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Regularity of certain vertex operator superalgebras

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ABSTRACT. We present our results on representation theory of certain vertex operator superalgebras. In particular, we consider vertex operator superalgebras associated to the minimal models for the Neveu-Schwarz algebra and the $N = 2$ superconformal algebra. We present the results on the classification of irreducible representation, regularity, rationality and fusion rules for these vertex operator superalgebras. The connections with the theory of affine Kac-Moody and Virasoro vertex operator algebras will be also discussed.

1. Introduction

The main problems in the theory of vertex operator (super)algebras are related to the constructions and classification of rational vertex operator (super)algebras. The rationality of certain well-known examples of vertex operator (super)algebras was proved in papers [**D**], [**DL**], [**Li1**], [**FZ**], [**Wn**], [**A1**], [**A2**], [**A3**].

In [**DLM**] was introduced the notion of regular vertex operator algebra, i.e., rational vertex operator algebra with the property that every weak module is completely reducible. So every regular vertex operator algebra has finitely many irreducible modules, and every V–module is completely reducible. Further development in theory of regular vertex operator algebras was made by H. Li in [**Li2**] by proving that every regular vertex operator algebra satisfies Zhu's C_2–finiteness condition and that the fusion rules for irreducible modules are finite. Zhu's C_2–finiteness condition is important for studying modular invariance of characters of vertex operator algebras (see [**Z**]).

The paper [**DLM**] also gave the proof of regularity of lattice vertex operator algebras, vertex operator algebras associated to integrable representations of affine Kac-Moody Lie algebras, and vertex operator algebras associated to minimal models for the Virasoro algebra.

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In this paper, as one of the main results, we will demonstrate the regularity of vertex operator superalgebras associated to minimal models for the Neveu-Schwarz algebra and for the $N = 2$ superconformal algebra.

Let us describe these results in more detail. Let $L^{Vir}(d_{p,q}, 0)$ be a minimal Virasoro vertex operator algebra, and $L^{\mathbf{ns}}(c_{p,q}, 0)$ a minimal Neveu-Schwarz vertex operator superalgebra. Let M^1 be the vertex operator superalgebra $L^{Vir}(\frac{1}{2}, 0) \oplus L^{Vir}(\frac{1}{2}, \frac{1}{2})$(cf. [**FRW**]). Then we show that the vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0)$ contains the minimal Virasoro vertex operator algebra $L^{Vir}(d_{p_1,q_1}, 0) \otimes L^{Vir}(d_{p_2,q_2}, 0)$ as a subalgebra, and find the decomposition of $M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0)$ as a $L^{Vir}(d_{p_1,q_1}, 0) \otimes L^{Vir}(d_{p_2,q_2}, 0)$–module. This fact implies that many results obtained earlier in the framework of Virasoro vertex operator algebras can be used in the study of Neveu-Schwarz vertex operator superalgebra. As a particular application, we prove that the minimal Neveu-Schwarz vertex operator superalgebras are regular. This result is new.

Next we consider vertex operator superalgebras associated to the minimal models for the N=2 superconformal algebra. These vertex operator superalgebras can be constructed using the Kazama-Suzuki mapping (cf. [**KS**], [**FST**], [**A2**]). Let $m, k \in \mathbb{Z}_{>0}$. Let $L(m, 0)$ be the simple vertex operator algebra associated to the affine Lie algebra $\hat{sl}_2$–module of level m, and F_k the vertex operator (super)algebra F_k associated to the lattice $\sqrt{k}\mathbb{Z}$. Set $c_m = \frac{3m}{m+2}$. Then the simple $N = 2$ vertex operator superalgebra L_{c_m} can be realized as a subalgebra of the vertex operator superalgebra $L(m, 0) \otimes F_1$. We present the results on the classification of irreducible L_{c_m}–modules, the regularity and the fusion rules from [**A2**] and [**A3**].

The construction of the $N = 2$ vertex operator superalgebra L_{c_m} motivates us to investigate a larger family of vertex operator superalgebras $D_{m,k}$, $k \in \mathbb{Z}_{>0}$, such that $D_{m,1} \cong L_{c_m}$. These vertex operator superalgebras are realized as subalgebras of the vertex operator superalgebra $L(m, 0) \otimes F_k$. We present the regularity result from [**A4**].

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2. Preliminaries

In this section we recall the definition of vertex operator superalgebras, their modules and intertwining operators (cf. [**FHL**], [**FLM**], [**K**], [**KWn**], [**Li1**], [**P**]).

Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be any $\mathbb{Z}_2$–graded vector space. Then any element $u \in V^{\bar{0}}$ (resp. $u \in V^{\bar{1}}$) is said to be even (resp. odd). We define $|u| = 0$ if u is even and $|u| = 1$ if u is odd. Elements in $V^{\bar{0}}$ or $V^{\bar{1}}$ are called homogeneous. Whenever $|u|$ is written, it is understood that u is homogeneous.

DEFINITION 2.1. *A vertex superalgebra is a quadruple $(V, Y, \mathbf{1}, D)$ where V is a $\mathbb{Z}_2$-graded vector space, D is an endomorphism of V, $\mathbf{1}$ is a specified element called the vacuum of V, and Y is a linear map*

$$\begin{aligned} Y(\cdot, z): \quad & V \to (End\ V)[[z, z^{-1}]]; \\ & a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (End\ V)[[z, z^{-1}]] \end{aligned}$$

satisfying the following conditions for $a, b \in V$:

(V1) $|a_n b| = |a| + |b|$.

(V2) $a_n b = 0$ *for n sufficiently large.*

(V3) $[D, Y(a, z)] = Y(D(a), z) = \frac{d}{dz} Y(a, z)$.

(V4) $Y(\mathbf{1}, z) = I_V$ *(the identity operator on V).*

(V5) $Y(a, z)\mathbf{1} \in (EndV)[[z]]$ *and* $\lim_{z \to 0} Y(a, z)\mathbf{1} = a$.

(V6) *The following Jacobi identity holds*

$$\begin{aligned} & z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1) Y(b, z_2) \\ & \quad -(-1)^{|a||b|} z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(b, z_2) Y(a, z_1) \\ & \quad = z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0) b, z_2). \end{aligned}$$

A vertex superalgebra V is called a vertex operator superalgebra if there is a special element $\omega \in V$ (called the Virasoro element*) whose vertex operator we write in the form $Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_n z^{-n-1} = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, such that*

(V7) $[L(m), L(n)] = (m - n) L(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} c, \quad c = rank\ V \in \mathbb{C}$.

(V8) $L(-1) = D$.

