Application of VOA to representation theory of $W(2,2)$-algebra and the twisted Heisenberg-Virasoro algebra

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Overview

1. G. R. "Application of vertex algebras to the structure theory of certain representations over Virasoro algebra", Algebras and Represent. Theory 16 (2013)
2. G. R. "Subsingular vectors in Verma modules, and tensor product of weight modules over the twisted Heisenberg-Virasoro algebra and $W(2, 2)$ algebra", Journal of Mathematical Physics 54 (2013)
3. D. Adamović, G. R. "Free fields realization of the twisted Heisenberg-Virasoro algebra at level zero and its applications" to appear

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- VOA, intertwining operators and tensor product modules

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- The twisted Heisenberg-Virasoro Lie algebra $\mathcal{H}$. We study representations at level zero, important in rep. theory of toroidal Lie algebras. Developed by Y. Billig in Representations of the twisted Heisenberg-Virasoro algebra at level zero, Canadian Math. Bulletin, 46 (2003)

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- $W(2,2)$-structure on $\mathcal{H}$-modules.

Algebra $W(2,2)$

Algebra $\mathcal{L} = W(2,2)$ is a complex Lie algebra with a basis $\{L_n, W_n, C_L, C_W : n \in \mathbb{Z}\}$ and a Lie bracket

$$[L_n, L_m] = (n-m)\, L_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} C_L,$$

$$[L_n, W_m] = (n-m)\, W_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} C_W,$$

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$\{L_n, C_L, : n \in \mathbb{Z}\}$ spans a copy of the Virasoro algebra.

$\{W_n : n \in \mathbb{Z}\}$ spans a Virasoro module $V'_{1,-1}$.

Algebra $W(2,2)$

Triangular decomposition:

$$\mathcal{L} = \mathcal{L}_- \oplus \mathcal{L}_0 \oplus \mathcal{L}_+$$

where

$$\mathcal{L}_+ = \bigoplus_{n>0} (\mathbb{C}L_n + \mathbb{C}W_n),$$

$$\mathcal{L}_- = \bigoplus_{n>0} (\mathbb{C}L_{-n} + \mathbb{C}W_{-n}),$$

$$\mathcal{L}_0 = \operatorname{span}\{L_0, W_0, C_L, C_W\}.$$

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$V(c_L, c_W, h, h_W)$ - the **Verma module** with highest weight (h, h_W) and central charge (c_L, c_W)

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$$L_0 v = hv, \quad W_0 v = h_W v,$$
$$C_L v = c_L v, \quad C_W v = c_W v, \quad \mathcal{L}_+ v = 0.$$

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However, W_0 does not act semisimply on rest of the module (unlike I_0 in the twisted Heisenberg-Virasoro algebra).

The Verma module

- PBW basis

$$\{W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v : \\ m_s \geq \cdots \geq m_1 \geq 1, n_t \geq \cdots \geq n_1 \geq 1\}$$

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- $\dim V(c_L, c_W, h, h_W)_{h+n} = P_2(n) = \sum_{i=0}^{n} P(n-i)P(i)$, where P is a partition function, with $P(0) = 1$

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Theorem (Zhang-Dong)

Verma module $V(c_L, c_W, h, h_W)$ is irreducible if and only if $h_W \neq \frac{1-m^2}{24} c_W$ for any $m \in \mathbb{N}$.

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- $y \in V(c_L, c_W, h, h_W)$ is called a **subsingular vector** if y is a singular vector in some quotient $V(c_L, c_W, h, h_W)/U$ i.e. if $\mathcal{L}_+ y \in U$ for a submodule $U \subset V(c_L, c_W, h, h_W)$

W-degree

W-degree on $\mathcal{L}_-$

$$\deg_W L_{-n} = 0, \qquad \deg_W W_{-n} = 1$$

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induces $\mathbb{Z}$-grading on $U(\mathcal{L})$ and on $V(c_L, c_W, h, h_W)$ (in a standard PBW basis)

$$\deg_W W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v = s$$

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$$\mathcal{W} = \mathbb{C}\left[W_{-1}, W_{-2}, \ldots\right] v$$
$$\mathcal{W}_{h+n} = \mathcal{W} \cap V(c_L, c_W, h, h_W)_{h+n}$$

W-degree

Lemma (Jiang-Pei (Y. Billig))

Let $0 \neq x \in V(c_L, c_W, h, h_W)$ *and* $\deg_W \overline{x} = k$.

