Vertex algebras and Whittaker modules for affine Lie algebras

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Supported by CSF, grant no. 2634

HRI, Allahabad, December 16, 2014

Dedicated to:

Senepati Eswara Rao 60th
Birthday

Abstract

In this talk we shall study Whittaker modules for affine Lie algebras as modules for universal affine vertex algebras. We shall discuss a role of Whittaker categories in the representation theory of affine vertex algebras. We will present a complete description of Whittaker modules for the affine Lie algebra $A_1^{(1)}$ at arbitrary level. A particular emphasis will be put on explicit bosonic realization of Whittaker modules at the critical level.

Abstract

- Based on the paper:
  D. Adamovic, R. Lu, K. Zhao, Whittaker modules for the affine Lie algebra $A_1^{(1)}$, http://arxiv.org/abs/1409.5354
- We shall also discuss related paper/projects.

Representations for Lie algebras

Let $\mathfrak{g}$ is a complex, simple Lie algebra.

Basic problem: Classify all irreducible $\mathfrak{g}$–modules.

Solved (in full generality) only for sl_2 by R. Block, Advances in Math. 39 (1) 1981

Representations for Lie algebras

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Let $\mathfrak{g} := sl_2$ be a simple, complex three dimensional Lie algebra with generators: e, f, h

and relations $[h, e] = 2e, \; [h, f] = -2f, \; [e, f] = h.$

Let $\mathfrak{h} = \mathbb{C}h$ (Cartan subalgebra) $\mathfrak{b} = \mathbb{C}e + \mathbb{C}h$ (Borel subalgebra) $\mathfrak{n}_+ = \mathbb{C}e$, $\mathfrak{n}_- = \mathbb{C}f$.

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+.$$

Highest weight vs. Whittaker modules

For every $\lambda \in \mathfrak{h}^*$ there is unique irreducible highest weight $\mathfrak{g}$–module $U(\lambda)$ generated by cyclic vector v_λ such that

$$hv_\lambda = \lambda(h)v_\lambda, \ \mathfrak{n}_+ v_\lambda = 0.$$

On highest weight modules Cartan subalgebra acts semisimple. They are not irreducible as $\mathfrak{b}$–modules.

If λ is generic, $U(\lambda) = U(\mathfrak{n}) \cong \mathbb{C}[f]$ as vector space.

For every $\lambda \in \mathfrak{n}^*$ there is a unique irreducible Whittaker $\mathfrak{g}$–module $W(\lambda)$ generated by vector w_λ such that

$$ew_\lambda = \lambda(e)w_\lambda.$$

If $\lambda \neq 0$, then $W(\lambda) \cong \mathbb{C}[h]$ as a vector space and $W(\lambda)$ is an irreducbile module for Borel subalgebra $\mathfrak{b}$.

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Whittaker modules : general definition

Assume that $\mathfrak{g}$ is a Lie algebra (possible infinite dimensional) with triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_{-} + \mathfrak{h} + \mathfrak{n}_{+}.$$

Let $\lambda : \mathfrak{n} \to \mathbb{C}$ be a Lie algebra homomorphism. The universal Whittaker modules with Whittaker function λ is defined as

$$\widetilde{W}(\lambda) := U(\mathfrak{g})/J(\lambda)$$

where $J(\lambda) := U(\mathfrak{g}).\{x - \lambda(x)1 \mid x \in \mathfrak{n}_{+}\}$.

Let $W(\lambda)$ be its simple quotient.

Problem: Determine the structure of $W(\lambda)$.

For $\mathfrak{g}$ be a semisimple complex Lie algebra, Whittaker modules are studied by Kostant, McDowell, Miličić, Soergel and others.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot, \cdot)$ be a nondegenerate symmetric bilinear form on $\mathfrak{g}$.

The affine Kac-Moody Lie algebra $\widetilde{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

where K is the canonical central element and the Lie algebra structure is given by

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n(x, y)\delta_{n+m,0}K,$$

$$[d, x \otimes t^n] = nx \otimes t^n$$

Let $\hat{\mathfrak{g}} = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$.

Set $x(n) = x \otimes t^n$, for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, and identify $\mathfrak{g}$ as the subalgebra $\mathfrak{g} \otimes t^0$.