(V9) $V = \oplus_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}$ *is a $\frac{1}{2}\mathbb{Z}$-graded so that* $V^{\bar{0}} = \oplus_{n \in \mathbb{Z}} V_{(n)}$, $V^{\bar{1}} = \oplus_{n \in \frac{1}{2} + \mathbb{Z}} V_{(n)}$, $L(0)\ |_{V(n)} = nI\ |_{V_{(n)}}$, $\dim V_{(n)} < \infty$, *and* $V_{(n)} = 0$ *for n sufficiently small.*

REMARK 2.1. If in the definition of vertex (operator) superalgebra the odd subspace $V^{\bar{1}} = 0$ we get the usual definition of vertex (operator) algebra.

We will say that the vertex operator superalgebra V is generated by the set S if

$$V = \mathrm{span}_{\mathbb{C}} \{ u^1_{n_1} \cdots u^r_{n_r} \mathbf{1} \mid u^1, \ldots, u^r \in S,\ n_1, \ldots, n_r \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0} \}.$$

A subspace $I \subset V$ is called an ideal in the vertex operator superalgebra V if $\mathbf{1}, \omega \notin I$, and

$$a_n I \subset I, \text{ for every } a \in V, n \in \mathbb{Z}.$$

A vertex operator superalgebra V is called simple if V doesn't contain any proper nontrivial ideal.

DEFINITION 2.2. *Let V be a vertex operator superalgebra. A weak V–module is a pair (M, Y_M), where $M = M^{\bar{0}} \oplus M^{\bar{1}}$ is a $\mathbb{Z}_2$–graded vector space, and $Y_M(\cdot, z)$ is a linear map*

$$Y_M : V \to End(M)[[z, z^{-1}]], \ a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

satisfying the following conditions for $a, b \in V$ and $v \in M$:

(M1) $|a_n v| = |a| + |v|$ *for any* $a \in V$.
(M2) $Y_M(\mathbf{1}, z) = I_M$.
(M3) $Y_M(L(-1)a, z) = \frac{d}{dz} Y_M(a, z)$.
(M4) $a_n v = 0$ *for n sufficiently large.*
(M5) *The following Jacobi identity holds*

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) & Y_M(a, z_1) Y_M(b, z_2) \\ & -(-1)^{|a||b|} z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b, z_2) Y_M(a, z_1) \\ & = z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(a, z_0)b, z_2). \end{aligned}$$

A weak V–module (M, Y_M) is called a V–module if

(M6) $M = \coprod_{n \in \mathbb{C}} M(n)$;
(M7) $L(0)u = nu$, $u \in M(n)$; $\dim M(n) < \infty$;
(M8) $M(n) = 0$ *for n sufficiently small.*

Let M_i $(i = 1, 2, 3)$ be V–modules. Then an intertwining operator of type $\left(\begin{array}{c} M_3 \\ M_1 M_2 \end{array} \right)$ is defined [**FHL**] to be a linear map $I(\cdot, z)$ from M_1 to $\mathrm{Hom}_{\mathbb{C}}(M_2, M_3)\{z\}$ such that $I(L(-1)u, z) = \frac{d}{dz} I(u, z)$ for $u \in M_1$ and a suitably adjusted Jacobi identity holds. Denote by $I_V \left(\begin{array}{c} M_3 \\ M_1 M_2 \end{array} \right)$ the space of all intertwining operators of the indicated type. The dimension of this vector space is called the *fusion rule* of this type.

EXAMPLE 2.1. Let $\mathfrak{g}$ be the Lie algebra sl_2 with generators e, f, h and relations $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the corresponding affine Lie algebra of type $A_1^{(1)}$. As usual we write $x(n)$ for $x \otimes t^n$ where $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Let Λ_0, Λ_1 denote the fundamental weights for $\hat{\mathfrak{g}}$. For any complex numbers m, j, let $L(m, j) = L((m-j)\Lambda_0 + j\Lambda_1)$ be the irreducible highest weight $\hat{sl_2}$–module with the highest weight $(m - j)\Lambda_0 + j\Lambda_1$. Then $L(m, 0)$ has a natural structure of a simple vertex operator algebra. Let $\mathbf{1}_m$ denote the vacuum vector in $L(m, 0)$. Then $L(m, 0)$ is generated by $e(-1)\mathbf{1}_m$, $f(-1)\mathbf{1}_m$ and $h(-1)\mathbf{1}_m$.

If $m \in \mathbb{Z}_{>0}$, then $L(m, 0)$ has $m + 1$ irreducible modules $L(m, j)$, $j = 0, \dots, m$. Let $0 \le j_1, j_2, j_3 \le m$. Then (cf. [**FZ**])

$$\dim I_{L(m,0)} \left(\begin{array}{c} L(m, j_3) \\ L(m, j_1) L(m, j_2) \end{array} \right) = 1$$

if $j_1 + j_2 + j_3 \in 2\mathbb{Z}$ and

$$|j_1 - j_2| \leq j_3 \leq \min\{j_1 + j_2, 2m - j_1 - j_2\},$$

and zero otherwise.

We recall the definition of regular vertex operator algebra introduced by C. Dong, H. Li and G. Mason in [**DLM**].

DEFINITION 2.3. *The vertex operator superalgebra V is called regular if every weak V–module is completely reducible.*

If vertex operator superalgebra V is regular, then V is also a rational vertex operator superalgebra, meaning that V has only finitely many irreducible modules and that every V–module is completely reducible. It was also conjectured in [**DLM**] that every rational vertex operator algebra is regular.

Further development in the theory of regular vertex operator algebras was made by H. Li in [**Li2**] by proving that every regular vertex operator algebra satisfies Zhu's C_2–finiteness condition and that the fusion rules for irreducible modules are finite. Familiar examples of regular vertex operator algebras are vertex operator algebras associated to the (vacuum) integrable highest weight representations of affine Lie algebras of positive integer level m (Wess-Zumino-Novikov-Witten models), vertex operator (super)algebra V_L associated to the positive definite lattice L, vertex operator algebras associated to minimal models for the Virasoro algebra. In particular, the Moonshine module is a regular vertex operator algebra.

3. Virasoro vertex operator algebras

In this section we recall some results on vertex operator algebras associated to minimal models for Virasoro algebra. Rationality of these vertex operator algebras was proved by W. Wang in [**Wn**], and regularity by C. Dong, H. Li and G. Mason in [**DLM**].