(a) If $\overline{x} \notin \mathcal{W}$ *and* $n \in \mathbb{N}$ *is the smallest, such that* L_{-n} *occurs as a factor in one of the terms in* $\overline{x}$*, then the part of* $W_n x$ *of the W-degree* k *is given by*

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$$n\left(2h_W + \frac{n^2-1}{12}c_W\right)\frac{\partial \overline{x}}{\partial L_{-n}}.$$

(b) *If* $\overline{x} \in \mathcal{W}$, $\overline{x} \notin \mathbb{C}v$ *and* $m \in \mathbb{N}$ *is maximal, such that* W_{-m} *occurs as a factor in one of the terms of* $\overline{x}$, *then the part of* $L_m x$ *of the* W*-degree* $k-1$ *is given by*

$$m\left(2h_W + \frac{m^2-1}{12}c_W\right)\frac{\partial \overline{x}}{\partial W_{-m}}.$$

Singular vectors

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Lemma (Jiang-Pei (Y. Billig))

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Theorem

Let $h_W = \frac{1-p^2}{24} c_W$, $p \in \mathbb{N}$. Then there is a singular vector $u' \in \mathcal{W}_{h+p}$, such that $\overline{u'} = W_{-p}v$. Moreover, $U(\mathcal{L})u'$ is isomorphic to Verma module $V(c_L, c_W, h+p, h_W)$.

Examples of singular vectors

module	u'
$V(c_L, c_W, h, 0)$	$W_{-1}v$
$V\left(c_L, c_W, h, -\frac{c_W}{8}\right)$	$\left(W_{-2} + \frac{6}{c_W} W_{-1}^2\right)v$
$V(c_L, c_W, h, -\frac{c_W}{3})$	$\left(W_{-3} + \frac{6}{c_W} W_{-2} W_{-1} + \frac{9}{c_W^2} W_{-1}^3\right)v$
$V(c_L, c_W, h, -\frac{5c_W}{8})$	$\left(W_{-4} + \frac{4}{c_W} W_{-3} W_{-1} + \frac{2}{3c_W} W_{-2}^2 + \right.$ $\left. + \frac{10}{c_W^2} W_{-2} W_{-1}^2 + \frac{15}{4c_W^2} W_{-1}^4\right)v$

Characters

From now on, u' denotes the singular vector from previous theorem.

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$$J'(c_L, c_W, h, h_W) := U(\mathcal{L})\, u'$$
$$L'(c_L, c_W, h, h_W) = V(c_L, c_W, h, h_W) / J'(c_L, c_W, h, h_W)$$

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Since

$$\operatorname{char} V(c_L, c_W, h, h_W) = q^h \sum_{n \geq 0} P_2(n) q^n,$$

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the theorem yields

$$\mathrm{char}\, J'(c_L, c_W, h, h_W) = q^{h+p} \sum_{n \geq 0} P_2(n) q^n,$$
$$\mathrm{char}\, L'(c_L, c_W, h, h_W) = \mathrm{char}\, V - \mathrm{char}\, J' =$$
$$= q^h (1 - q^p) \sum_{n \geq 0} P_2(n) q^n.$$

Reducibility of a quotient module

Is $L'(c_L, c_W, h, h_W)$ irreducible?

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i) $L_{-1}v$ is a singular vector in $L'(c_L, c_W, 0, 0) = V(c_L, c_W, 0, 0) / U(\mathcal{L}) W_{-1}$.

ii) $\left(L_{-2} + \frac{12}{c_W} W_{-1} L_{-1} - \frac{6(14+c_L)}{c_W} W_{-1}^2\right) v$ is a singular vector in $L'\left(c_L, c_W, \frac{18-c_L}{8}, -\frac{c_W}{8}\right) =$
$V\left(c_L, c_W, \frac{18-c_L}{8}, -\frac{c_W}{8}\right) / U(\mathcal{L}) (W_{-2} + \frac{6}{c_W} W_{-1}^2) v$.

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iii) $\left(L_{-1}^2 + \frac{6}{c_W}W_{-2}\right)v$ is a singular vector in $L'(c_L, c_W, -\frac{1}{2}, 0) = V(c_L, c_W, -\frac{1}{2}, 0)/U(\mathcal{L})W_{-1}v$.

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iii) $\left(L_{-1}^2 + \frac{6}{c_W} W_{-2}\right) v$ is a singular vector in $L'(c_L, c_W, -\frac{1}{2}, 0) = V(c_L, c_W, -\frac{1}{2}, 0) / U(\mathcal{L}) W_{-1} v$.

Problem

What is the structure of $L'(c_L, c_W, h, h_W)$*?*

Structure of a quotient module L'

Lemma (Jiang, Pei (Y. Billig))

Let $0 \neq x \in J'(c_L, c_W, h, h_W)$. *Then there exist terms in* $\overline{x}$, *containing factor* W_{-p}.

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Proposition

The set of all PBW vectors $W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v$ *modulo* $J'(c_L, c_W, h, h_W)$ *with* $m_i \neq p$ *forms a basis for* $L'(c_L, c_W, h, h_W)$*.*

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Theorem

Assume that $L'(c_L, c_W, h, h_W)$ is reducible. Then there is a singular vector $u \in L'(c_L, c_W, h, h_W)$ such that $\overline{u} = L_{-p}^q v$ for some $q \in \mathbb{N}$.