Then $\widetilde{\mathfrak{g}}$ is a Kac-Moody Lie algebra with triangular decomposition

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_+$$

$$\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \ \widetilde{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}].$$

Define the field $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$ which acts on restricted $\hat{\mathfrak{g}}$–modules of level k.

Let $V_k(\mathfrak{g})$ be the universal vertex algebra generated by fields $x(z)$, $x \in \mathfrak{g}$.

As a $\hat{\mathfrak{g}}$–module, $V_k(\mathfrak{g})$ can be realized as a generalized Verma module.

Modules for affine vertex algebras

A $\hat{\mathfrak{g}}$–module M is called restricted if

$$x(z)w \in M((z)), \quad (\forall x \in \mathfrak{g},\ w \in M).$$

Every restricted $\hat{\mathfrak{g}}$-module M of level k is a module for $V_k(\mathfrak{g})$. In particular:

$\hat{\mathfrak{g}}$–modules from the category $\mathcal{O}_k$ are $V_k(\mathfrak{g})$–modules

Whittaker modules of level k are $V_k(\mathfrak{g})$–modules.

Our approach: Description of Whittaker modules for affine Lie algebra using theory of vertex algebras.

Replace the role of $U(\mathfrak{g})$ by universal affine vertex algebra.

Level $k = -h^\vee$ is called **critical level**

($h^\vee$ denotes dual Coxeter number).

Let x_i, y_i, $i = 1, \ldots, \dim \mathfrak{g}$ be dual bases of $\mathfrak{g}$ with respect to form $(\cdot, \cdot)$, and let

$$sug = \sum_{i=1}^{\dim \mathfrak{g}} x_i(-1)y_i(-1)\mathbf{1} \in V_k(\mathfrak{g}).$$

$$Y(sug, z) = \sum_{n \in \mathbb{Z}} S(n) z^{-n-1}.$$

Then

$$S(n) \quad \text{are in the center of } V_{-h^\vee}(\mathfrak{g}).$$

Let now $\mathfrak{g} = sl_2(\mathbb{C})$

with generators e, f, h

and relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

The corresponding affine Lie algebra $\widetilde{\mathfrak{g}}$ is of type $A_1^{(1)}$.

$\widetilde{\mathfrak{n}}_+$ is generated by

$$e_0 = e \otimes t^0; e_1 = f \otimes t^1.$$

Lie algebra homomorphism $\lambda : \widetilde{\mathfrak{n}}_+ \to \mathbb{C}$ is uniquely determined by

$$(\lambda_0, \lambda_1) = (\lambda(e_0), \lambda(e_1)).$$

Let $\widetilde{W}(k, \lambda_1, \lambda_2)$ and $W(k, \lambda_1, \lambda_2)$ denote the universal and simple Whittaker modules of level k and type (λ_1, λ_2).

Modules outside the critical level

$$\omega = \frac{1}{2(k+2)}(e(-1)f(-1)+f(-1)e(-1)+1/2h(-1)^2)\mathbf{1} \in V_k(sl_2).$$

Assume that M is a $V_k(sl_2)$–module for $k \neq -2$. Then M is a module for the Virasoro algebra at central charge $c = 3k/(k+2)$ (Sugawara construction)

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.$$

M can be treated as $\hat{\mathfrak{g}} \rtimes Vir$–module and as $\widehat{\mathfrak{b}} \rtimes Vir$–module.

Whittaker modules outside the critical level

Theorem (D.A; R. L, K. Z, 2014)

Assume that $k \neq -2$ and $\lambda_1 \cdot \lambda_2 \neq 0$. Then

- *The universal Whittaker module $\widetilde{W}(k, \lambda_1, \lambda_2)$ is an irreducible $\widehat{\mathfrak{b}} \rtimes Vir$–module.*
- *The universal Whittaker module $\widetilde{W}(k, \lambda_1, \lambda_2)$ is an irreducible $\widetilde{sl_2}$–module.*

Whittaker modules at the critical level

Assume that M is $V_{-2}(sl_2)$–module. M is also a module for the center of $V_{-2}(sl_2)$ generated by $S(n)$, $n \in \mathbb{Z}$.

