k)})^2 - H_s^2)\, ds\right] + \\
+ \lim_{k\to\infty}\mathbb{E}\left[\int_0^t \left|\frac{\partial^2 f}{\partial x^2}(s, X_s^{(n_k)}) - \frac{\partial^2 f}{\partial x^2}(s, X_s)\right| H_s^2\, ds\right] = 0.
\end{aligned}$$

Dakle, na podnizu integral trećeg člana u (3) konvergira g.s. prema $\int_0^t \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, X_s)H_s^2\, ds$.

Preostaje dokazati Itôvu formulu za jednostavne procese H i V. Neka je

$$\Pi = \{0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t\}$$

particija segmenta $[0, t]$ koja sadrži particije na kojoj pomoću kojih su definirani jednostavni procesi H i V. Tada je

$$H_s = \sum_{j=1}^{n} h_{t_{j-1}} 1_{[t_{j-1}, t_j)}(s) \qquad \text{i} \qquad V_s = \sum_{j=1}^{n} v_{t_{j-1}} 1_{[t_{j-1}, t_j)}(s),\ 0 \leq s \leq t\,,$$

gdje su h_{t_j} i v_{t_j} $\mathcal{F}_{t_j}$-izmjerive omeđene slučajne varijable za $j \in \{0, 1, \ldots, n-1\}$.

Koristeći Taylorovu formulu (na svakom intervalu $[t_{j-1}, t_j)$) dobijemo

$$
\begin{aligned}
f(t, X_t) =& f(0, X_0) + \sum_{j=1}^{n} (f(t_j, X_{t_j}) - f(t_{j-1}, X_{t_{j-1}})) \\
=& f(0, X_0) + \sum_{j=1}^{n} \frac{\partial f}{\partial t}(t_{j-1}, X_{t_{j-1}})(t_j - t_{j-1}) && (S_1) \\
&+ \sum_{j=1}^{n} \frac{\partial f}{\partial x}(t_{j-1}, X_{t_{j-1}})(X_{t_j} - X_{t_{j-1}}) && (S_2) \\
&+ \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial t^2}(t_{j-1}, X_{t_{j-1}})(t_j - t_{j-1})^2 && (S_3) \\
&+ \sum_{j=1}^{n} \frac{\partial^2 f}{\partial t \partial x}(t_{j-1}, X_{t_{j-1}})(t_j - t_{j-1})(X_{t_j} - X_{t_{j-1}}) && (S_4) \\
&+ \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x^2}(t_{j-1}, X_{t_{j-1}})(X_{t_j} - X_{t_{j-1}})^2 && (S_5) \\
&+ \sum_{i=1}^{n} R_{j-1}, && (S_6)
\end{aligned}
$$

gdje su R_{j-1} slučajne varijable takve da je

$$
|R_{j-1}| \leq C \left((t_j - t_{j-1})^2 + (X_{t_j} - X_{t_{j-1}})^2 \right), \quad \text{za sve} \quad j \in \{1, 2, \ldots, n\} \tag{4}
$$

za neku konstantu $C > 0$, jer su druge parcijalne derivacije funkcije f omeđene te vrijedi

$$
\lim_{\|\Pi\| \to 0} \frac{R_{j-1}}{(t_{j-1} - t_{j-1})^2 + (X_{t_j} - X_{t_{j-1}})^2} = 0\,. \tag{5}
$$

Budući je $s \mapsto \frac{\partial f}{\partial t}(s, X_s)$ $\mathbb{P}$-g.s. neprekidna, S_1 konvergira kao Riemannova suma prema pripadnom integralu:

$$
S_1 = \sum_{j=1}^{n} \frac{\partial f}{\partial t}(t_{j-1}, X_{t_{j-1}})(t_j - t_{j-1}) \underset{\|\Pi\| \to 0}{\longrightarrow} \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds \quad \mathbb{P} - \text{g.s.}\,.
$$