Necessary condition

Equating certain coefficients in relation $L_p u \in J'(c_L, c_W, h, h_W)$ we get the following result:

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Theorem (Necessary condition for the existence of a subsingular vector)

Let $h_W = \frac{1-p^2}{24} c_W$. If $L'(c_L, c_W, h, h_W)$ contains a singular vector u such that $\overline{u} = L_{-p}^q v$, for some $q \in \mathbb{N}$, then

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Necessary condition

Equating certain coefficients in relation $L_p u \in J'(c_L, c_W, h, h_W)$ we get the following result:

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For a PBW monomial $x = W_{-m_s} \cdots W_{-m_1} L_{-n_t} \cdots L_{-n_1} v$ define L_{-p}-degree $\deg_{L_{-p}} x$ as a number of factors $L_{-n_i} = L_{-p}$.

Irreducibility of a quotient module

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is the maximal submodule.

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and a character

$$\text{char}\, L(c_L, c_W, h, h_W) = q^h (1 - q^p)(1 - q^{qp}) \sum_{n \geq 0} P_2(n) q^n.$$

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Suppose $h_W = \frac{1-p^2}{24} c_W$ *for some* $p \in \mathbb{N}$. *Then* $L'(c_L, c_W, h, h_W)$ *is reducible if and only if*

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Using determinant formula one can prove

Theorem

Module $L'(c_L, c_W, \frac{1-q}{2}, 0)$ *is reducible for every* $q \in \mathbb{N}$, *i.e. there is a subsingular vector* $u \in V(c_L, c_W, \frac{1-q}{2}, 0)$ *such that* $\overline{u} = L_{-1}^q$.

Examples

Subsingular vectors u in $V(c_L, c_W, \frac{1-q}{2}, 0)$:

$V(c_L, c_W, 0, 0)$	$L_{-1}v$
$V(c_L, c_W, -\frac{1}{2}, 0)$	$\left(L_{-1}^2 + \frac{6}{c_W} W_{-2}\right) v$
$V(c_L, c_W, -1, 0)$	$\left(L_{-1}^3 + \frac{12}{c_W} W_{-3} + \frac{24}{c_W} W_{-2} L_{-1}\right) v$
$V(c_L, c_W, -\frac{3}{2}, 0)$	$\Big(L_{-1}^4 + \frac{60}{c_W} W_{-2} L_{-1}^2 + \frac{60}{c_W} W_{-3} L_{-1} + \frac{36}{c_W} W_{-4} + \frac{324}{c_W^2} W_{-2}^2\Big) v$
$V(c_L, c_W, -2, 0)$	$\Big(L_{-1}^5 + \frac{120}{c_W} W_{-2} L_{-1}^3 + \frac{180}{c_W} W_{-3} L_{-1}^2 + \frac{48}{c_W} W_{-4} L_{-1} + \frac{3312}{c_W^2} W_{-2}^2 L_{-1} + \frac{144}{c_W} W_{-5} + \frac{2304}{c_W^2} W_{-3} W_{-2}\Big) v$

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It can be shown that $u = (L_{-1}^q + \sum_{i=0}^{q-1} w_i L_{-1}^i) v$ for some $w_i \in \mathcal{W}$.

Intermediate series

For $\alpha, \beta \in \mathbb{C}$ take Vir-modules

$$V_{\alpha,\beta} = \operatorname{span}_{\mathbb{C}} \{v_n : n \in \mathbb{Z}\}$$

with

$$L_k v_n = -(n + \alpha + \beta + k\beta)\, v_{n+k},$$
$$C_L v_n = 0, \qquad k, n \in \mathbb{Z}.$$

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Define $\mathcal{L}$-modules

$$\begin{aligned} V_{\alpha,\beta,0} &:= V_{\alpha,\beta} \quad \text{with} \\ C_W v_n &= W_k v_n = 0, \qquad k, n \in \mathbb{Z}. \end{aligned}$$

Intermediate series

$$V_{\alpha,\beta,0} \cong V_{\alpha+k,\beta,0} \qquad \text{for} \qquad k \in \mathbb{Z}$$

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$$V'_{0,0,0} := V_{0,0,0}/\mathbb{C}v_0,$$

$$V'_{0,1,0} := \bigoplus_{m \neq -1} \mathbb{C}v_m \subseteq V_{0,1,0},$$

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$\left\{ V'_{\alpha,\beta,0} : \alpha, \beta \in \mathbb{C} \right\}$ - all irreducible modules belonging to intermediate series.

Irreducible Harish-Chandra modules

Theorem (Liu, D., Zhu, L.)