Theorem (D.A; R. L, K. Z, 2014)

Assume that $k = -2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \neq 0$. Let $c(z) = \sum_{n \leq 0} c_n z^{-n-2} \in \mathbb{C}((z))$.

(1) $W(-2, \lambda_1, \lambda_2, c(z)) = \widetilde{W}(-2, \lambda_1, \lambda_2) / \langle (S(n) - c_n) v_{-2,\lambda_1,\lambda_2} \mid n \leq 0 \rangle$ *is an irreducible* $\widehat{sl_2}$*–module.*

(2) $W(-2, \lambda_1, \lambda_2, c(z))$ *is irreducible* $\widehat{\mathfrak{b}}$*–module.*

(3) *Assume that* $\lambda_2 \neq 0$. *Then*

$$Ind_{\widehat{sl_2}}^{\widetilde{sl_2}} W(-2, \lambda_1, \lambda_2, c(z))$$

is an irreducible Whittaker module at the critical level.

Theorem

Basis of the universal Whittaker module $\widetilde{W}(k, \lambda_1, \lambda_2)$ is given by vectors

$$e(-n_1)\cdots e(-n_r)h(-m_1)\cdots h(-m_s)X(-p_1)\cdots X(-p_t)v_{k,\lambda_1,\lambda_2}$$

for

$$n_1 \geq \cdots n_r \geq 1; m_1 \geq \cdots m_s \geq 0; p_1 \geq \cdots p_t \geq 0, ...$$

where $X(n) = L(n)$ if $k \neq -2$ and $X(n) = S(n)$ at the critical level.

This suggests that a higher rank generalization of our result should use $\mathcal{W}$–algebra $\mathcal{W}(\mathfrak{g})$.

Weyl vertex algebra

The Weyl vertex algebra W is generated by the fields

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n},$$

whose components satisfy the commutation relations for infinite-dimensional Weyl algebra

$$[a(n), a(m)] = [a^*(n), a^*(m)] = 0, \quad [a(n), a^*(m)] = \delta_{n+m,0}.$$

Assume that $\chi(z) \in \mathbb{C}((z))$.

On the vertex algebra W exists the structure of the $\hat{\mathfrak{g}}$–module at the critical level defined by

$$\begin{aligned} e(z) &= a(z), \\ h(z) &= -2 : a^*(z)a(z) : -\chi(z) \\ f(z) &= - : a^*(z)^2 a(z) : -2\partial_z a^*(z) - a^*(z)\chi(z). \end{aligned}$$

This module is called the Wakimoto module and it is denoted by $W_{-\chi(z)}$.

Theorem (D.A., CMP 2007; Contemporary Math. 2014.)

The Wakimoto module $W_{-\chi}$ is irreducible if and only if $\chi(z)$ satisfies one of the following conditions:

(i) *There is $p \in \mathbf{Z}_{>0}$, $p \geq 1$ such that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_p \neq 0.$$

(ii) $\chi(z) = \sum_{n=0}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad$ *and* $\quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z})$.

(iii) *There is $\ell \in \mathbb{Z}_{\geq 0}$ such that*

$$\chi(z) = \frac{\ell + 1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

and $S_\ell(-\chi) \neq 0$, where $S_\ell(-\chi) = S_\ell(-\chi_{-1}, -\chi_{-2}, \dots)$ is a Schur polynomial.

Explicit realization of Whittaker modules

Every restricted module for the Weyl algebra is a module for Weyl vertex algebra W.

For $(\lambda, \mu) \in \mathbb{C}^2$ let $M_1(\lambda, \mu)$ be the module for the Weyl algebra generated by the Whittaker vector v_1 such that

$$a(0)v_1 = \lambda v_1, \; a^*(1)v_1 = \mu v_1, \; a(n+1)v_1 = a^*(n+2)v_1 = 0 \quad (n \geq 0).$$

$M_1(\lambda, \mu)$ is a W-module.