Kod S_2, primijetimo da jednostavni proces

$$
\sum_{j=1}^{n} \frac{\partial f}{\partial x}(t_{j-1}, X_{t_{j-1}}) h_{t_{j-1}} 1_{[t_{j-1}, t_j)} \quad \text{aproksimira} \quad \frac{\partial f}{\partial x}(s, X_s) H_s = \sum_{j=1}^{n} \frac{\partial f}{\partial x}(s, X_{t_{j-1}}) h_{t_{j-1}} 1_{[t_{j-1}, t_j)}
$$

kada $\|\Pi\| \to 0$, jer iz teorema srednje vrijednosti slijedi

$$\sum_{j=1}^{n} \mathbb{E}\left[\int_{t_{j-1}}^{t_j} \left|\frac{\partial f}{\partial x}(t_{j-1}, X_{t_{j-1}}) - \frac{\partial f}{\partial x}(s, X_{t_{j-1}})\right|^2 h_{t_{j-1}}^2\right] \leq M^2 \mathbb{E}\left[\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |s - t_{i-1}|^2 h_{t_{j-1}}^2 \, ds\right]$$

$$\leq M^2 \|\Pi\|^2 \mathbb{E}\left[\int_0^t H_s^2 \, ds\right] \underset{\|\Pi\|\to 0}{\longrightarrow} 0\,.$$

Tada je po definiciji Itôvog integrala

$$L^2 - \lim_{\|\Pi\|\to 0} \sum_{j=1}^{n} \frac{\partial f}{\partial x}(t_{j-1}, X_{t_{j-1}}) h_{t_{j-1}} (B_{t_j} - B_{t_{j-1}}) = \int_0^t \frac{\partial f}{\partial x}(s, X_s) H_s \, dB_s\,.$$

Na sličan način, iz $\frac{\partial f}{\partial x}(s, X_s) V_s = \sum_{j=1}^{n} \frac{\partial f}{\partial x}(s, X_{t_{j-1}}) v_{t_{j-1}} 1_{[t_{j-1}, t_j)}$ slijedi

$$\left|\sum_{j=1}^{n} \frac{\partial f}{\partial x}(t_{j-1}, X_{t_{j-1}}) v_{t_{j-1}} - \int_0^t \frac{\partial f}{\partial x}(s, X_{t_{j-1}}) V_s \, ds\right| \leq$$

$$\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left|\frac{\partial f}{\partial x}(t_{j-1}, X_{t_{j-1}}) - \frac{\partial f}{\partial x}(s, X_{t_{j-1}})\right| v_{t_{j-1}} \, ds$$

$$\leq M \|\Pi\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_1} v_{t_{j-1}} \, ds = M \|\Pi\| \int_0^t V_s \, ds \underset{\|\Pi\|\to 0}{\longrightarrow} 0 \quad \mathbb{P} - \text{g.s.}\,.$$

Koristeći omeđenost $\frac{\partial^2 f}{\partial t^2}$, S_3 ocijenimo na sljedeći način

$$|S_3| \leq \frac{M}{2} \|\Pi\| \sum_{j=1}^{n} (t_j - t_{j-1}) = \frac{M}{2} \|\Pi\| t \underset{\|\Pi\|\to 0}{\longrightarrow} 0 \quad \mathbb{P} - \text{g.s.}\,.$$

Kod ocjene S_4, svaki član ocijenimo posebno. Član vezan uz proces V je

$$\sum_{j=1}^{n} \frac{\partial^2 f}{\partial t \partial x}(t_{j-1}, X_{t_{j-1}})(t_j - t_{j-1}) v_{t_{j-1}} (t_j - t_{j-1}) \leq M \|\Pi\| \sum_{j=1}^{n} v_{t_{j-1}} (t_j - t_{j-1})$$

$$= M \|\Pi\| \int_0^t V_s \, ds \underset{\|\Pi\|\to 0}{\longrightarrow} 0 \quad \mathbb{P} - \text{g.s.}\,.$$

Drugi član ocijenimo u L^2-normi. Koristeći nezavisnost prirasta, slijedi da je

$$\mathbb{E}\left[\left|\sum_{j=1}^{n} \frac{\partial^2 f}{\partial t \partial x}(t_{j-1}, X_{t_{j-1}})(t_j - t_{j-1}) h_{t_{j-1}} (B_{t_j} - B_{t_{j-1}})\right|^2\right] \leq$$

$$\leq M^2 \|\Pi\|^2 \sum_{j=1}^{n} \mathbb{E}\left[h_{t_{j-1}}^2 (t_j - t_{j-1})\right] = M^2 \|\Pi\|^2 \mathbb{E}\left[\int_0^t H_s^2 \, ds\right] \underset{\|\Pi\|\to 0}{\longrightarrow} 0\,.$$

Uzimajući podniz, slijedi $S_4 \underset{\|\Pi\|\to 0}{\longrightarrow} 0$ $\mathbb{P}-$g.s.