An irreducible weight $\mathcal{L}$-module with finite-dimensional weight spaces is isomorphic either to a highest (or lowest) weight module, or to $V'_{\alpha,\beta,0}$ for some $\alpha, \beta \in \mathbb{C}$.

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What about modules with infinite-dimensional weight spaces?

Tensor product modules

$V'_{\alpha,\beta,0} \otimes L(c_L, c_W, h, h_W)$ is $\mathcal{L}$-module:

$$
\begin{aligned}
L_k(v_n \otimes x) &= L_k v_n \otimes x + v_n \otimes L_k x, \\
W_m(v_n \otimes x) &= v_n \otimes W_m x, \\
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All weight subspaces are infinite-dimensional:

$$\left(V'_{\alpha,\beta,0} \otimes L(c_L, c_W, h, h_W)\right)_{h+m-\alpha-\beta} = \\ = \bigoplus_{n \in \mathbb{Z}_+} \mathbb{C} v_{n-m} \otimes L(c_L, c_W, h, h_W)_{h+n}$$

(Ir)reducibility of the tensor product modules

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Theorem (Irreducibiliy criterion)

$V'_{\alpha,\beta,0} \otimes L(c_L, c_W, h, h_W)$ is irreducible if and only if it is cyclic on every $v_n \otimes v$, i.e., if $U_n = U_{n+1}$ for $n \in \mathbb{Z}$.

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Theorem

Let $h \neq h_{p,q}$ for all q. Then module $V'_{\alpha,\beta,0} \otimes L(c_L, c_W, h, h_W)$ is reducible for any $\alpha, \beta \in \mathbb{C}$. Moreover:

$$U_n \supsetneq U_{n+1}, \qquad \forall n \in \mathbb{Z}.$$

Irreducibility of the tensor product modules

Theorem

Let $h = h_{p,q}$ and let $u \in V(c_L, c_W, h, h_W)$ be a subsingular vector such that $\overline{u} = L_{-p}^{q}$. If $\alpha + (1-p)\beta \notin \mathbb{Z}$ then module $V'_{\alpha,\beta,0} \otimes L(c_L, c_W, h, h_W)$ is irreducible.

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Proof.

[Sketch of proof] Using subsingular vector u we find $x \in U(\mathcal{L})$ such that

$$x(v_n \otimes v) =$$

$$= \left(\prod_{j=0}^{q-1} (n - 1 + (q-j)p + \alpha + (1-p)\beta) \right) v_{n-1} \otimes v$$

□

Irreducible submodules

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$$U_{1-p} \text{ is irreducible.}$$

Weight (0,0)

Corollary

(i) $V'_{\alpha,\beta,0} \otimes L(c_L, c_W, 0, 0)$ *is irreducible if and only if* $\alpha \notin \mathbb{Z}$.

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If $1-\beta \neq \frac{1-q}{2}$ *for* $q \in \mathbb{N}$ *then*

$$\left(V'_{0,\beta,0} \otimes L(c_L, c_W, 0, 0)\right) / U_0 \cong L(c_L, c_W, 1-\beta, 0),$$

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If $q \in \mathbb{N} \setminus \{1\}$

$$\left(V'_{0,\frac{1+q}{2},0} \otimes L(c_L, c_W, 0, 0)\right) / U_0 \cong L'(c_L, c_W, \frac{1-q}{2}, 0).$$

VOA

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Theorem (Zhang-Dong)

Let $c_L, c_W \neq 0$. Then

1. *There is a unique VOA structure on $L(c_L, c_W, 0, 0)$ which we denote $L^W(c_L, c_W)$, with the vacuum vector v, and the Virasoro element $\omega = L_{-2}v$. $L^W(c_L, c_W)$ is generated with ω and $x = W_{-2}v$ and $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, $Y(x, z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-2}$.*

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Theorem (Zhang-Dong)

Let $c_L, c_W \neq 0$. Then

1. *There is a unique VOA structure on $L(c_L, c_W, 0, 0)$ which we denote $L^W(c_L, c_W)$, with the vacuum vector v, and the Virasoro element $\omega = L_{-2}v$. $L^W(c_L, c_W)$ is generated with ω and $x = W_{-2}v$ and $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, $Y(x, z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-2}$.*
2. *Any quotient of $V(c_L, c_W, h, h_W)$ is an $L^W(c_L, c_W)$-module, and $\{L(c_L, c_W, h, h_W) : h, h_W \in \mathbb{C}\}$ gives a complete list of irreducible $L^W(c_L, c_W)$-modules.*

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- Recall that $W_0 v = W_{-1} v = 0$
- $\mathcal{I}(v, z) = z^{-\alpha} \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1}$ for $\alpha = h_1 + h_2 - h_3$

Intertwining operators

$$\left[L_m, v_{(n)}\right] = \sum_{i\geq 0}\binom{m+1}{i}\left(L_{i-1}v\right)_{(m+n-i+1)} =$$
$$= \left(L_{-1}v\right)_{(m+n+1)} + (m+1)\left(L_0 v\right)_{(m+n)} =$$
$$= -\left(\alpha + n + m + 1\right) v_{(m+n)} + (m+1)\, h_1 v_{(m+n)} =$$
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so components $v_{(n)}$ span $V'_{\alpha,1-h_1,0}$.