Let $Vir = \oplus_{n\in\mathbb{Z}}\mathbb{C}L(n) \oplus \mathbb{C}C$ be the Virasoro algebra. For any $(c,h) \in \mathbb{C}^2$ let $L^{Vir}(c,h)$ be the irreducible highest weight Vir–module with central charge c and highest weight h (cf. [**FZ**], [**Wn**], [**DMZ**], [**H**]). Then $L^{Vir}(c,0)$ is a simple vertex operator algebra. Set

$$d_{p,q} = 1 - 6\frac{(p-q)^2}{pq}, \quad k_{p,q}^{m,n} = \frac{(np-mq)^2 - (p-q)^2}{4pq}.$$

Whenever we mention $d_{p,q}$ again, we always assume that p and q are relatively prime positive integers larger than 1. Define

$$S_{p,q}^{Vir} = \{k_{p,q}^{m,n} \mid 0 < m < p,\ 0 < n < q\}.$$

THEOREM 3.1. [**Wn**], [**DLM**] *The vertex operator algebra $L^{Vir}(d_{p,q},0)$ is regular, and the set*

$$\{L^{Vir}(d_{p,q},h) \mid h \in S_{p,q}^{Vir}\}$$

provides all irreducible $L^{Vir}(d_{p,q},0)$*–modules.*

$L^{Vir}(d_{p_1,q_1},0)\otimes\cdots\otimes L^{Vir}(d_{p_n,q_n},0)$ *is also a regular vertex operator algebra.*

In this paper we shall use the properties of vertex operator superalgebra $M^1 = L^{Vir}(\frac{1}{2},0)\oplus L^{Vir}(\frac{1}{2},\frac{1}{2})$, which can be constructed using a neutral fermion. So let us first recall the fermionic construction of the vertex operator superalgebra M^N (see [**FFR**], [**KWn**] and [**Li1**]).

Let N be a positive integer. The Clifford algebra CL_N is an associative algebra generated by $\{e^i(n),\ i=1,\dots,N; n\in Z\}\cup\{1\}$ and relations

$$\{e^i(n),e^j(m)\} = 2\delta_{i,j}\delta_{n,-m},\quad n,m\in\tfrac{1}{2}+\mathbb{Z},\ i,j\in\{1,\dots,N\}.$$

Let M^N be the CL_N–module generated by the vector $\mathbf{1}$ such that

$$e^i(n)\mathbf{1}=0,\ n>0.$$

Then the fields

$$Y(e^i(-\tfrac{1}{2})\mathbf{1},z)=e^i(z)=\sum_{n\in\frac{1}{2}+\mathbb{Z}} e^i(n)z^{-n-\frac{1}{2}},\ (i=1,\dots,N)$$

generate the unique vertex operator superalgebra structure on M^N. The vector

$$\omega^{(N)}=\frac{1}{4}\sum_{i=1}^{N} e^i(-\tfrac{3}{2})e^i(-\tfrac{1}{2})\mathbf{1}$$

is a Virasoro element of central charge $\frac{N}{2}$. Moreover, M^N is a rational vertex operator superalgebra, and M^N is the unique irreducible M^N–module (see [**FFR**], [**KWn**], [**Li1**]).

REMARK 3.1. Let $L_{(N)}$ be the following lattice $L_{(N)}=\sum_{i=1}^N \mathbb{Z}x^i,\quad \langle x^i,x^j\rangle = \delta_{i,j}$. Using a boson fermion correspondence, one can see that the lattice vertex operator superalgebra $V_{L_{(N)}}$ is isomorphic to the fermionic vertex operator superalgebra M^{2N} (see [**Fr**], [**Fe**], [**K**], [**T**]).

We will now consider some special cases. Let $N=1$. Then the vertex operator superalgebra M^1 is isomorphic to the Virasoro vertex operator superalgebra $L^{Vir}(\frac{1}{2},0)\oplus L^{Vir}(\frac{1}{2},\frac{1}{2})$ (cf. [**FRW**]).

Let $N=3$. The vertex operator superalgebra M^3 is isomorphic to the $\hat{sl}_2$ vertex operator superalgebra $L(2,0)\oplus L(2,2)$ (cf. [**GKO**], [**KWn**]).

Finally, let $N=4$. Clearly $M^4\cong M^1\otimes M^3$. Using Remark 3.1, we see that $M^4\cong V_{L_{(2)}}$. Set $L=L_{(2)}$. For $i\in\{0,1\}$, let

$$L^i=\{w\in L \mid \langle w,w\rangle \in i+2\mathbb{Z}\}.$$

Then we have $V_L \cong V_{L^0} \oplus V_{L^1}$, and V_{L^0} is a vertex operator algebra, which is a subalgebra of V_L, and V_{L^1} is a V_{L^0}–module. In our case we have that

$$L^0 = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2,\ L^1 = \frac{\alpha_1+\alpha_2}{2} + L^0,\ \text{where } \alpha_1 = x^1 + x^2,\ \alpha_2 = x^1 - x^2.$$

Now the results from Chapter 13 of [**DL**] imply that V_{L^0} is isomorphic to the vertex operator algebra associated to the $\hat{sl}_2$–module $L(1,0)\otimes L(1,0)$, and $V_{L^1} \cong L(1,1)\otimes L(1,1)$. Thus, we have obtained the following lemma.

LEMMA 3.1. *On the $L(1,0)\otimes L(1,0)$–module $L(1,0)\otimes L(1,0) \bigoplus L(1,1)\otimes L(1,1)$, there exists a natural structure of a vertex operator superalgebra isomorphic to M^4.*

4. Neveu-Schwarz vertex operator superalgebras

In this section we recall the result on vertex operator superalgebras associated to the minimal models for the Neveu-Schwarz algebra from [**A1**].

The Neveu-Schwarz algebra is the Lie superalgebra

$$\mathbf{ns} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L(n) \bigoplus \bigoplus_{m\in\frac{1}{2}+\mathbb{Z}} \mathbb{C}G(m) \bigoplus \mathbb{C}C$$

with commutation relations $(m,n\in\mathbb{Z})$:

$$\begin{aligned}
&[L(m),L(n)] = (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}C,\\
&[G(m+\tfrac{1}{2}),L(n)] = (m+\tfrac{1}{2}-\tfrac{n}{2})G(m+n+\tfrac{1}{2}),\\
&\{G(m+\tfrac{1}{2}),G(n-\tfrac{1}{2})\} = 2L(m+n) + \tfrac{1}{3}m(m+1)\delta_{m+n,0}C,\\
&[L(m),C] = 0,\quad [G(m+\tfrac{1}{2}),C] = 0.
\end{aligned}$$

For any $(c,h)\in\mathbb{C}^2$ let $L^{\mathbf{ns}}(c,h)$ be the corresponding irreducible highest weight $\mathbf{ns}$–module with central charge c and highest weight h (cf. [**KWn**], [**Li1**], [**A1**]). $L^{\mathbf{ns}}(c,0)$ is a simple vertex operator superalgebra.

