Laplace transform

Lecture notes *

Ante Mimica

Department of Mathematics

Univeristy of Zagreb

Croatia

*These lecture notes follow the course given in period April 27 - May 01 2015. at Technische Universität Dresden .

Contents

1 Definition and basic properties

Definition 1.1 Let μ be a measure on $[0,\infty)$. The *Laplace transform* $\mathcal{L}\mu$ of μ is defined by

$$\mathcal{L}\mu(\lambda) = \int_{[0,\infty)} e^{-\lambda x}\mu(dx) \quad \text{for} \quad \lambda > \sigma_0\,,$$

where $\sigma_0 = \inf\{\lambda \in \mathbb{R} : \int_{[0,\infty)} e^{-\lambda x}\mu(dx) < \infty\}$.

Remark 1.2 (i) If μ is a finite measure, then $\sigma_0 \le 0$.

(ii) Assume that $\mathcal{L}\mu(a) < \infty$ for some $a \in \mathbb{R}$. Then

$$\nu(dx) = \frac{e^{-ax}}{\mathcal{L}\mu(a)}\mu(dx)$$

defines a probability measure on $[0,\infty)$ with the Laplace transform

$$\int_{[0,\infty)} e^{-\lambda x}\nu(dx) = \frac{1}{\mathcal{L}\mu(a)}\int_{[0,\infty)} e^{-(\lambda+a)x}\,\mu(dx) = \frac{\mathcal{L}\mu(\lambda+a)}{\mathcal{L}\mu(a)}\,.$$

(iii) Some special cases:

(iiia) $\mu(dx) = f(x)\,dx$

$$\mathcal{L}\mu(\lambda) = \mathcal{L}f(\lambda) = \int_0^\infty e^{-\lambda x} f(x)\,dx\,.$$

(iiib) let X be a non-negative random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by μ the law of X; that is $\mu(B) = \mathbb{P}(X \in B)$. Then

$$\mathcal{L}\mu(\lambda) = \mathbb{E}[e^{-\lambda X}] \quad \text{for} \quad \lambda > \sigma_0\,.$$

Example 1.3 (a) $f(x) = x^\alpha$, $\alpha > 0$

$$\mathcal{L}f(\lambda) = \int_0^\infty e^{-x} \left(\frac{x}{\lambda}\right)^\alpha \frac{dx}{\lambda} = \frac{\Gamma(\alpha+1)}{\lambda^{1+\alpha}}, \quad \sigma_0 = 0\,.$$

(b) $f(x) = e^{-ax}$, $a \in \mathbb{R}$

$$\mathcal{L}f(\lambda) = \int_0^\infty e^{-(a+\lambda)x}\,dx = \frac{1}{a+\lambda}, \quad \sigma_0 = -a\,.$$

(c) $f(x) = e^{-x^2}$, $\sigma_0 = -\infty$, since

$$\int_0^\infty e^{-\lambda x - x^2}\,dx \quad \text{converges for any } \lambda \in \mathbb{R}\,.$$

(d) $f(x) = e^{x^2}$, $\sigma_0 = +\infty$, since

$$\int_0^\infty e^{-\lambda x + x^2}\,dx \quad \text{does not converge for any } \lambda \in \mathbb{R}\,.$$

Proposition 1.4 (Properties of LT) Let μ be a measure on $[0,\infty)$ ans assume that $\mathcal{L}\mu$ is finite $(0,\infty)$ (i.e. $\sigma_0 \le 0$) .

(i) Then $\mathcal{L}\mu \in C^\infty((0,\infty))$ and for all $n \in \mathbb{N}$

$$(\mathcal{L}\mu)^{(n)}(\lambda) = (-1)^n \int_{[0,\infty)} x^n e^{-\lambda x}\,\mu(dx)\,.$$

(ii) μ has finite n-th moment if and only if $(\mathcal{L}\mu)^{(n)}(0+)$ exists and it is finite. In particular, μ is finite if and only if $\mathcal{L}\mu(0+)$ exists and it is finite.

(iii) If $\gamma : [0,\infty) \to [0,\infty)$ is defined by $\gamma(x) = ax$, $a > 0$, then

$$\mathcal{L}(\mu \circ \gamma^{-1})(\lambda) = \mathcal{L}\mu(a\lambda), \quad \lambda > 0\,.$$

(iv) If ν is a measure such that $\mathcal{L}\nu(\lambda)$ exists for $\lambda > 0$, then

$$\mathcal{L}(\mu \star \nu) = \mathcal{L}\mu\,\mathcal{L}\nu\,,$$

where the *convolution* of measures μ and ν is defined by

$$(\mu \star \nu)(B) := \int\int_{[0,\infty)\times[0,\infty)} 1_B(x+y)\mu(dx)\nu(dy)$$

for a Borel measurable set $B \subset [0,\infty)$.

Proof. (i)

$$\frac{\mathcal{L}\mu(\lambda+h)-\mathcal{L}\mu(\lambda)}{h}=\int_{[0,\infty)}\frac{e^{-hx}-1}{h}\,e^{-\lambda x}\,\mu(dx)\,.$$

Since

$$\left|\frac{e^{-hx}-1}{h}\right|\le\frac{|hx|e^{|hx|}}{|h|}\le xe^{\frac{\lambda}{4}x}\le ce^{\frac{\lambda}{2}x}\quad\text{for}\quad|h|\le\frac{\lambda}{4}$$

for some constant $c>0$ depending only on $\lambda>0$, we can use dominated convergence theorem to conclude that

$$(\mathcal{L}\mu)'(\lambda)=\int_{[0,\infty)}\frac{e^{-hx}-1}{h}\,e^{-\lambda x}\,\mu(dx)=-\int_{[0,\infty)}xe^{-\lambda x}\mu(dx)\,.$$