Theorem (D.A; R. Lu, K. Zhao, 2014)

For every $\chi(z) \in \mathbb{C}((z))$, $(\lambda, \mu) \in \mathbb{C}^2$, $\lambda \neq 0$ *there exists irreducible* $\widehat{sl_2}$*–module* $\overline{M_{Wak}}(\lambda, \mu, -2, \chi(z))$ *realized on the* W*–module* $M_1(\lambda, \mu)$ *such that*

$$\begin{aligned} e(z) &= a(z); \\ h(z) &= -2 : a^*(z)a(z) : +\chi(z); \\ f(z) &= - : a^*(z)^2 a(z) : -2\partial_z a^*(z) + a^*(z)\chi(z) \end{aligned}$$

Whittaker vs. Wakimoto modules

- We see that Whittaker modules from previous theorem are realized using same formulas.
- Wakimoto modules are realized on vertex algebra W, but Whittaker are realized on W–module $M_1(\lambda, \mu)$.
- Whitteker modules are always irreducible, but for Wakimoto modules we have non–trivial criteria.

As a special case, the previous theorem provides a realization of degenerate Whittaker modules at the critical level.

But it does not cover non-degenerate Whittaker modules at the critical level.

We need to modify construction of Wakimoto modules.

Method: Use vertex algebra $\Pi(0)$, a localization of Weyl vertex algebra.

The vertex algebra $\Pi(0)$

$$L = \mathbb{Z}\alpha + \mathbb{Z}\beta, \ \langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1, \quad \langle \alpha, \beta \rangle = 0,$$

and $V_L = M_{\alpha,\beta}(1) \otimes \mathbb{C}[L]$ the associated lattice vertex superalgebra, where $M_{\alpha,\beta}(1)$ is the Heisenbeg vertex algebra generated by fields $\alpha(z)$ and $\beta(z)$ and $\mathbb{C}[L]$ is the group algebra of L. We have its subalgebra

$$\Pi(0) = M_{\alpha,\beta}(1) \otimes \mathbb{C}[\mathbb{Z}(\alpha + \beta)] \subset V_L.$$

The vertex algebra $\Pi(0)$

The Weyl vertex algebra W can be realized as a subalgebra of $\Pi(0)$ generated by

$$a = e^{\alpha+\beta}, \; a^* = -\alpha(-1)e^{-\alpha-\beta}.$$

$$M = \mathrm{Ker}_{\Pi(0)} e_0^{\alpha}.$$

$\Pi(0)$ is a localization of Weyl vertex algebra with respect to $a(-1)$, $\Pi(0) = M[(a(-1)^{-1}]$. Let $a^{-1} := e^{-\alpha-\beta}$ and $a^{-1}(n) := e_{n-2}^{-\alpha-\beta}$. We have the expansion

$$a^{-1}(z) = Y(a^{-1}, z) = \sum_{n\in\mathbb{Z}} a^{-1}(n) z^{-n+1}.$$

$$a^{-1}(z)a(z) = Id.$$

Theorem

Assume that $\lambda \neq 0$. There is a $\Pi(0)$–module Π_λ generated by the cyclic vector w_λ such that

$$a(0)w_\lambda = \lambda w_\lambda, \quad a^{-1}(0)w_\lambda = \frac{1}{\lambda} w_\lambda, \quad a(n)w_\lambda = a^{-1}(n)w_\lambda = 0 (n \geq 1).$$

As a vector space

$$\Pi_\lambda \cong \mathbb{C}[d(-n), c(-n-1) \mid n \geq 0] = \mathbb{C}[d(0)] \otimes M_{\alpha,\beta}(1),$$

where $c = \alpha + \beta$, $d = \alpha - \beta$.
Π_λ is $\mathbb{Z}_{\geq 0}$–graded

$$\Pi_\lambda = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Pi_\lambda(n)$$

and lowest component is isomorphic to $\mathbb{C}[d(0)]$.