Set

$$\begin{aligned}
c_{p,q} &= \frac{3}{2}(1-\frac{2(p-q)^2}{pq}),\\
h^{r,s}_{p,q} &= \frac{(sp-rq)^2-(p-q)^2}{8pq}.
\end{aligned}$$

Whenever we mention $c_{p,q}$ again, we always assume that $p,q\in\{2,3,4,\cdots\}, p-q\in 2\mathbb{Z}$, and that $(p-q)/2$ and q are relatively prime to each other. Set

$$S^{\mathbf{ns}}_{p,q} = \{h^{r,s}_{p,q} \mid 0<r<p, 0<s<q, r-s\in 2\mathbb{Z}\}.$$

THEOREM 4.1. [**A1**] *The vertex operator superalgebra $L^{\mathbf{ns}}(c_{p,q},0)$ is rational. The set*

$$\{L^{\mathbf{ns}}(c_{p,q},h),\ h\in S^{\mathbf{ns}}_{p,q}\}$$

provides all irreducible modules for the vertex operator superalgebra $L^{\mathbf{ns}}(c_{p,q},0)$.

REMARK 4.1. The rationality of minimal Neveu-Schwarz vertex operator superalgebras was conjectured by V. Kac and W. Wang in [**KWn**], and their conjecture was proved in [**A1**].

The results from [**A1**] and [**KWn**] give that Zhu's algebra (cf. [**Z**]) is

$$A(L^{\mathbf{ns}}(c_{p,q},0)) \cong \frac{\mathbb{C}[x]}{\langle \prod_{h\in S^{\mathbf{ns}}_{p,q}}(x-h)\rangle}.$$

In particular, it is semisimple and finite-dimensional.

5. Connections between Neveu-Schwarz and Virasoro vertex operator superalgebras

In this section, we shall investigate the vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$. This structure from the physics point of view was studied in [**La**]. We will show that $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ contains a minimal Virasoro algebra $L^{Vir}(d_{p_1,q_1},0) \otimes L^{Vir}(d_{p_2,q_2},0)$ as a subalgebra. We shall also discuss some applications of this result. In particular, we will prove the regularity of the vertex operator superalgebras $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ and $L^{\mathbf{ns}}(c_{p,q},0)$

Let Y be the vertex operator defining the vertex operator superalgebra structure on $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$.

Define

$$p_1 = \frac{p+q}{2},\ q_1 = q; \quad p_2 = p,\ q_2 = \frac{p+q}{2}. \tag{5.1}$$

Set $m = 2\frac{2q-p}{p-q}$, and $d_m = 1 - \frac{6}{(m+2)(m+3)}$. Then $d_m = d_{p_1,q_1}$ and $d_{m+1} = d_{p_2,q_2}$.

Let $\omega^s_{(m)}$ be the Virasoro element in $L^{\mathbf{ns}}(c_{p,q},0)$, and let $\omega^{(1)}$ be the Virasoro element in M^1. Motivated by formulae (16) in [**La**], we define the vectors $\omega_{(m)}$, $\omega_{(m+1)}$ as follows:

$$\begin{aligned}
\omega_{(m)} = \quad & \frac{1}{2}\frac{m}{m+3}\omega^{(1)} \otimes \mathbf{1} + \frac{1}{2}\frac{m+4}{m+3}\mathbf{1} \otimes w^s_{(m)} \\
& + \frac{\sqrt{(m+2)(m+4)}}{\sqrt{8}(m+3)} e^1(-\frac{1}{2})\mathbf{1} \otimes G(-\frac{3}{2})\mathbf{1}, \\
\omega_{(m+1)} = \quad & \frac{1}{2}\frac{m+6}{m+3}\omega^{(1)} \otimes \mathbf{1} + \frac{1}{2}\frac{m+2}{m+3}\mathbf{1} \otimes \omega^s_{(m)} \\
& - \frac{\sqrt{(m+2)(m+4)}}{\sqrt{8}(m+3)} e^1(-\frac{1}{2})\mathbf{1} \otimes G(-\frac{3}{2})\mathbf{1}.
\end{aligned}$$

The following proposition can be proved by a straightforward calculation.

PROPOSITION 5.1. *We have:*

(1) $\omega_{(m)}$ *is a Virasoro element in the vertex operator superalgebra* $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ *and the components of the fields* $Y(\omega_{(m)},z)$ *provide a representation of the Virasoro algebra with central charge* d_m.

(2) $\omega_{(m+1)}$ *is a Virasoro element in the vertex operator superalgebra* $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ *and the components of the fields* $Y(\omega_{(m+1)},z)$ *provide a representation of the Virasoro algebra with central charge* d_{m+1}.

(3) *The Virasoro actions in (1) and (2) commute.*
(4) $\mathbf{1} \otimes \omega^s_{(m)} + \omega^{(1)} \otimes \mathbf{1} = \omega_{(m)} + \omega_{(m+1)}$.

The next result gives that the subalgebra of the vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0)$ generated by the vectors $\omega_{(m)}$ and $\omega_{(m+1)}$ is isomorphic to the minimal Virasoro vertex operator algebra $L^{Vir}(d_{p_1,q_1}, 0) \otimes L^{Vir}(d_{p_2,q_2}, 0)$.

THEOREM 5.1.
(i) *The vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0)$ has a vertex subalgebra (with the same Virasoro element) isomorphic to the vertex operator algebra $L^{Vir}(d_{p_1,q_1}, 0) \otimes L^{Vir}(d_{p_2,q_2}, 0)$.*
(ii) *As a $L^{Vir}(d_{p_1,q_1}, 0) \otimes L^{Vir}(d_{p_2.q_2}, 0)$-module $M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0)$ has the following decomposition:*

$$M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0) \cong \bigoplus_{0 \le n' \le p_1 - 2} L^{Vir}(d_{p_1,q_1}, k^{n'+1,1}_{p_1,q_1}) \otimes L^{Vir}(d_{p_2,q_2}, k^{1,n'+1}_{p_2,q_2}).$$

Proof of Theorem 5.1 will be given in Section 6.

REMARK 5.1. The vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0)$ has the same representation theory (= classification of irreducible modules, semisimplicity of certain categories of representations and tensor product theory) as the vertex operator superalgebra $L^{\mathbf{ns}}(c_{p,q}, 0)$.

REMARK 5.2. It was shown in [**H**], [**HM**] that the irreducible representations of $L^{Vir}(d_{p,q}, 0)$ and $L^{\mathbf{ns}}(c_{p,q}, 0)$ can be organized in a general structure called intertwining operator (super)algebra.

Theorem 5.1 has the following generalization.