For higher derivatives we proceed in a similar manner.
(ii) An application of monotone convergence theorem yields

$$(-1)^n(\mathcal{L}\mu)^{(n)}(0+)=\lim_{\lambda\to0+}\int_{[0,\infty)}x^ne^{-\lambda x}\mu(dx)=\int_{[0,\infty)}x^n\mu(dx)$$

and the claim follows.
(iii)

$$\mathcal{L}(\mu\circ\gamma^{-1})(\lambda)=\int_{[0,\infty)}e^{-\lambda x}(\mu\circ\gamma^{-1})(dx)=\int_{[0,\infty)}e^{-\lambda\overbrace{\gamma(x)}^{=ax}}\mu(dx)=\mathcal{L}\mu(ax)\,.$$

(iv)

$$\begin{aligned}\mathcal{L}(\mu\star\nu)(\lambda)&=\int_{[0,\infty)}e^{-\lambda x}(\mu\star\nu)(dx)=\int_{[0,\infty)}\int_{[0,\infty)}e^{-\lambda(x+y)}\mu(dx)\nu(dy)\\&=\mathcal{L}\mu(\lambda)\,\mathcal{L}\nu(\lambda)\,.\end{aligned}$$

□

Definition 1.5 For a measure μ on $\mathbb{R}$ its *distribution function* $F:\mathbb{R}\to[0,\infty)$ is defined by

$$F(x)=\mu((-\infty,x]),\quad x\in\mathbb{R}\,.$$

We call $x\in\mathbb{R}$ a *continuity point* of μ (or F) if F is continuous at x.

Remark 1.6 Note that F is always right continuous with left limits and non-decreasing function and it has at most countable number of discontinuities. If μ is a measure on $[0,\infty)$, then $F(x)=0$ for $x<0$ and $F(x)=\mu([0,x])$ for $x\geq 0$.

Proposition 1.7 Let μ be a measure on $[0,\infty)$ with the distribution function F. Then for any $\lambda>\sigma_0\vee 0$ and $x>0$ following integration by parts formula holds

$$\int_{[0,x]} e^{-\lambda y}\mu(dy) = e^{-\lambda x}F(x) + \int_{[0,x]} \lambda e^{-\lambda y}F(y)\,dy\,.$$

In particular,

$$\int_0^\infty e^{-\lambda y}F(y)\,dy = \frac{\mathcal{L}\mu(\lambda)}{\lambda}\,.$$

If μ is a probability measure, then

$$\int_0^\infty e^{-\lambda x}(1-F(x))\,dx = \frac{1-\mathcal{L}\mu(\lambda)}{\lambda} \qquad \text{for all}\ \ \lambda>0\,.$$

Proof. By Fubini theorem

$$\begin{aligned}
\int_{[0,x]} e^{-\lambda y}\mu(dy) &= \int_{[0,\infty)}\int_0^\infty \overbrace{1_{[0,x]}(y)1_{[y,\infty)}(z)}^{1_{[0,x\wedge z]}(y)}\,\lambda e^{-\lambda z}\,dz\mu(dy)\\
&= \int_0^\infty \lambda e^{-\lambda z}\mu([0,x\wedge z])\,dz\\
&= e^{-\lambda x}\mu([0,x]) + \int_0^x \lambda e^{-\lambda z}\mu([0,z])\,dz\\
&= e^{-\lambda x}F(x) + \int_0^x \lambda e^{-\lambda z}F(z)\,dz\,.
\end{aligned}$$

Second claim follows by letting $x\longrightarrow\infty$ and using monotone convergence theorem, since we can find $\varepsilon>0$ so that $\lambda-\varepsilon>\sigma_0\vee 0$ and hence

$$\infty > \mathcal{L}\mu(\lambda-\varepsilon) \geq \int_{[0,x]} e^{-(\lambda-\varepsilon)y}\mu(dy) \geq e^{\varepsilon x}e^{-\lambda x}F(x)$$

yielding $e^{-\lambda x}F(x)\underset{x\to\infty}{\longrightarrow}0$. The last claim follows directly from the previous. □

Proposition 1.8 (Uniqueness and inversion of LT) A finite measure μ on $[0,\infty)$ is uniquely determined by its Laplace transform. More precisely, for any $x \geq 0$ we have

$$\mu([0,x]) - \frac{1}{2}\mu(\{x\}) = \lim_{n\to\infty} \sum_{k\leq nx} (-1)^k \frac{n^k}{k!} (\mathcal{L}\mu)^{(k)}(n)\,. \tag{1.1}$$

In particular, at every continuity point $x \geq 0$ of μ we have

$$\mu([0,x]) = \lim_{n\to\infty} \sum_{k\leq nx} (-1)^k \frac{n^k}{k!} (\mathcal{L}\mu)^{(k)}(n)\,. \tag{1.2}$$

Proof. By Proposition 1.4 (a), for any $x > 0$ and $n \in \mathbb{N}$,

$$\sum_{k\leq nx} (-1)^k \frac{n^k}{k!} (\mathcal{L}\mu)^{(k)}(\lambda) = \int_{[0,\infty)} \sum_{k\leq nx} \frac{(nt)^k}{k!} e^{-nt} \mu(dt)\,.$$

Note,

$$\sum_{k\leq nx} \frac{(nt)^k}{k!} e^{-nt} = \mathbb{P}(X \leq nx)\,,$$

where X has Poisson distribution with mean nt. First we consider case $x = t$. Let $\{Y_i : i \geq 1\}$ be a sequence of independent Poisson random variables with mean t. By the central limit theorem,

$$\mathbb{P}(X \leq nt) = \mathbb{P}\left(\frac{Y_1 + \ldots + Y_n - nt}{\sqrt{nt}} \leq 0 \right) \underset{n\to\infty}{\longrightarrow} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = \frac{1}{2}\,.$$