Intertwining operators and reducibility

We get a nontrivial $\mathcal{L}$-homomorphism

$$\Phi : V'_{\alpha,1-h_1,0} \otimes M(c_L, c_W, h_2, h_W) \longrightarrow M(c_L, c_W, h_3, h'_W),$$
$$\Phi(v_{(n)} \otimes x) = v_{(n)}x.$$

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dimensions of weight spaces $\Rightarrow$
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In particular, operators of type

$$\binom{L(c_L, c_W, h, 0)}{L(c_L, c_W, h, 0) \quad L(c_L, c_W, 0, 0)}$$

and

$$\binom{L'(c_L, c_W, h, 0)}{L'(c_L, c_W, h, 0) \quad L(c_L, c_W, 0, 0)}$$

exist for all h.

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exist, there are nontrivial $\mathcal{L}$-homomorphisms

$$V'_{0,\beta,0} \otimes L(c,0,0) \to L(c_L, c_W, 1-\beta, 0),$$

$$V'_{0,\frac{1+q}{2},0} \otimes L(c,0,0) \to L'(c_L, c_W, \frac{1-q}{2}, 0).$$

The twisted Heisenberg-Virasoro algebra

Algebra $\mathcal{H}$ is a complex Lie algebra with a basis $\{L_n, I_n, C_L, C_I, C_{L,I} : n \in \mathbb{Z}\}$ and a Lie bracket

$$\begin{aligned}
[L_n, L_m] &= (n-m)\, L_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} C_L, \\
[L_n, I_m] &= -mI_{n+m} - \delta_{n,-m}(n^2 + n) C_{LI}, \\
[I_n, I_m] &= n\delta_{n,-m} C_I, \\
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The Verma module

- $V(c_L, c_I, c_{L,I}, h, h_I)$ - the **Verma module** with highest weight (h, h_I) and central charge $(c_L, c_I, c_{L,I})$.

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- Note that I_0 acts semisimply on entire module.

The Verma module

Theorem (Y. Billig)

Assume that $c_I = 0$ *and* $c_{LI} \neq 0$.
(i) *If* $\frac{h_I}{c_{LI}} \notin \mathbb{Z}$ *or* $\frac{h_I}{c_{LI}} = 1$, *then the Verma module* $V(c_L, c_{LI}, 0, h, h_L)$ *is irreducible.*

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The quotient module $L(c_L, 0, c_{L,I}, h, h_I) = V(c_L, 0, c_{L,I}, h, h_I)/U(\mathcal{H})u$ *is irreducible and its character is*

$$\mathrm{char}\, L(c_L, 0, c_{L,I}, h, h_I) = q^h(1-q^p)\prod_{j\geq 1}(1-q^j)^{-2}.$$

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(iii) If $\frac{h_I}{c_{LI}} = 1 - p$, then $\overline{u} = L_{-p}$.

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Let $\alpha, \beta, F \in \mathbb{C}$ define $V_{\alpha,\beta,F} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C} v_n$ with Lie bracket

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- $V'_{0,0,0} := V/\mathbb{C}v_0$, $V'_{0,1,0} := \bigoplus_{n\neq -1} \mathbb{C} v_n$ and $V'_{\alpha,\beta,F} := V_{\alpha,\beta,F}$ otherwise.

Tensor product modules

Consider $V'_{\alpha,\beta,F} \otimes L\left(c_L, 0, c_{L,I}, h, h_I\right)$ module:

$$\begin{aligned}
L_k(v_n \otimes x) &= L_k v_n \otimes x + v_n \otimes L_k x, \\
I_m(v_n \otimes x) &= F v_n \otimes x + v_n \otimes I_m x, \\
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- Generated by $\{v_n \otimes v : n \in \mathbb{Z}\}$.
- Set $U_n = U(\mathcal{H})(v_n \otimes v)$.

Reducibility of a tensor product module

Theorem

$V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ is irreducible if and only if $U_n = U_{n+1}$ for all $n \in \mathbb{Z}$.

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$V'_{\alpha,\beta,F} \otimes V\left(c_L, 0, c_{L,I}, h, h_I\right)$ is reducible. Modules $V\left(c_L, 0, c_{L,I}, h - \alpha - \beta - n, h_I\right)$, $n \in \mathbb{Z}$ occur as subquotients.