COROLLARY 5.1. *Let n be a positive integer, and (p^i, q^i), $i = 1, \ldots, n$, n pairs of integers larger than 1 such that $p^i - q^i \in 2\mathbb{Z}$, and $\frac{p^i - q^i}{2}$ and q^i are relatively prime to each other. Define*

$$p^i_1 = \frac{p^i + q^i}{2},\ q^i_1 = q^i; \quad p^i_2 = p^i,\ q^i_2 = \frac{p^i + q^i}{2}.$$

Then the vertex operator superalgebra

$$M^n \otimes L^{\mathbf{ns}}(c_{p^1,q^1}, 0) \otimes \cdots \otimes L^{\mathbf{ns}}(c_{p^n,q^n}, 0)$$

has a vertex subalgebra isomorphic to

$$L^{Vir}(d_{p^1_1,q^1_1}, 0) \otimes L^{Vir}(d_{p^1_2,q^1_2}, 0) \otimes \cdots \otimes L^{Vir}(d_{p^n_1,q^n_1}, 0) \otimes L^{Vir}(d_{p^n_2,q^n_2}, 0).$$

In [**A1**], we proved that the vertex operator superalgebra $L^{\mathbf{ns}}(c_{p,q}, 0)$ is rational. Since M^1 is a holomorphic vertex operator superalgebra, we conclude that the vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0)$ is also rational (see also Lemma 5.2.6. in [**Li1**]). Moreover, Theorem 5.1 gives that $M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, 0)$ contains a subalgebra isomorphic to the minimal Virasoro vertex operator algebra

$L^{Vir}(d_{p_1,q_1},0) \otimes L^{Vir}(d_{p_2,q_2},0)$. So $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ is a rational vertex operator superalgebra containing a regular subalgebra. Now applying the theory from [**DLM**] to our case, we get the following regularity result.

LEMMA 5.1. *The tensor product vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ is regular, i.e., every weak $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$–module is completely reducible.*

THEOREM 5.2.
(a) *The vertex operator superalgebra $L^{\mathbf{ns}}(c_{p,q},0)$ is regular.*
(b) *The vertex operator superalgebra $L^{\mathbf{ns}}(c_{p^1,q^1},0) \otimes \cdots \otimes L^{\mathbf{ns}}(c_{p^n,q^n},0)$ is regular.*

Proof. First we notice that every irreducible $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$–module has the form $M^1 \otimes N$, where N is an irreducible $L^{\mathbf{ns}}(c_{p,q},0)$–module. Assume that W is any weak $L^{\mathbf{ns}}(c_{p,q},0)$–module. Then $M^1 \otimes W$ is a weak $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$–module, and regularity of $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ (see Lemma 5.1) gives that

$$M^1 \otimes W \cong \bigoplus_i M^1 \otimes W_i$$

where W_i is an irreducible $L^{\mathbf{ns}}(c_{p,q},0)$–module. Let $L(-1)$ be the element of the Virasoro algebra in the vertex operator superalgebra M^1. Since $\ker_{L(-1)} M^1 = \mathbb{C}\mathbf{1}$ we have that

$$\ker_{L(-1)\otimes Id}(M^1 \otimes W) \cong W \quad \text{and} \quad \ker_{L(-1)\otimes Id}(M^1 \otimes W_i) \cong W_i.$$

This immediately implies that

$$W \cong \bigoplus_i W_i,$$

and we have that W is a completely reducible $L^{\mathbf{ns}}(c_{p,q},0)$–module. This proves (a). The proof of (b) is now standard (cf. [**DLM**]). □

6. Proof of Theorem 5.1

In order to prove Theorem 5.1 we shall study certain coset constructions for representations of the vertex operator algebra $L(m,0)$ associated to the affine Kac-Moody Lie algebra $\hat{\mathfrak{g}} = \hat{sl}_2$.

DEFINITION 6.1. *A rational number $m = t/u$ is called admissible if $u \in \mathbb{Z}_{>0}$, $t \in \mathbb{Z}$, $(t,u) = 1$ and $2u + t - 2 \geq 0$.*

Let $m = t/u \in \mathbb{Q}$ be admissible, and let

$$\begin{aligned} P^m = \{\lambda_{m,k,n} = (m - n + k(m+2))\Lambda_0 + (n - k(m+2))\Lambda_1, \\ k, n \in \mathbb{Z}_{\geq 0},\ n \leq 2u + t - 2,\ k \leq u - 1\}. \end{aligned}$$

The modules $L(\lambda)$, $\lambda \in P^m$ are all modular invariant modules for affine Lie algebra $\hat{sl}_2$ of a level m (cf. [**KW**]).

When m is admissible then $\hat{sl}_2$–module $L(m\Lambda_0) = L(m,0)$ carries a structure of a vertex operator algebra. The classification of irreducible $L(m,0)$–modules was given in [**AM**]. It was proved that the set $\{L(\lambda) \mid \lambda \in P^m\}$ provides all irreducible

$L(m,0)$–modules from the category $\mathcal{O}$. So the admissible representations of level m for $\hat{\mathfrak{g}}$ can be identified with the irreducible $L(m,0)$–modules in the category $\mathcal{O}$.

Let $m = \frac{t}{u} \in \mathbb{Q}$ be admissible. Set $p = t + 4u$, $\quad q = t + 2u$, and define p_1, q_1, p_2, q_2 by (5.1). Then $d_m = d_{p_1,q_1}$, $d_{m+1} = d_{p_2,q_2}$ and $c_m = \frac{3}{2} - \frac{12}{(m+2)(m+4)} = c_{p,q}$.

Recall that the vertex operator superalgebra M^3 is isomorphic to the $\hat{\mathfrak{g}}$–module $L(2,0) \oplus L(2,2)$. Now using the classification of irreducible $L(m,0)$–modules, the GKO construction [**GKO**], and the results from [**KW**] we get the following theorem.

THEOREM 6.1.
(1) *The $\hat{\mathfrak{g}}$–module $L(\Lambda_i) \otimes L(\lambda_{m,n,k})$, $i = 0,1$, is a module for the vertex operator algebra $L^{Vir}(d_{p_1,q_1},0) \otimes L(m+1,0)$, and the following decomposition holds:*

$$L(\Lambda_i) \otimes L(\lambda_{m,n,k}) \cong \bigoplus_{\substack{0 \le n' \le p_1 - 2 \\ n' \equiv n+i \ mod\ 2}} L^{Vir}(d_{p_1,q_1}, k_{p_1,q_1}^{n'+1,n+1}) \otimes L(\lambda_{m+1,k,n'}).$$

(2) *The $\hat{\mathfrak{g}}$–module $M^3 \otimes L(\lambda_{m,n,k})$ is a module for the vertex operator superalgebra $L^{\mathbf{ns}}(c_{p,q},0) \otimes L(m+2,0)$, and the following decomposition holds:*

$$M^3 \otimes L(\lambda_{m,n,k}) \cong \bigoplus_{\substack{0 \le n' \le p - 2 \\ n' \equiv n \ mod\ 2}} L^{\mathbf{ns}}(c_{p,q}, h_{p,q}^{n'+1,n+1}) \otimes L(\lambda_{m+2,k,n'}).$$

REMARK 6.1. The second statement in Theorem 6.1 appeared in [**A1**], and it was used in a proof of rationality of the vertex operator superalgebra $L^{\mathbf{ns}}(c_{p,q},0)$. A similar version of the first statement was used in [**DMZ**] for a proof of rationality of the vertex operator algebra $L^{Vir}(d_{p,q},0)$ in the case $p = m+3$, $q = m+2$, $m \in \mathbb{Z}_{>0}$.