For $x < t$,

$$\begin{aligned} \mathbb{P}(X \leq nx) &= \mathbb{P}(n(t-x) \leq nt - X) \leq \mathbb{P}(n(t-x) \leq |X - nt|) \\ &\leq \frac{\mathrm{Var} X}{n^2(t-x)^2} = \frac{t}{n(t-x)^2} \underset{n\to\infty}{\longrightarrow} 0\,. \end{aligned}$$

and, similarly, for $x > t$,

$$\begin{aligned} \mathbb{P}(X \leq nx) &= 1 - \mathbb{P}(X - nt > n(x-t)) \geq 1 - \mathbb{P}(|X - nt| \geq n(x-t)) \\ &\geq 1 - \frac{t}{n(t-x)^2} \underset{n\to\infty}{\longrightarrow} 1\,. \end{aligned}$$

The last three displays show that

$$\sum_{k \le nx} \frac{(nt)^k}{k!} e^{-nt} \underset{n\to\infty}{\longrightarrow} 1_{[0,x)}(t) + \frac{1}{2} 1_{\{x\}}(t)$$

and, thus by the dominated convergence theorem

$$\begin{aligned}\lim_{n\to\infty} (-1)^k \frac{n^k}{k!} (\mathcal{L}\mu)^{(k)}(n) &= \int_{[0,\infty)} \left(1_{[0,x)}(t) + \frac{1}{2} 1_{\{x\}}(t) \right) \mu(dt) \\ &= \mu([0,x]) - \frac{1}{2}\mu(\{x\})\,.\end{aligned}$$

□

Remark 1.9 If $f : [0,\infty) \to \mathbb{R}$ is a continuous function such that for some $b \in \mathbb{R}$

$$\sup_{x\ge 0} e^{-bx} f(x) < \infty\,,$$

then the Post-Widder inversion formula holds

$$f(x) = \lim_{n\to\infty} \tfrac{(-1)^n}{n!} \left(\tfrac{n}{x}\right)^{n+1} (\mathcal{L}f)^{(n)}(\tfrac{n}{x}), \;\; x > 0\,.$$

2 Continuity theorem and applications

Theorem 2.1 (Continuity theorem) Let (μ_n) be a sequence of measures on $[0,\infty)$ and denote by F_n their distribution functions.

(i) If μ is a measure on $[0,\infty)$ with the distribution function F such that $F_n \underset{n\to\infty}{\longrightarrow} F$ at continuity points of F and if there is $a \geq 0$ such that $\sup\limits_{n\geq 1} \mathcal{L}\mu_n(a) < \infty$, then

$$\mathcal{L}\mu_n(\lambda) \underset{n\to\infty}{\longrightarrow} \mathcal{L}\mu(\lambda) \quad \text{for all} \;\; \lambda > a\,.$$

(ii) If there is $a \in \mathbb{R}$ such that $\phi(\lambda) = \lim\limits_{n\to\infty} \mathcal{L}\mu_n(\lambda)$ exists for all $\lambda > a$, then ϕ is the Laplace transform of a measure μ and if F is its distribution function, then $F_n \underset{n\to\infty}{\longrightarrow} F$ at all continuity points of F.

Proof. (i) Let $A := \sup\limits_{n\geq 1} \mathcal{L}\mu_n(a) < \infty$. Using Proposition 1.7 and dominated convergence theorem, for any $\lambda > a$ and point of continuity $x > 0$,

$$\int_{[0,x]} e^{-\lambda y}\mu_n(dy) = \int_0^x \lambda e^{-\lambda y} F_n(y)\,ds + e^{-\lambda x}F_n(x) \underset{n\to\infty}{\longrightarrow} \int_{[0,x]} e^{-\lambda y}\mu(dy)\,,$$

since $F_n(y) \leq e^{ax}\mathcal{L}\mu_n(a) \leq Ae^{ax}$ for all $0 \leq y \leq x$.

Let $\lambda > a$ and $\varepsilon > 0$. For any point of continuity $x > 0$ of F satisfying $Ae^{-(\lambda-a)x} \leq \varepsilon$ we have

$$\begin{aligned}
&\int_{[0,x]} e^{-\lambda y}\mu_n(dy) \leq \mathcal{L}\mu_n(\lambda) \leq \\
&\qquad \leq \int_{[0,x]} e^{-\lambda y}\mu_n(dy) + e^{-(\lambda-a)x}\int_{(x,\infty)} e^{-ay}\mu_n(dy) \leq \int_{[0,x]} e^{-\lambda y}\mu_n(dy) + \varepsilon\,.
\end{aligned}$$

This implies

$$\int_{[0,x]} e^{-\lambda y}\mu(dy) \leq \liminf_{n\to\infty} \mathcal{L}\mu_n(\lambda) \leq \limsup_{n\to\infty} \mathcal{L}\mu_n(\lambda) \leq \int_{[0,x]} e^{-\lambda y}\mu(dy) + \varepsilon$$

and, letting $x \to \infty$ (along points of continuity of F) we obtain

$$\mathcal{L}\mu(\lambda) \le \liminf_{n\to\infty} \mathcal{L}\mu_n(\lambda) \le \limsup_{n\to\infty} \mathcal{L}\mu_n(\lambda) \le \mathcal{L}\mu(\lambda) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude that $\lim_{n\to\infty} \mathcal{L}\mu_n(\lambda)$ exists and equals $\mathcal{L}\mu(\lambda)$ for any $\lambda > a$.
(ii) Take $\lambda_0 > a$. By Remark 1.2 (ii),

$$\nu_n(dx) = \frac{e^{-\lambda_0 x}}{\mathcal{L}\mu_n(\lambda_0)} \mu_n(dx)$$

are probability measures with the Laplace transforms

$$\frac{\mathcal{L}\mu_n(\lambda_0 + \lambda)}{\mathcal{L}\mu_n(\lambda_0)} \quad \text{for} \quad n \in \mathbb{N}.$$