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For a complete solution of irreducibility problem for $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$ we need more detailed formulas for singular vectors.

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Irreducible $\mathcal{H}$-module $L(c_L, 0, c_{L,I}, 0, 0)$ has the structure of vertex operator algebra which we denote $L^{\mathcal{H}}(c_L, c_{L,I})$.

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Let $c_{L,I} \neq 0$. Then $L^{\mathcal{H}}(c_L, c_{L,I})$ is a simpe VOA, and $V(c_L, 0, c_{L,I}, h, h_I)$ and $L(c_L, 0, c_{L,I}, h, h_I)$ are $L^{\mathcal{H}}(c_L, c_{L,I})$-modules.

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- $L^{\mathcal{H}}(c_L, c_{L,I})$ can be realized as a subalgebra of the Heisenberg vertex algebra $M(1)$.
- Moreover, $M(1)$-modules $M(1, \gamma)$ become $L^{\mathcal{H}}(c_L, c_{L,I})$-modules, and also $\mathcal{H}$-modules.

The Heisenberg-Virasoro vertex-algebra

Irreducible $\mathcal{H}$-module $L(c_L, 0, c_{L,I}, 0, 0)$ has the structure of vertex operator algebra which we denote $L^{\mathcal{H}}(c_L, c_{L,I})$.

Theorem (Y. Billig)

Let $c_{L,I} \neq 0$. Then $L^{\mathcal{H}}(c_L, c_{L,I})$ is a simpe VOA, and $V(c_L, 0, c_{L,I}, h, h_I)$ and $L(c_L, 0, c_{L,I}, h, h_I)$ are $L^{\mathcal{H}}(c_L, c_{L,I})$-modules.

- $L^{\mathcal{H}}(c_L, c_{L,I})$ can be realized as a subalgebra of the Heisenberg vertex algebra $M(1)$.
- Moreover, $M(1)$-modules $M(1, \gamma)$ become $L^{\mathcal{H}}(c_L, c_{L,I})$-modules, and also $\mathcal{H}$-modules.
- (Joint work with D. Adamović)

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- We get the twisted Heisenberg-Virasoro Lie algebra $\mathcal{H}$ such that

$$c_L = 2 - 12(\lambda^2 - \mu^2),\ c_{L,I} = \lambda - \mu$$

i.e.

$$\lambda = \frac{2 - c_L}{24 c_{L,I}} + \frac{1}{2}c_{L,I},\ \mu = \frac{2 - c_L}{24 c_{L,I}} - \frac{1}{2}c_{L,I}.$$

Free-field realization

- For every $r, s \in \mathbb{C}$ let $e^{r\alpha + s\beta}$ is a $\mathcal{H}$-singular vector and $U(\mathcal{H})e^{r\alpha+s\beta}$ is a highest weight module with the highest weight (h, h_I) where

$$h = \Delta_{r,s} = \frac{1}{2}r^2 - \frac{1}{2}s^2 - \lambda r + \mu s, \quad h_I = r - s$$

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Proposition

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(ii) *For every $r, s \in \mathbb{C}$ such that $r - s = \lambda - \mu = c_{L,I}$, $e^{r\alpha+s\beta}$ is a highest weight vector of weight*

$$(h, h_I) = (\frac{c_L - 2}{24}, c_{L,I}).$$

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Proposition

Assume that $\frac{h_I}{c_{L,I}} - 1 \notin -\mathbb{Z}_{>0}$. *Then* $\mathcal{F}_{r,s} \cong V(c_L, 0, c_{L,I}, h, h_I)$ *as* $L^{\mathcal{H}}(c_L, c_{L,I})$*-modules.*

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Proposition

Assume that $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$. As a $L^{\mathcal{H}}(c_L, c_{L,I})$–module $\mathcal{F}_{r,s}$ is generated by $e^{r\alpha+s\beta}$ and a family of subsingular vectors $\{v_{n,p} : n \geq 1\}$ of weights $h + np$. There is a filtration $\mathcal{F}_{r,s} = \cup_{n\geq 0} Z_n$ such that

$$Z_n / Z_{n-1} \cong L^{\mathcal{H}}(c_L, 0, c_{L,I}, h + np, h_I).$$

Schur polynomials

- Schur polynomials $S_r(x_1, x_2, \cdots)$ in variables $x_1, x_2, \ldots$ are defined by the following equation:

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n} y^n\right) = \sum_{r=0}^{\infty} S_r(x_1, x_2, \cdots) y^r.$$

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$$S_r(x_1, x_2, \cdots) = \frac{1}{r!} \begin{vmatrix} x_1 & x_2 & \cdots & & x_r \\ -r+1 & x_1 & x_2 & \cdots & x_{r-1} \\ 0 & -r+2 & x_1 & \cdots & x_{r-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & x_1 \end{vmatrix}$$

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- Schur polynomials naturally appear in formulas for vertex operator for lattice vertex algebras.