Define now the $\hat{\mathfrak{g}}$–module

$$D(m, \lambda_{m,k,n}) = M^4 \otimes L(\lambda_{m,k,n}).$$

Then $D(m, \lambda_{m,k,n})$ is an (irreducible) module for the vertex operator superalgebra $D(m) = M^4 \otimes L(m,0)$.

We shall now identify two important subalgebras in the vertex operator superalgebra $D(m)$.

LEMMA 6.1. *The vertex operator algebra $V_0 = L^{Vir}(d_{p_1,q_1},0) \otimes L^{Vir}(d_{p_2,q_2},0) \otimes L(m+2,0)$ and the vertex operator superalgebra $W_0 = M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0) \otimes L(m+2,0)$ are isomorphic to certain subalgebras of $D(m)$.*

Proof. First we notice that the vertex operator superalgebra $D(m)$ has a subalgebra $V_2 \cong L(1,0) \otimes L(1,0) \otimes L(m,0)$. But, Theorem 6.1 (1) implies that V_2 has a subalgebra $V_1 \cong L^{Vir}(d_{p_1,q_1},0) \otimes L(1,0) \otimes L(m+1,0)$. Applying again Theorem 6.1 (1), we find a subalgebra $V_0 \cong L^{Vir}(d_{p_1,q_1},0) \otimes L^{Vir}(d_{p_2,q_2},0) \otimes L(m+2,0)$ inside the subalgebra V_1.

On the other hand, Theorem 6.1 (2) implies that $M^3 \otimes L(m,0)$ has a subalgebra $W_1 \cong L^{\mathbf{ns}}(c_{p,q},0) \otimes L(m+2,0)$, which gives that $W_0 \cong M^1 \otimes W_1$ is a subalgebra of $D(m)$ isomorphic to $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0) \otimes L(m+2,0)$. □

Theorem 6.1 and Lemma 6.1 imply the following result.

LEMMA 6.2.

(1) *$D(m,\lambda_{m,k,n})$ is a module for the vertex operator algebra $L^{Vir}(d_{p_1,q_1},0) \otimes L^{Vir}(d_{p_2,q_2},0) \otimes L(m+2,0)$, and the following decomposition holds:*

$$D(m,\lambda_{m,k,n}) \cong \bigoplus_{\substack{0 \le n'' \le p_2-2 \\ n'' \equiv n \bmod 2}} \bigoplus_{0 \le n' \le p_1-2}$$

$$L^{Vir}(d_{p_1,q_1}, k_{p_1,q_1}^{n'+1,n+1}) \otimes L^{Vir}(d_{p_2,q_2}, k_{p_2,q_2}^{n''+1,n'+1}) \otimes L(\lambda_{m+2,k,n''}).$$

(2) *$D(m,\lambda_{m,k,n})$ is a module for the vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0) \otimes L(m+2,0)$, which is isomorphic to*

$$\bigoplus_{\substack{0 \le n'' \le p-2 \\ n'' \equiv n \bmod 2}} M^1 \otimes L^{\mathbf{ns}}(c_{p,q}, h_{p,q}^{n''+1,n+1}) \otimes L(\lambda_{m+2,k,n''}).$$

Now we shall finish the proof of Theorem 5.1. Let

$$W(m) = \{v \in D(m) \mid (\mathfrak{g} \otimes t^n)v = 0 \text{ for } n \ge 0\}.$$

Then $W(m)$ is a subalgebra of the vertex operator superalgebra $D(m)$ which commutes with the action of $\hat{\mathfrak{g}}$. Now Lemma 6.2 (2) (see also [**A1**]) gives that

$$W(m) \cong M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0). \tag{6.2}$$

On the other hand, from Lemma 6.2 (1) we get that

$$W(m) \cong \bigoplus_{0 \le n' \le p_1-2} L^{Vir}(d_{p_1,q_1}, k_{p_1,q_1}^{n'+1,1}) \otimes L^{Vir}(d_{p_2,q_2}, k_{p_2,q_2}^{1,n'+1}). \tag{6.3}$$

Thus, we get that the vertex operator superalgebra $W(m)$ has a subalgebra isomorphic to $L^{Vir}(d_{p_1,q_1},0) \otimes L^{Vir}(d_{p_2,q_2},0)$.

Now Proposition 5.1, relations (6.2) and (6.3), imply that $L^{Vir}(d_{p_1,q_1},0) \otimes L^{Vir}(d_{p_2,q_2},0)$ is a subalgebra of the vertex operator superalgebra $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ with the same Virasoro element. Moreover, the $L^{Vir}(d_{p_1,q_1},0) \otimes L^{Vir}(d_{p_2,q_2},0)$–module $M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0)$ has the following decomposition:

$$M^1 \otimes L^{\mathbf{ns}}(c_{p,q},0) \cong \bigoplus_{0 \le n' \le p_1-2} L^{Vir}(d_{p_1,q_1}, k_{p_1,q_1}^{n'+1,1}) \otimes L^{Vir}(d_{p_2,q_2}, k_{p_2,q_2}^{1,n'+1}).$$

This proves Theorem 5.1 . □

REMARK 6.2. Theorem 5.1 can be also proved using Proposition 5.1 and the results from [**KW**] on characters of minimal models.

7. $N=2$ vertex operator superalgebra

The $N=2$ superconformal algebra is the infinite-dimensional Lie superalgebra with basis $L(n), T(n), G(r)^{\pm}, C$, $n \in \mathbb{Z}$, $r \in \frac{1}{2}+\mathbb{Z}$ and (anti)commutation relations given by

$$\begin{aligned}
&[L(m), L(n)] = (m-n)L(m+n) + \frac{C}{12}(m^3-m)\delta_{m+n,0} \\
&[L(m), G(r)^{\pm}] = (\tfrac{1}{2}m - r)G(m+r)^{\pm} \\
&[L(m), T(n)] = -nT(n+m) \\
&[T(m), T(n)] = \frac{C}{3} m\delta_{m+n,0} \\
&[T(m), G(r)^{\pm}] = \pm G(m+r)^{\pm} \\
&\{G(r)^{+}, G(s)^{-}\} = 2L(r+s) + (r-s)T(r+s) + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s,0} \\
&[L(m), C] = [T(n), C] = [G(r)^{\pm}, C] = 0 \\
&\{G(r)^{+}, G(s)^{+}\} = \{G(r)^{-}, G(s)^{-}\} = 0
\end{aligned}$$

for all $m, n \in \mathbb{Z}$, $r, s \in \frac{1}{2}+\mathbb{Z}$.