Denoting by G_n the distribution function of ν_n for $n \in \mathbb{N}$, we can apply Helly's selection theorem to obtain a right-continuous non-decreasing function G and a subsequence such that $G_{n_k} \underset{k\to\infty}{\longrightarrow} G$ at all continuity points of G. Let ν be the measure on $[0, \infty)$ corresponding to G (such that G is the distribution function of ν). Since $\mathcal{L}\nu_n(0) = 1$ for all $n \in \mathbb{N}$, we can apply (i) to conclude that

$$\frac{\phi(\lambda_0 + \lambda)}{\phi(\lambda_0)} = \lim_{k\to\infty} \frac{\mathcal{L}\mu_{n_k}(\lambda_0 + \lambda)}{\mathcal{L}\mu_{n_k}(\lambda_0)} = \mathcal{L}\nu(\lambda_0 + \lambda).$$

In the same way, we obtain that any subsequence (G_{n_k}) of (G_n) has a subsequence $(G_{n_{k_l}})$ converging to G at all continuity points of G. This implies that (G_n) converges to G at all continuity points of G(see [1]). Define μ by

$$\mu(dx) = \phi(\lambda_0) e^{\lambda_0 x} \nu(dx),$$

Note that it has the same continuity points as ν since, by Proposition 1.7,

$$\mu([0, x]) = \phi(\lambda_0) \left[e^{\lambda_0 x} G(x) + \int_0^x \lambda_0 e^{\lambda_0 t} G(t)\, dt \right].$$

[1] Here we use the following reasoning from analysis. If (a_n) is a sequence of real numbers such that every sequence has a subsequence converging to some fixed number $L \in \mathbb{R}$, then (a_n) converges to L as well. If this were not true, we could find $\varepsilon > 0$ and a subsequence (a_{n_k}) such that

$$|a_{n_k} - L| \ge \varepsilon \quad \text{for all} \quad k \in \mathbb{N}$$

which would lead to a contradiction, since we could not find any further subsequence of (a_{n_k}) converging to L.

Also, for any $\lambda > \lambda_0$

$$\mathcal{L}\mu(\lambda) = \phi(\lambda) \int_{[0,\infty)} e^{-(\lambda-\lambda_0)x} \nu(dx) = \phi(\lambda_0) \frac{\phi(\lambda_0 + (\lambda - \lambda_0))}{\phi(\lambda_0)} = \phi(\lambda)\,.$$

Then for any continuity point $x > 0$ of the distribution function F of μ, by the dominated convergence theorem and Proposition 1.7, it follows that

$$\begin{aligned}
F_n(x) = \int_{[0,x]} \mathcal{L}\mu_n(\lambda_0) e^{\lambda_0 y} \nu_n(dy) &= \mathcal{L}\mu_n(\lambda_0) e^{\lambda_0 x} G_n(x) - \int_{[0,x]} \mathcal{L}\mu_n(\lambda_0) \lambda_0 e^{\lambda_0 y} G_n(y)\, dy \\
&\underset{n\to\infty}{\longrightarrow} \phi(\lambda_0) e^{\lambda_0 x} G(x) - \int_{[0,x]} \phi(\lambda_0) \lambda_0 e^{\lambda_0 y} G(y)\, dy = \int_{[0,x]} \phi(\lambda_0) e^{\lambda_0 y} \nu(dy) \\
&= \mu([0,x]) = F(x)\,.
\end{aligned}$$

□

Corollary 2.2 Let $(\mu_n)_n$ be a sequence of probability measures such that

$$\mathcal{L}\mu_n(\lambda) \underset{n\to\infty}{\longrightarrow} \phi(\lambda) \quad \text{for all} \quad \lambda > 0\,.$$

If $\phi(0+) = 1$, then there exists a probability measure μ on $[0,\infty)$ such that

$$\mathcal{L}\mu = \phi \quad \text{and} \quad \mu_n([0,x]) \underset{n\to\infty}{\longrightarrow} \mu([0,x]) \ \text{ at continuity points } x > 0 \text{ of } \mu\,.$$

Proof. Since $\sup_{n\geq 1} \mathcal{L}\mu_n(0) = \sup_{n\geq 1} \mu_n([0,\infty) = 1$, Theorem 2.1 (ii) yiedls a measure μ on $[0,\infty)$ such that $\mu_n([0,x]) \underset{n\to\infty}{\longrightarrow} \mu([0,x])$ at continuity points of μ satisfying

$$\mathcal{L}\mu(\lambda) = \phi(\lambda) \text{ for all } \ \lambda > 0\,.$$

Using Proposition 1.4 (i) we get that $1 = \phi(0+) = \mu([0,\infty))\,.$ □

2.1 Completely monotone functions

In this section we explore range of the Laplace transform.

Definition 2.3 A function $f : (0,\infty) \to (0,\infty)$ is *completely monotone function* if it has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(\lambda) \geq 0 \quad \text{for all} \;\; \lambda > 0 \;\; \text{and} \;\; n \in \mathbb{N} \cup \{0\}\,.$$

Theorem 2.4 (Bernstein-Hausdorff-Widder) A function $f\colon (0,\infty) \to (0,\infty)$ is completely monotone if and only if there exists a measure μ on $[0,\infty)$ such that

$$f(\lambda) = \mathcal{L}\mu(\lambda) = \int_{[0,\infty)} e^{-\lambda x} \mu(dx) \quad \text{for all} \;\; \lambda > 0\,.$$

Proof. $\boxed{\Leftarrow}$ By Proposition 1.4 (i), for any $n \in \mathbb{N}$ and $\lambda > 0$ we have

$$(-1)^n f^{(n)}(\lambda) = \int_{[0,\infty)} x^n e^{-\lambda x} \mu(dx) \geq 0\,.$$