Schur polynomials and singular vectors

Lemma

If $v \in \mathcal{I} \subset V(c_L, 0, c_{L,I}, h, h_I)$ is such that $\Phi(v) \in \mathcal{F}_{r,s}$ is a non-trivial singular vector, then v is a singular vector in $V(c_L, 0, c_{L,I}, h, h_I)$.

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Since $S_p\left(-\frac{I_{-1}}{c_{L,I}}, -\frac{I_{-2}}{c_{L,I}}, \ldots, -\frac{I_{-p}}{c_{L,I}}\right) e^{r\alpha+s\beta}$ is a singular vector in $U(\mathcal{H})e^{r\alpha+s\beta}$ we have:

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Theorem

Assume that $c_{L,I} \neq 0$ and $p = \frac{h_I}{c_{L,I}} - 1 \in \mathbb{Z}_{>0}$. Then $\Omega v_{h,h_I}$ where

$$\Omega = S_p\left(-\frac{I_{-1}}{c_{L,I}}, -\frac{I_{-2}}{c_{L,I}}, \ldots, -\frac{I_{-p}}{c_{L,I}}\right)$$

is a singular vector of weight p in the Verma module $V(c_L, 0, c_{L,I}, h, (1+p)\,c_{L,I})$.

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- Using technical lemma and some calculation with $e^{r\alpha+s\beta}$ in $\mathcal{F}_{r,s}$ we get:

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Theorem

Assume that $c_{L,I} \neq 0$ *and* $p = 1 - \frac{h_I}{c_{L,I}} \in \mathbb{Z}_{>0}$. *Then* $\Lambda v_{h,h_I}$ *where*

$$\Lambda = \sum_{i=0}^{p-1} S_i\left(\frac{I_{-1}}{c_{L,I}}, \ldots, \frac{I_{-i}}{c_{L,I}}\right) L_{i-p} +$$

$$\sum_{i=0}^{p-1}\left(\frac{h}{p} + \frac{c_L - 2}{24}\frac{(p-1)^2 - pi}{p}\right) S_i\left(\frac{I_{-1}}{c_{L,I}}, \ldots, \frac{I_{-i}}{c_{L,I}}\right)\frac{I_{i-p}}{c_{L,I}}$$

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Intertwining operators and tensor product modules

As with Virasoro and $W(2,2)$ algebras, the existence of a nontrivial intertwining operator of type

$$\binom{L(c_L, 0, c_{L,I}, h'', h_I'')}{L(c_L, 0, c_{L,I}, h, h_I) \quad L(c_L, 0, c_{L,I}, h', h_I')}$$

yields a nontrivial $\mathcal{H}$-homomorphism

$$\varphi : V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h', h_I') \to L(c_L, 0, c_{L,I}, h'', h_I'')$$

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Again, by dimension argument, we get reducibility of $V'_{\alpha,\beta,F} \otimes L(c_L,0,c_{L,I},h',h_I')$.

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- Next we use formulas for Ω and Λ to get irreducibility criterion for $V'_{\alpha,\beta,F} \otimes L\left(c_L, 0, c_{L,I}, h, h_I\right)$.

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for $u \in U\left(\mathcal{H}\_\right)_{-k}$.

- $V'_{\alpha,\beta,F} \otimes L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ is irreducible if and only if $\phi_n(\Omega) \neq 0$ $(\phi_n(\Lambda) \neq 0)$ for every $n \in \mathbb{Z}$.

Irreducibility criterion

- If $p = \frac{h_I}{c_{L,I}} - 1 \in \mathbb{Z}_{>0}$, then for every $n \in \mathbb{Z}$ we have

$$\phi_n(\Omega) = (-1)^p \binom{-\frac{F}{c_{L,I}}}{p}.$$

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- This expands the list of reducible tensor products realized with intertwining operators.

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for a certain polynomial $g_p \in \mathbb{C}[x]$.

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  for a certain polynomial $g_p \in \mathbb{C}[x]$.
- If $F/c_{L,l} \notin \{1, \ldots, p-1\}$, then for every $n \in \mathbb{Z}$ there is a unique $\alpha := \alpha_n \in \mathbb{C}$ such that $\phi_n(\Lambda) = 0$.

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- This, along with previous results on existence of intertwining operators result with the following:

Irreducibiliy criterion

Theorem

Let $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$. *We write* V *short for* $V'_{\alpha,\beta,F} \otimes L(c_L, 0, c_{L,I}, h, h_I)$.

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(i) Let $F/c_{L,I} \notin \{1, \ldots, p-1\}$ and let $\alpha_0 \in \mathbb{C}$ be such that $\phi_0(\Lambda) = 0$. Then V is reducible if and only if $\alpha \equiv \alpha_0 \mod \mathbb{Z}$.