Koristeći oznake $a_{t_{j-1}} = \frac{\partial^2 f}{\partial x^2}(t_{j-1}, X_{t_{j-1}})h^2_{t_{j-1}}$, $b_{t_{i-1}} = \frac{\partial^2 f}{\partial x^2}(t_{j-1}, X_{t_{j-1}})h_{t_{j-1}}v_{t_{j-1}}$ i $c_{t_{i-1}} = \frac{\partial^2 f}{\partial x^2}(t_{j-1}, X_{t_{j-1}})v^2_{t_{j-1}}$ slijedi

$$S_5 = \frac{1}{2}\sum_{j=1}^{n} a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})^2 + \sum_{j=1}^{n} b_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})(t_j - t_{j-1}) + \frac{1}{2}\sum_{j=1}^{n} c_{t_{j-1}}(t_j - t_{j-1})^2$$

Budući su H i V omeđeni procesi kao jednostavni procesi, slijedi da postoji $\widetilde{M} > 0$ (koji ne ovisi o n) takav da vrijedi $|a_{t_j}|, |b_{t_j}|, |c_{t_j}| \le \widetilde{M}$ za sve $j \in 0, 1, \ldots, n-1$ pa je

$$\left|\sum_{j=1}^{n} c_{t_{j-1}}(t_j - t_{j-1})^2\right| \le \widetilde{M}t\|\Pi\| \underset{\|\Pi\|\to 0}{\longrightarrow} 0\,.$$

i, zbog nezavisnosti prirasta,

$$\mathbb{E}\left[\left(\sum_{j=1}^{n} b_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})(t_j - t_{j-1})\right)^2\right] = \sum_{j=1}^{n} \mathbb{E}\left[b^2_{t_{j-1}}\right](t_j - t_{j-1})^3 \le \widetilde{M}^2 t\|\Pi\|^2 \underset{\|\Pi\|\to 0}{\longrightarrow} 0\,.$$

Dokazat ćemo da

$$L^2 - \lim_{\|\Pi\|\to 0} \sum_{j=1}^{n} a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})^2 = \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)H_s^2\, ds\,,$$

a za to je dovoljno pokazati (zbog neprekidnosti funkcije $\frac{\partial^2 f}{\partial x \partial t}(s, X_s)$) da je

$$\lim_{\|\Pi\|\to 0} \mathbb{E}\left[\left|\sum_{j=1}^{n} a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})^2 - \sum_{j=1}^{n} a_{t_{j-1}}(t_j - t_{j-1})\right|^2\right] = 0\,.$$

Vrijedi

$$\begin{aligned}
\mathbb{E}&\left[\left|\sum_{j=1}^{n} a_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})^2 - \sum_{j=1}^{n} a_{t_{j-1}}(t_j - t_{j-1})\right|^2\right] \\
&= \sum_{i,j=1}^{n} \mathbb{E}[a_{t_{i-1}}a_{t_{j-1}}((B_{t_i} - B_{t_{i-1}}) - (t_i - t_{i-1}))((B_{t_j} - B_{t_{j-1}}) - (t_j - t_{j-1}))]\,.
\end{aligned}$$