Let $M_{h,q,c}$ be the Verma module generated from a highest weight vector $|h,q,c\rangle$ with $L(0)$ eigenvalue h, $T(0)$ eigenvalue q and central charge c. Let $J_{h,q,c}$ be the maximal proper submodule in $M_{h,q,c}$. Then $L_{h,q,c} = \frac{M_{h,q,c}}{J_{h,q,c}}$ is an irreducible highest weight module for the $N=2$ superconformal algebra.

Let $L_c = L_{0,0,c}$. Then for every $c \in \mathbb{C}$, L_c is a simple vertex operator superalgebra (cf. [**A2**], [**EG**]).

We will now present the classification result from [**A2**].

For $m \in \mathbb{C} \setminus \{-2\}$, set $c_m = \frac{3m}{m+2}$. For $j, k \in \mathbb{C}$ define

$$h_{j,k} = \frac{jk - \frac{1}{4}}{m+2}, \quad q_{j,k} = \frac{j-k}{m+2}; \quad L_{c_m}^{j,k} = L_{h_{j,k}, q_{j,k}, c_m}.$$

THEOREM 7.1. [**A2**] *Let $m \in \mathbb{Z}_{>0}$. Then the set*

$$\{L_{c_m}^{j,k} \mid j,k \in \mathbb{N}_{\frac{1}{2}}, 0 < j,k, j+k < m+2\}$$

provides all irreducible modules for the vertex operator superalgebra L_{c_m}. So, irreducible L_{c_m}-modules are exactly all unitary modules for the $N=2$ superconformal algebra with central charge c_m. (Here $\mathbb{N}_{\frac{1}{2}} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$).

REMARK 7.1. Theorem 7.1 shows that vertex operator superalgebra L_{c_m} for $m \in \mathbb{Z}_{>0}$ has exactly $\frac{(m+2)(m+1)}{2}$ non-isomorphic irreducible modules. In the non-unitary case the vertex operator superalgebra L_{c_m} has uncountably many irreducible modules (cf. [**EG**], [**A2**]).

Now we shall recall the results on regularity and fusion rules obtained in the paper [**A3**].

Let $m \in \mathbb{Z}_{>0}$. Then the affine Kac-Moody vertex operator algebra $L(m,0)$ is regular.

Let F_n, $n \in \mathbb{Z}$ be the lattice vertex (super)algebra $V_{\mathbb{Z}\alpha}$, associated to the lattice $\mathbb{Z}\alpha$, where $\langle \alpha, \alpha \rangle = n$. Let $\mathbf{1}$ be the vacuum vector in F_n. As usual, denote the generators of F_n by $\iota(e_\alpha), \iota(e_{-\alpha})$ (cf. [**DL**], [**A3**]).

Every F_n–module is completely reducible (cf. [**DLM**]). Let $n = -2(m+2)$.

Set $MF_{-2(m+2)} = V_{\frac{\alpha}{2}+\mathbb{Z}\alpha}$.

The following fundamental isomorphism of vertex superalgebras was proved in [**A3**] :

$$L_{c_m} \otimes F_{-1} \cong L(m,0) \otimes F_{-2(m+2)} \oplus L(m,m) \otimes MF_{-2(m+2)}. \tag{7.4}$$

Relation (7.4) shows that the vertex superalgebra $L_{c_m} \otimes F_{-1}$ is an extension of the vertex algebra $L(m,0) \otimes F_{-2(m+2)}$ by its module $L(m,m) \otimes MF_{-2(m+2)}$. Since regularity and the fusion rules for vertex algebras $L(m,0)$ and $F_{-2(m+2)}$ are known (cf. [**FZ**], [**DL**]), we get the following results.

THEOREM 7.2. [**A3**] *The vertex operator superalgebra L_{c_m} is regular.*

THEOREM 7.3. [**A3**] *Assume that $L_{c_m}^{j_1,k_1}$, $L_{c_m}^{j_2,k_2}$ and $L_{c_m}^{j_3,k_3}$ are L_{c_m}–modules. Then we have:*

(1)

$$\dim I_{L_{c_m}} \begin{pmatrix} L_{c_m}^{j_3,k_3} \\ L_{c_m}^{j_1,k_1} \; L_{c_m}^{j_2,k_2} \end{pmatrix} \leq 1.$$

(2)

$$\dim I_{L_{c_m}} \begin{pmatrix} L_{c_m}^{j_3,k_3} \\ L_{c_m}^{j_1,k_1} \; L_{c_m}^{j_2,k_2} \end{pmatrix} = 1$$

if and only if one of the conditions (F1) and (F2) holds, where

$$\begin{aligned} (F1) \qquad & (j_1 + j_2 - j_3) - (k_1 + k_2 - k_3) = 0, \\ & |j_2 + k_2 - j_1 - k_1| < j_3 + k_3, \\ & j_3 + k_3 < min\{j_1 + k_1 + j_2 + k_2, \\ & 2m + 4 - (j_1 + k_1 + j_2 + k_2)\}, \end{aligned}$$

$$\begin{aligned} (F2) \qquad & (j_1 + j_2 - j_3) - (k_1 + k_2 - k_3) = \pm(m+2), \\ & |j_2 + k_2 - j_1 - k_1| < m + 2 - j_3 - k_3, \\ & m + 2 - j_3 - k_3 < min\{j_1 + k_1 + j_2 + k_2, \\ & 2m + 4 - (j_1 + k_1 + j_2 + k_2)\}. \end{aligned}$$

REMARK 7.2. The fusion rules from Theorem 7.3 are identical to the fusion rules for the Neveu-Schwarz sector $NS^{(m+2)}$ obtained by M. Wakimoto using a modified Verlinde formula (cf. [**W**]).

8. A Generalization. Family of vertex operator superalgebras $D_{m,k}$

In this section we recall the results from [**A4**] on the family of vertex operator (super)algebras $D_{m,k}$.

Let $m \in \mathbb{C} \setminus \{0, -2\}$, and let k be a nonnegative integer. Let α be the lattice element in the definition of F_k such that $\langle \alpha, \alpha \rangle = k$. The vertex operator (super)algebra $D_{m,k}$ is defined to be the subalgebra of the vertex operator (super)algebra $L(m,0) \otimes F_k$ generated by vectors:

$$X = e(-1)\mathbf{1}_m \otimes \iota(e_\alpha) \quad \text{and} \quad Y = f(-1)\mathbf{1}_m \otimes \iota(e_{-\alpha}).$$

$D_{m,k}$ is a vertex operator algebra if k is even, and a vertex operator superalgebra if k is odd. These algebras have rank c_m.