$\boxed{\Rightarrow}$ Let f be a completely monotone function such that $f(0+) < \infty$. Since $(-1)^n f^{(n)}$ is nonincreasing for any $n \in \mathbb{N}$, we have

$$(-1)^n f^{(n)} \leq \frac{2}{\lambda} \int_{2/\lambda}^{\lambda} (-1)^n f^{(n)}(y)\,dy \leq \frac{2}{\lambda} (-1)^{n-1} f^{(n-1)}(\tfrac{\lambda}{2})\,.$$

Iterating this inequality we obtain

$$(-1)^n f^{(n)}(y) \leq \frac{2^{\frac{n(n+1)}{2}}}{y^n} f(\tfrac{y}{2^n}) \leq \frac{2^{\frac{n(n+1)}{2}}}{y^n} f(0+), \quad y > 0\,.$$

Having this inequality we can use integration by parts and change of variable $y = \frac{n}{t}$ to get

$$\begin{aligned} f(\lambda) - f(\infty) &= -\int_{\lambda}^{\infty} f'(y)\,dy = \int_{\lambda}^{\infty} (y-\lambda) f''(y)\,dy \\ &= \dots = \frac{(-1)^n}{(n-1)!} \int_{\lambda}^{\infty} (y-\lambda)^{n-1} f^{(n)}(y)\,dy \\ &= \int_{n/\lambda}^{\infty} (\tfrac{n}{t} - \lambda)^{n-1} \frac{(-1)^n f^{(n)}(\frac{n}{t})}{(n-1)!} \frac{n\,dt}{t^2} \\ &= \int_0^{\infty} \left(1 - \frac{\lambda t}{n}\right)^{n-1} 1_{[0,n]}(\lambda t) \frac{(-1)^n f^{(n)}(\frac{n}{t})(\frac{n}{t})^{n+1}}{n!}\,dt\,. \end{aligned}$$

Define
$$\mu_n(dt) = \frac{(-1)^n f^{(n)}(\frac{n}{t})(\frac{n}{t})^{n+1}}{n!}\,dt\,.$$
By monotone convergence theorem
$$\mu_n([0,\infty)) = \lim_{\lambda\to 0+}\int_0^\infty \left(1-\frac{\lambda t}{n}\right)^{n-1} 1_{[0,n]}(\lambda t)\mu_n(dt) = f(0+)-f(\infty)$$
for all $n \in \mathbb{N}$ and so we may use Helly's selection principle to obtain a nondecreasing right continuous function F and a subsequence (μ_{n_k}) such that
$$\mu_{n_k}([0,x]) \underset{k\to\infty}{\longrightarrow} F(x)$$
at continuity points x of F. Let μ be the measure corresponding to μ_0. By the continuity theorem (Theorem 2.1),
$$\mathcal{L}\mu_{n_k}(\lambda) \underset{k\to\infty}{\longrightarrow} \mathcal{L}\mu(\lambda)$$
It can be shown that
$$\left(1-\frac{x}{n}\right)^{n-1} 1_{[0,n]}(x) \underset{n\to\infty}{\longrightarrow} e^{-\lambda x} \quad \text{uniformly in } \ x>0\,,$$
hence,
$$\mathcal{L}\mu_{n_k}(\lambda) - \int_0^\infty \left(1-\frac{\lambda t}{n_k}\right)^{n_k-1} 1_{[0,n_k]}(\lambda t)\mu_{n_k}(dt) \underset{k\to\infty}{\longrightarrow} 0$$
implying that
$$f(\lambda)-f(\infty) = \mathcal{L}\mu_0(\lambda)\,.$$
Therefore, in this case it is enough to put
$$\mu = \mu_0 + f(\infty)\delta_0\,,$$
where δ_0 is point mass at 0. In general case we set $f_a(\lambda) := f(a+\lambda)$ for $a>0$ and note that f_a is also a completely monotone function such that $f_a(0+)<\infty$ and so there is a measure μ_a such that $f_a = \mathcal{L}\mu_a$. Since it is possible do this for any $a>0$, uniqueness theorem for the Laplace transform (Proposition 1.8) implies that for $0<a<b$ from
$$\mathcal{L}\mu_b(\lambda) = f(a+(\lambda+b-a)) = \int_{[0,\infty)} e^{-\lambda x}e^{-\lambda(b-a)}\mu_a(dx)\,,$$

we deduce that $e^{ax}\mu_a(dx) = e^{bx}\mu_b(dx)$. Hence, for any $a > 0$ we may consistently define measure

$$\mu(dx) = e^{ax}\mu_a(dx)\,.$$

Then for $\lambda > 0$ and $a = \frac{\lambda}{2}$ we get

$$f(\lambda) = f(\tfrac{\lambda}{2} + \tfrac{\lambda}{2}) = \int_{[0,\infty)} e^{-\frac{\lambda}{2}x}\mu_{\lambda/2}(dx) = \int_{[0,\infty)} e^{-\lambda x}\mu(dx)\,.$$

□

Proposition 2.5 Let f be a completely monotone function.

(a) If g is a completely monotone function, then fg is also completely monotone.

(b) If $h : (0,\infty) \to (0,\infty)$ is such that h' is completely monotone, then $f \circ h$ is also completely monotone.