Modified Wakimoto construction

There is an embedding of vertex algebras

$$V_{-2}(sl_2) \to M_T(0) \otimes \Pi(0)$$

such that

$$\begin{aligned} e &= a, & (1)\\ h &= -2\beta(-1) = -2a^*(0)a(-1)\mathbf{1} & (2)\\ f &= \left[T(-2) - (\alpha(-1)^2 + \alpha(-2))\right] a^{-1} & (3)\\ &= -a^*(0)^2 a(-1)\mathbf{1} - 2a^*(-1)\mathbf{1} + T(-2)a^{-1} & (4) \end{aligned}$$

For any $\chi(z) = \sum_{n\in\mathbb{Z}} \chi(n) z^{-n-2} \in \mathbb{C}((z))$ let $M_T(\chi(z))$ be 1-dimensional $M_T(0)$–module such that $T(n)$ acts as multiplication with $\chi(n) \in \mathbb{C}$.

Theorem

Let $\lambda \neq 0$. Let

$$\chi(z) = \frac{\lambda\mu}{z^3} + c(z), \quad c(z) = \sum_{n\leq 0} \chi(n) z^{-n-2} \in \mathbb{C}((z)).$$

Then we have:

$$V_{\widehat{sl_2}}(\lambda, \mu, -2, c(z)) \cong M_T(\chi(z)) \otimes \Pi_\lambda.$$

Higher rank generalizations

- Consider Whittaker modules as modules for related $\mathcal{W}$–algebras.
- In particular, Whittaker modules for $\widehat{sl_n}$ can be considered as modules for $W(sl_n)$–algebras obtained as higher rank generalization of Sugawara construction.
- At the critical level consider Whittaker modules as (Whittaker) modules for the Feigin-Frenkel center.

More comments on vertex algebra $\Pi(0)$ and its modules

- (Most important) $\Pi(0)$ appeared in works of S. Rao, Y. Billig, S. Tan, S. Berman, C. Dong, S. Futorny, M. Lau ... for constructions of modules for toroidal algebras.
- So in the analysis of Whittaker modules for toroidal algebras, module Π_λ probably must appear.

More comments on vertex algebra $\Pi(0)$ and its modules

- In *D. Adamović, A realization of certain modules for the $N = 4$ superconformal algebra and the affine Lie algebra $A_2^{(1)}$ (2014)*

  the vertex algebra $\Pi(0)$ appears in these new realizations

  In particular, the simple affine vertex algebra

  $$L_{A_2}(-\frac{3}{2}\Lambda_0) \subset \Pi(0)^{\otimes 2}$$

  We know that $\Pi_\lambda \otimes \Pi_\mu$ is an irreducible (degenerate) Whittaker $A_2^{(1)}$–module.

More related works

- In *D. A., G. Radobolja, Free field realization of the twisted Heisenberg-Virasoro algebra at level zero and its applications (2014)*,

  the representation theory of $\Pi(0)$ was used to construct screening operators for the twisted Heisnberg-Virasoro algebra and determination of fusion rules.
- Principal representations of Wakimoto modules by using twisted modules for vertex algebras will be subject of our forthcoming paper

  D.Adamovic, N. Jing; K. Misra, to appear,

Whittaker modules in Kazhdan-Lusztig corrspondence (preliminary)

- Idea: To suitable vertex algebra, obtained as kernel of screening operators, one can associate quantum group, so called Kazhdan-Lusztig dual.
- Conjecture is that the representation categories of vertex algebras and associated quantum groups should be closely related (equivalent ?)
- Whittaker modules can appear as modules on the quantum group side, so one can expect that they naturally appear in the vertex-algebra side

Whittaker modules in Kazhdan-Lusztig corrspondence (preliminary)

1. The triplet vertex algebra $\mathcal{W}(p)$ (studied by D.A, A. Milas in series of works) is C_2–cofinite and irrational. Its Kazhdan-Lusztig dual is restricted quantum group $\overline{U}_q(sl_2)$, when $q = e^{2\pi i/p}$. There are no Whittaker modules in this case.
2. Kazhdan-Lusztig dual of the Virasoro vertex algebra of central charge $c = 1 - 6(p-1)^2/p$ is the quantum group $\mathcal{LU}_q(sl_2)$ (introduced by Bushlanov, Feigin, Gainutdinov , Tipunin) which is extension of $\overline{U}_q(sl_2)$ by sl_2 triple e, f, h.

   $\mathcal{LU}_q(sl_2)$ is extension of $\overline{U}_q(sl_2)$ by sl_2 triple e, f, h.

   There are Whittaker modules on both sides.

Thank you