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$$h'' = -\alpha_0 + h + (1 - \beta), \qquad h''_I = F + h_I.$$

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(ii) Let $F/c_{L,I} \in \{2, \dots, p-1\}$. Then V is reducible.

(iii) Let $p > 1$ and $F/c_{L,I} = 1$. Then V is reducible if and only if $1 - \beta = \frac{c_L - 2}{24}$.

Fusion rules

Theorem

Let $(h, h_I) = (\Delta_{r_1,s_1}, r_1 - s_1), (h', h'_I) = (\Delta_{r_2,s_2}, r_2 - s_2)$ *such that*

$$\frac{h_I}{c_{L,I}} - 1 = q, \ \frac{h'_I}{c_{L,I}} - 1 = p, \quad p, q \in \mathbb{Z} \setminus \{0\}.$$

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Then $d = 1$ *if and only if* $h_I'' = h_I + h_I'$ *and one of the following holds:*

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$$d = \dim I \begin{pmatrix} & L^{\mathcal{H}}(c_L, 0, c_{L,I}, h'', h''_I) & \\ L^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I) & & L^{\mathcal{H}}(c_L, 0, c_{L,I}, h', h'_I) \end{pmatrix}.$$

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$d = 0$ otherwise.

Nontrivial intertwining operators

$$\begin{pmatrix} & (\Delta_{r_1+r_2,s_1+s_2}, (1-(p+q-1)c_{L,l}) & \\ (\Delta_{r_1,s_1}, (1-q)c_{L,l}) & & (\Delta_{r_2,s_2}, (1-p)c_{L,l}) \end{pmatrix}$$

for $p, q \geq 1$

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Vertex-algebra homomorphism

- Vertex-algebra $L^W(c_L, c_W)$ is generated by

$$Y(L_{-2}, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \; Y(W_{-2}, z) = \sum_{n\in\mathbb{Z}} W_n z^{-n-2}.$$

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Theorem

There is a non-trivial homomorphism of vertex algebras

$$\begin{aligned} \Psi : L^W(c_L, c_W) &\to L^{\mathcal{H}}(c_L, c_{L,I}) \\ L_{-2} &\mapsto L_{-2}\mathbf{1} \\ W_{-2} &\mapsto (I_{-1}^2 + 2c_{L,I} I_{-2})\mathbf{1} \end{aligned}$$

where

$$c_W = -24c_{L,I}^2.$$

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- Every $L^{\mathcal{H}}(c_L, c_{L,I})$-module becomes a $L^W(c_L, c_W)$-module.

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- $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$ is a $L^W(c_L, c_W)$-module and v_{h,h_I} is a $W(2,2)$ highest weight vector such that

$$L(0)v_{h,h_I} = hv_{h,h_I}, \quad W(0)v_{h,h_I} = h_W v_{h,h_I}$$

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- There is a nontrivial $W(2,2)$-homomorphism

$$\Psi : V^{W(2,2)}(c, c_W, h, h_W) \to V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, h_I)$$

Highest weight H-modules as W(2,2)-modules

Example

Let $h_W = \frac{1-p^2}{24} c_W = (p^2-1)c_{L,I}^2 = h_I(h_I - 2c_{L,I})$ as above. Then there are nontrivial $W(2,2)$-homomorphisms

$$V^{W(2,2)}(c, c_W, h, \tfrac{1-p^2}{24} c_W)$$

$$\Psi_+ \swarrow \qquad \searrow \Psi_-$$

$$V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, (1+p)\, c_{L,I}) \qquad V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, (1-p)\, c_{L,I})$$

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Theorem

(i) *Let* $\frac{h_I}{c_{L,I}} - 1 \notin -\mathbb{Z}_{>0}$. *Then* Ψ *is an isomorphism of* $W(2,2)$*-modules.*

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$$\Psi^{-1}\left(S_p\left(-\frac{I(-1)}{c_{L,I}}, -\frac{I(-2)}{c_{L,I}}, \cdots\right) v_{h,h_I}\right) = u'$$

is a singular vector in $V^{W(2,2)}(c_L, c_W, h, h_W)_{h+p}$.

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(iii) *If* $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ *then* $\Psi(u') = 0$.

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(iv) *Let* $\frac{h_I}{c_{L,I}} - 1 = -p \in -\mathbb{Z}_{>0}$ *and let* u *be a subsingular vector in* $V^{W(2,2)}(c_L, c_W, h_{pq}, h_W)_{h+pq}$. *Then* $\Psi(u)$ *is a singular vector in* $V^{\mathcal{H}}(c_L, 0, c_{L,I}, h, (1-p)c_{L,I})$.

The End

THANK YOU!

The End

THANK YOU!

...if you're still awake... :)