Budući su za $1 \le i < j \le n-1$ slučajne varijable $a_{t_{i-1}}a_{t_{j-1}}((B_{t_i} - B_{t_{i-1}}) - (t_i - t_{i-1}))$ i $((B_{t_j} - B_{t_{j-1}}) - (t_j - t_{j-1}))$ nezavisne (i analogno za $1 \le j < i \le n-1$), slijedi da su pripadni članovi u gornjoj sumi jednaki 0 pa je ona zapravo jednaka

$$
\begin{aligned}
&\sum_{j=1}^{n}\mathbb{E}[a_{t_{j-1}}^2((B_{t_j}-B_{t_{j-1}})-(t_j-t_{j-1}))^2] \\
&\quad=\sum_{j=1}^{n}\mathbb{E}[a_{t_{j-1}}^2]\left(\mathbb{E}[(B_{t_j}-B_{t_{j-1}})^4]-2(t_j-t_{j-1})E[(B_{t_j}-B_{t_{j-1}})^2]+(t_j-t_{j-1})^2\right) \\
&\quad=\sum_{j=1}^{n}\mathbb{E}[a_{t_{j-1}}^2]\left(3(t_j-t_{j-1})^2-2(t_j-t_{j-1})^2+(t_j-t_{j-1})^2\right)\le 2\widetilde{M}^2\|\Pi\|t \underset{\|\Pi\|\to 0}{\longrightarrow} 0\,.
\end{aligned}
$$

Konačno, zbog omeđenosti jednostavnih procesa H i V je

$$
\begin{aligned}
|S_6| &\le \sum_{j=1}^{n}\frac{|R_{j-1}|}{(t_j-t_{j-1})^2+(X_{t_j}-X_{t_{j-1}})^2}\left((t_j-t_{j-1})^2+(X_{t_j}-X_{t_{j-1}})^2\right) \\
&\le \widetilde{C}\sum_{j=1}^{n}\frac{|R_{j-1}|}{(t_j-t_{j-1})^2+(X_{t_j}-X_{t_{j-1}})^2}\left((t_j-t_{j-1})^2+(B_{t_j}-B_{t_{j-1}})^2\right)
\end{aligned}
$$

za neku konstantu $\widetilde{C}$. Tada je, zbog (4),

$$
\sum_{j=1}^{n}\frac{|R_{j-1}|}{(t_j-t_{j-1})^2+(X_{t_j}-X_{t_{j-1}})^2}(t_j-t_{j-1})^2\le Ct\|\Pi\| \underset{\|\Pi\|\to 0}{\longrightarrow} 0\,.
$$

S druge strane, koristeći (4), teorem o kvadratnoj varijaciji Brownovog gibanja i Lebesgueov teorem o dominiranoj konvergenciji, dobijemo

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=1}^{n}\ \frac{|R_{j-1}|}{(t_j-t_{j-1})^2+(X_{t_j}-X_{t_{j-1}})^2}((B_{t_j}-B_{t_{j-1}})^2-(t_j-t_{j-1}))\right)^2\right]\le \\
&\qquad\le C^2\mathbb{E}\left[\left(\sum_{j=1}^{n}(B_{t_j}-B_{t_{j-1}})^2-t\right)^2\right] \underset{\|\Pi\|\to 0}{\longrightarrow} 0\,.
\end{aligned}
$$

Zato je, zbog (5) i Lebesgueovog teorema o dominiranoj konvergenciji, na podnizu $\mathbb{P}$-g.s.

$$
\begin{aligned}
&\lim_{\|\Pi\|\to 0}\sum_{j=1}^{n}\frac{|R_{j-1}|}{(t_j-t_{j-1})^2+(X_{t_j}-X_{t_{j-1}})^2}(B_{t_j}-B_{t_{j-1}})^2 \\
&\quad=\lim_{\|\Pi\|\to 0}\sum_{j=1}^{n}\frac{|R_{j-1}|}{(t_j-t_{j-1})^2+(X_{t_j}-X_{t_{j-1}})^2}(t_j-t_{j-1}) \\
&\quad=\lim_{\|\Pi\|\to 0}\int_0^t\sum_{j=1}^{n}\frac{|R_{j-1}|}{(t_j-t_{j-1})^2+(X_{t_j}-X_{t_{j-1}})^2}1_{[t_{j-1},t_j)}(s)\,ds=0\,,
\end{aligned}
$$

jer je podintegralna funkcija dominirana konstantom C, koja je integrabilna na $[0, t]$. $\square$

Zadatak. Neka je $f \in C^2(\mathbb{R})$ takva da su f, f' i f'' omeđene funkcije. Dokažite da je

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2} \int_0^t f''(B_s)\, ds, \quad t \geq 0\,.$$