Proof. (a) By Theorem 2.4, there exist measures μ and ν such that $f = \mathcal{L}\mu$ and $g = \mathcal{L}\nu$. Then Proposition 1.4 (iv) implies that $fg = \mathcal{L}\mu\mathcal{L}\nu = \mathcal{L}(\mu \star \nu)$. Now it is enough to observe that $\mathcal{L}(\mu \star \nu)$ is completely monotone by Theorem 2.4.
(b) We prove this by mathematical induction. First,

$$(h \circ f)' = \underbrace{(f' \circ h)}_{\leq 0}\underbrace{h'}_{\geq 0} \leq 0\,.$$

Assume that for some $n \in \mathbb{N}$

$$(-1)^k(\tilde{f} \circ \tilde{h})^{(k)} \geq 0 \text{ for all } \ k \in \{1, \dots, n\} \tag{2.1}$$

and *all* functions $\tilde{f}, \tilde{h} : (0,\infty) \to (0,\infty)$ such that $\tilde{h}'$ and $\tilde{f}$ are completely monotone. Since $-f'$ and h' are completely monotone, by Leibniz formula for higher derivatives and (2.1),

$$\begin{aligned}
(-1)^{(n+1)}(h \circ f)^{(n+1)} &= (-1)^n \left[((-f') \circ h) \cdot h'\right]^{(n)} \\
&= (-1)^n \sum_{k=0}^{n} \binom{n}{k} ((-f)' \circ h)^{(k)} (h')^{(n-k)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \underbrace{(-1)^k((-f') \circ h)^{(k)}}_{\geq 0} \underbrace{(-1)^{n-k}(h')^{(n-k)}}_{\geq 0} \geq 0\,.
\end{aligned}$$

□

2.2 Tauberian theorems

We have seen that continuity theorem enables us to deduce convergence of distribution functions from the convergence of the Laplace transforms of the corresponding measures. In this section we will see that under certain conditions behavior of the Laplace transform around the origin determines the behavior of the distribution function near the infinity. Such type of relation, describing behavior of measure μ in terms of transform $\mathcal{L}\mu$ is often called Tauberian. Reverse relation is known as Abelian.

Theorem 2.6 Let $\rho \geq 0$ and let μ be a measure on $[0,\infty)$ such that its Laplace transform $\mathcal{L}\mu$ is defined on $(0,\infty)$. The following claims are equivalent

(i) $$\frac{\mathcal{L}\mu(\frac{\lambda}{t})}{\mathcal{L}\mu(\frac{1}{t})} \underset{t\to\infty}{\longrightarrow} \lambda^{-\rho} \quad \text{for all} \quad \lambda > 0 \ ,$$

(ii) $$\frac{\mu([0,tx])}{\mu([0,t])} \underset{t\to\infty}{\longrightarrow} x^{\rho} \quad \text{for all} \quad x > 0 \, .$$

Moreover, if the relations above hold, then

$$\frac{\mathcal{L}\mu(\frac{1}{t})}{\mu([0,t])} \underset{t\to\infty}{\longrightarrow} \Gamma(1+\rho) \, . \tag{2.2}$$

Proof. $\boxed{\text{(i)} \Longrightarrow \text{(ii)}}$ (Tauberian theorem) By Proposition 1.4 (iii) with $\gamma(y) = \frac{y}{t}$, for any $\lambda > 0$ and $t > 0$ we obtain

$$\frac{\mathcal{L}\mu(\frac{\lambda}{t})}{\mathcal{L}\mu(\frac{1}{t})} = \mathcal{L}\left(\frac{\mu\circ\gamma^{-1}}{\mathcal{L}\mu(\frac{1}{t})}\right)(\lambda) \, .$$

Hence,

$$\mathcal{L}\left(\frac{\mu\circ\gamma^{-1}}{\mathcal{L}\mu(\frac{1}{t})}\right)(\lambda) \underset{t\to\infty}{\longrightarrow} \lambda^{-\rho} = \begin{cases} \int_0^\infty e^{-\lambda y}\frac{y^{\rho-1}}{\Gamma(\rho)}\,dy\,, & \rho > 0 \\ \mathcal{L}\delta_0(\lambda)\,, & \rho = 0 \end{cases}$$

and so, by the continuity theorem (Theorem 2.1), we conclude that

$$\frac{\mu([0,tx])}{\mathcal{L}\mu(\frac{1}{t})} \underset{t\to\infty}{\longrightarrow} \begin{cases} \int_0^x \frac{y^{\rho-1}}{\Gamma(\rho)}\,dy\,, & \rho > 0 \\ \int_{[0,x]}\delta_0(dy)\,, & \rho = 0 \end{cases} = \frac{x^\rho}{\Gamma(1+\rho)} \quad \text{for any} \quad x > 0 \, . \tag{2.3}$$

In particular, taking $x = 1$ in (2.3) we get (2.2) and then

$$\frac{\mu([0,tx])}{\mu([0,t])} = \frac{\frac{\mu([0,tx])}{\mathcal{L}\mu(\frac{1}{t})}}{\frac{\mu([0,t])}{\mathcal{L}\mu(\frac{1}{t})}} \xrightarrow[t\to\infty]{} x^\rho \qquad \text{for any} \quad x > 0\,.$$

$\boxed{\text{(ii)} \implies \text{(i)}}$ (Abelian theorem) Similarly as before, by Proposition 1.4 (iii) with $\gamma(y) = \frac{y}{t}$

$$\mathcal{L}\left(\frac{\mu \circ \gamma^{-1}}{\mu([0,t])}\right)(\lambda) = \frac{\mathcal{L}\mu(\frac{\lambda}{t})}{\mu([0,t])} \qquad \text{for} \quad \lambda, t > 0\,.$$

In order to apply continuity theorem first we are going to prove that for some $t_0 > 0$

$$\sup_{t > t_0} \frac{\mathcal{L}\mu(\frac{1}{t})}{\mu([0,t])} < \infty\,.$$

From (ii) it follows that there is $t_0 > 0$ such that

$$\mu([0,2t]) \le 2 \cdot 2^\rho \mu([0,t]) \qquad \text{for every} \quad t \ge t_0\,. \tag{2.4}$$