For $k = 0$, $D_{m,0}$ is isomorphic to the $\hat{sl_2}$ vertex operator algebra $L(m,0)$. For $k = 1$, $D_{m,1}$ is the vertex operator superalgebra associated to the vacuum representation of the $N = 2$ superconformal algebra (cf. [**A2**]). This construction can be considered in the context of the Kazama-Suzuki mapping investigated in [**FST**] and [**KS**].

Next result provides a generalization of Theorem 7.2.

THEOREM 8.1. [**A4**] *Let $m \in \mathbb{C} \setminus \{0, -2\}$, and let k be a nonnegative integer. Then the vertex operator (super)algebra $D_{m,k}$ is rational if and only if $m \in \mathbb{Z}_{>0}$.*

Furthermore, if $m \in \mathbb{Z}_{>0}$, $D_{m,k}$ is a regular vertex operator (super)algebra.

REMARK 8.1. Proof of Theorem 8.1 was given in [**A4**], and it uses the following isomorphism of vertex (super)algebras:

$$D_{m,k} \otimes F_{-k} \cong L(m,0) \otimes F_{-\frac{k}{2}(mk+2)}, \quad (k \text{ even}),$$
$$D_{m,k} \otimes F_{-k} \cong L(m,0) \otimes F_{-k(mk+2)} \oplus L(m,m) \otimes MF_{-k(mk+2)}, \quad (k \text{ odd});$$

generalizing the isomorphism (7.4).

References

[A1] D. Adamović, Rationality of Neveu-Schwarz vertex operator superalgebras, Internat. Math. Res. Notices No. **17** (1997), 865-874.

[A2] D. Adamović, Representations of the $N = 2$ superconformal vertex algebra, Internat. Math. Res. Notices **2** (1999), 61-79.

[A3] D. Adamović, Vertex algebra approach to fusion rules for $N = 2$ superconformal minimal models, J. Algebra **239** (2001), 549-572.

[A4] D. Adamović, Regularity of certain vertex operator algebras with two generators, preprint, math.QA/0111055.

[AM] D. Adamović and A. Milas, Vertex operator algebras associated to the modular invariant representations for $A_1^{(1)}$, Math. Res. Lett. **2** (1995), 563-575.

[D] C. Dong, Vertex algebras associated with even lattices, J. Algebra **160** (1993), 245-265.

[DL] C. Dong and J. Lepowsky, Generalized vertex algebras and relative vertex operators, Birkhäuser, Boston, 1993.

[DLM] C. Dong, H. Li and G. Mason, Regularity of rational vertex operator algebras, Adv. Math. **132** (1997), 148-166.

[DMZ] C. Dong, G. Mason and Y. Zhu, Discrete Series of the Virasoro Algebra and the Moonschine Module, Proc. Symp. Pure Math., Amer. Math. Soc. **56**, (1994), part 2, 295-316.

[EG] W. Eholzer and M. R. Gaberdiel, Unitarity of rational $N = 2$ superconformal theories, Comm. Math. Phys. **186** (1997) 61-85.

[Fe] A. J. Feingold, Constructions of vertex operator algebras, Proc. Symp. Pure Math., Amer. Math. Soc. **56**, (1994), part 2, 317-336.

[Fr] I. B. Frenkel, Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory, J. Funct. Anal. 44 (1981), 259-327.

[FFR] A. J. Feingold, I. B. Frenkel and J. Ries, Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$, Contemporary Math. 121, Amer. Math. Soc., Providence, RI, 1991.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. **104**, 1993.

[FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Math., Vol. **134**, Academic Press, New York, 1988.

[FRW] A. J. Feingold, J. Ries and M. Weiner, Spinor construction of the $c = \frac{1}{2}$ minimal model, Moonshine, The Monster and related topics, Contemporary Math. 193, (Chongying Dong and Geoffrey Mason, eds.), Amer. Math. Soc., Providence, RI, 1995, pp. 45-92.

[FST] B. L. Feigin, A. M. Semikhatov and I. Yu. Tipunin, Equivalence between chain categories of representations of affine $sl(2)$ and $N = 2$ superconformal algebras, J. Math. Phys. **39** (1998), 3865-390.

[FZ] I. B. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. **66** (1992), 123-168.

[GKO] P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro and Super-Virasoro algebras, Comm. Math. Phys, **103** 105-119, 1986.

[H] Y.-Z. Huang, Virasoro vertex operator algebras, (nonmeromorphic) operator product expansion and the tensor product theory, J. Algebra **182** (1996), 201–234.

[HM] Y.-Z. Huang and A. Milas, Intertwining operator superalgebras and vertex tensor categories for superconformal algebras I, Commun. Contemp. Math. 4 (2002), no.2, 327-355.

[K] V. G. Kac, Vertex Algebras for Beginners, University Lecture Series, Second Edition, Amer. Math. Soc., 1998, Vol. 10.

[KS] Y. Kazama and H. Suzuki, New N=2 superconformal field theories and superstring compactifications, Nucl. Phys. B **321** (1989), 232-268.

[KW] V. G. Kac and M. Wakimoto, Modular invariant representations of infinite dimensional Lie algebras and superalgebras, Proc. Natl. Acad. Sci. USA **85**,1988, 4956-4960.

[KWn] V. G. Kac and W. Wang, Vertex operator superalgebras and their representations, Contemporary Math., **175** (1994), 161-191.

[La] M. Yu. Lashkevich, Superconformal 2D minimal models and an unusual coset constructions, Modern Physics Letter A (1993) 851-860, hep-th/9301093.

[Li1] H. Li, Local systems of vertex operators, vertex superalgebras and modules, J. Pure Appl. Algebra **109** (1996), 143-195.

[Li2] H. Li, Some finiteness properties of regular vertex operator algebras, J. Algebra **212** (1999), 495-514.

[P] M. Primc, Vertex algebras generated by Lie algebras, J. Pure Appl. Algebra **135** (1999), 253-293.

[T] H. Tsukada, Vertex operator superalgebras, Comm. Algebra **18** (1990), 2249-2274.

[W] M. Wakimoto, Fusion rules for $N = 2$ superconformal modules, hep-th/9807144

[Wn] W.Wang, Rationality of Virasoro Vertex operator algebras, Internat. Math. Res. Notices, Vol **71**, No.1 (1993),197-211.

[Z] Y. Zhu, Vertex operator algebras, elliptic functions and modular forms, Ph. D. thesis, Yale University, 1990; Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. **9** (1996), 237-302.

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