Then for any $a > 0$

$$\mathcal{L}\mu(\tfrac{a}{t}) \le \mu([0,t]) + \sum_{n=1}^{\infty} \int_{[2^{n-1}t, 2^n t]} e^{-a\frac{y}{t}} \mu(dy) \le \mu([0,t]) + \sum_{n=1}^{\infty} e^{-a2^{n-1}} \mu([0, 2^n t])$$

yielding together with multiple application of (2.4)

$$\frac{\mathcal{L}\mu(\frac{a}{t})}{\mu([0,t])} \le 1 + \sum_{n=1}^{\infty} e^{-a2^{n-1}} \frac{\mu([0,2^n t])}{\mu([0,t])} \le 1 + \sum_{n=1}^{\infty} e^{-a2^{n-1}} 2^{n(1+\rho)} < \infty\,.$$

Therefore, we may use continuity theorem (Theorem 2.1) to conclude that

$$\frac{\mathcal{L}\mu(\frac{\lambda}{t})}{\mu([0,t])} = \mathcal{L}\left(\frac{\mu \circ \gamma^{-1}}{\mu([0,t])}\right)(\lambda) \xrightarrow[t\to\infty]{} \begin{cases} \int_0^\infty e^{-\lambda y} \rho y^{\rho-1}\, dy\,, & \rho > 0 \\ \mathcal{L}\delta_0(\lambda)\,, & \rho = 0 \end{cases} = \frac{\Gamma(1+\rho)}{\lambda^\rho}$$

for all $\lambda > a$, implying that the convergence actually holds for all $\lambda > 0$, since $a > 0$ was arbitrary. As in the first part of the proof, by taking $\lambda = 1$ we obtain (2.2) and then

$$\frac{\mathcal{L}\mu(\frac{\lambda}{t})}{\mathcal{L}\mu(\frac{1}{t})} = \frac{\frac{\mathcal{L}\mu(\frac{\lambda}{t})}{\mu([0,t])}}{\frac{\mathcal{L}\mu(\frac{1}{t})}{\mu([0,t])}} \xrightarrow[t\to\infty]{} \lambda^{-\rho} \qquad \text{for all} \quad \lambda > 0\,.$$

□

Example 2.7 By (2.2),

$$\mu([0,t]) \sim \ln t \quad \text{as} \quad t \to \infty \qquad \Longleftrightarrow \qquad \mathcal{L}\mu(\lambda) \sim \ln \tfrac{1}{\lambda} \text{ as } \lambda \to 0+ \ .$$

Definition 2.8 A function $g\colon (0,\infty) \to (0,\infty)$ *varies regularly* at infinity with index $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{g(tx)}{g(t)} = t^{\rho} \qquad \text{for all} \quad x > 0\,.$$

If $\ell\colon (0,\infty) \to (0,\infty)$ varies regluarly with index $\rho = 0$, then it is said to *vary slowly* at infinity, i.e.

$$\lim_{t\to\infty} \frac{\ell(tx)}{\ell(t)} = 1 \qquad \text{for all} \quad x > 0\,.$$

Using this terminology we can restate Theorem 2.6 as follows.

Theorem 2.9 Let μ be a measure on $[0,\infty)$, $\ell\colon (0,\infty) \to (0,\infty)$ a slowly varying function at infinity and $\rho \geq 0$. The following relations are equivalent:

(i) $$\mathcal{L}\mu(\lambda) \sim \lambda^{-\rho}\ell(\tfrac{1}{\lambda}), \quad \lambda \to 0+$$

(ii) $$\mu([0,t]) \sim \frac{1}{\Gamma(1+\rho)} t^{\rho}\ell(t), \quad t \to \infty\,.$$

Proof. If (i) holds, then

$$\frac{\mathcal{L}\mu(\frac{\lambda}{t})}{\mathcal{L}\mu(\frac{1}{t})} \sim \lambda^{-\rho}\frac{\ell(\frac{t}{\lambda})}{\ell(t)} \sim \lambda^{-\rho} \quad \text{as} \quad t \to \infty,$$

which is (i) in Theorem 2.6. Hence, from (2.2) we deduce

$$\mu([0,t]) \sim \frac{1}{\Gamma(1+\rho)}\mathcal{L}\mu(\tfrac{1}{t}) \sim \frac{1}{\Gamma(1+\rho)} t^{\rho}\ell(t) \quad \text{as} \quad t \to \infty\,.$$

In the other case we proceed similarly. □

Proposition 2.10 (Monotone density theorem) Let μ be a measure on $[0,\infty)$ such that

$$\mu([0,t]) = \int_0^t m(s)\,ds, \quad t > 0\,,$$

where $m : (0,\infty) \to (0,\infty)$ is *ultimately monotone*, i.e. there exists $t_0 > 0$ such that m is monotone on (t_0,∞). If there exist $\rho \in \mathbb{R}$ and a slowly varying function $\ell : (0,\infty) \to (0,\infty)$ such that

$$\mu([0,t]) \sim t^\rho \ell(t), \quad \text{as} \quad t \to \infty\,, \tag{2.5}$$

then

$$m(t) \sim \rho t^{\rho-1}\ell(t), \quad \text{as} \quad t \to \infty\,. \tag{2.6}$$

Proof. Let us assume that m is eventualy nondecreasing and let $0 < a < b$. Then for $t > 0$ large enough,

$$\frac{(b-a)tm(at)}{t^\rho \ell(t)} \le \frac{\mu((at,bt])}{t^\rho \ell(t)} \le \frac{(b-a)tm(bt)}{t^\rho \ell(t)}\,. \tag{2.7}$$

Since ℓ varies slowly at infinity we have

$$\frac{\mu((at,bt])}{t^\rho \ell(t)} = \frac{\mu([0,bt])}{(bt)^\rho \ell(bt)} b^\rho \frac{\ell(bt)}{\ell(t)} - \frac{\mu([0,at])}{(at)^\rho \ell(at)} a^\rho \frac{\ell(at)}{\ell(t)} \underset{t\to\infty}{\longrightarrow} b^\rho - a^\rho\,.$$

Therefore, it follows from (2.7) that

$$\limsup_{t\to\infty} \frac{m(at)}{t^{\rho-1}\ell(t)} \le \frac{b^\rho - a^\rho}{b-a}$$

and by taking $a = 1$ and letting $b \to 1+$ we get

$$\limsup_{t\to\infty} \frac{m(t)}{t^{\rho-1}\ell(t)} \le \lim_{b\to 1+} \frac{b^\rho - 1}{b-1} = \rho\,.$$

Similarly,

$$\liminf_{t\to\infty} \frac{m(t)}{t^{\rho-1}\ell(t)} \ge \lim_{a\to 1-} \frac{1-a^\rho}{1-a} = \rho$$

showing that

$$\lim_{t\to\infty} \frac{m(t)}{t^{\rho-1}\ell(t)} = \rho\,.$$

□

Remark 2.11 It can be proved that (2.5) and (2.6) in Proposition 2.10 are equivalent when $\rho > 0$. The direction that we have not proved is known as Karamata theorem (see [1, Proposition 1.5.8]).

Example 2.12 Let μ be a probability measure on $[0, \infty)$ with the distribution function. Then, by Proposition 1.7,

$$\int_0^\infty e^{-\lambda x}[1 - F(x)]\, dx = \frac{1 - \mathcal{L}\mu(\lambda)}{\lambda}, \quad \lambda > 0\,.$$

Since $1-F$ is monotone, Theorem 2.9 and Proposition 2.10 and previous remark imply that for a slowly varying function ℓ and $\rho > 0$ the following relations are equivalent

$$1 - \mathcal{L}\mu(\lambda) \sim \lambda^{1-\rho}\ell(\tfrac{1}{\lambda}),\ \lambda \to 0+ \qquad \text{and} \qquad 1 - F(t) \sim \underbrace{\frac{\rho}{\Gamma(1+\rho)}}_{=\frac{1}{\Gamma(\rho)}} t^{\rho-1}\ell(t),\ t \to \infty\,. \tag{2.8}$$

Example 2.13 (Stable distributions) Let $\alpha \in (0,1)$ and $\phi(\lambda) = e^{-\lambda^\alpha}$. Then ϕ is a completely monotone function by Proposition 2.5 (b) as a composition of a completely monotone function and a function with a completely monotone derivative. Moreover, since $\phi(0) = 1$, by Theorem 2.4 ϕ is the Laplace transform of a probability measure μ.

Let (X_n) be a sequence of independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with law μ. Then the Laplace transform of the random variable $\frac{X_1+\ldots+X_n}{n^{1/\alpha}}$ is

$$\mathbb{E}\left[e^{-\lambda\frac{X_1+\ldots+X_n}{n^{1/\alpha}}}\right] = \phi(\lambda n^{-1/\alpha})^n = \phi(\lambda),$$

showing by the uniqueness of the Laplace transform (Proposition 1.8) that the law of $\frac{X_1+\ldots+X_n}{n^{1/\alpha}}$ is again μ. Note that

$$1 - \mathcal{L}\mu(\lambda) = 1 - e^{-\lambda^\alpha} \sim \lambda^\alpha,\ \lambda \to 0+\ \ .$$

If X has the law μ, then (2.8) with $\rho = 1 - \alpha$ yields

$$\mathbb{P}(X > t) \sim \frac{1}{\Gamma(1-\alpha)} t^{1-\alpha-1} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},\ t \to \infty\,.$$

2.3 Further developments

If we consider example of the gamma distribution

$$\gamma(dx) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}\, dx \quad (\alpha > 0)\,,$$

then

$$\mathcal{L}\mu(\lambda) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(1+\lambda)x} x^{\alpha-1}\, dx = \frac{1}{(1+\lambda)^\alpha}$$

implying that $\sigma_0 = -1$. It can be proved that the tail of the measure $\mu(t,\infty) = \int_t^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}\, dx$ satisfies

$$\lim_{t\to\infty} \frac{\ln \mu(t,\infty)}{t} = -1 = \sigma_0\,.$$

More generally, we have the following result.

Theorem 2.14 Let $f : (0,\infty) \to (0,\infty)$ be a completely monotone function and let μ be the measure such that $f = \mathcal{L}\mu$ (see Theorem 2.4) . Assume that

$$\limsup_{\lambda\to 0+} \lambda \ln f(\sigma_0 + \lambda) = 0 \quad \text{and} \quad \limsup_{\lambda\to 0+} \frac{f(\sigma_0 + 2\lambda)}{f(\sigma_0+\lambda)} < 1\,. \tag{2.9}$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \ln \mu(t,\infty) = \sigma_0\,.$$

Proof. See [3, Theorem 1.2] . □

Remark 2.15 Condition (2.9) is satisfied if function $\lambda \mapsto f(\sigma_0 + \lambda)$ varies regularly at 0 with index $\rho < 0$.

Bibliography

[1] N. H. Bingham, C. M. Goldie, J. L. Teugels. *Regular variation*, Cambridge University Press, Cambridge, 1987.

[2] W. Feller. *An Introduction to Probability Theory and Its Applications*, Volume II, John Wiley & Sons, New York, 1971.

[3] A. Mimica. *Exponential decay of measures and Tauberian theorems*, preprint, 2014., `http://arxiv.org/abs/1301.2690`

[4] R. L. Schilling, R. Song, Z. Vondraček. *Bernstein functions: Theory and applications*, 2nd edition, De Gruyter, Berlin, 2012.