Spectral information contained in Dirichlet-to-Neumann-type maps

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Biograd, September 2013

Dirichlet-to-Neumann type maps

Motivation:

- Titchmarsh-Weyl m-function for Sturm-Liouville problems
- Dirichlet-to-Neumann map for PDEs

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Method: make use of abstract theory of boundary triples to

- introduce M-function,
- relate resolvent to operators on the boundary,
- compare resolvents of different realisations.

Boundary triples

(contributions from Kochubei, Gorbachuk & Gorbachuk, Derkach & Malamud, Vainerman, Lyantze & Storozh, Malamud & Mogilievski...)

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- $A \subseteq (A')^* =: A_{\max}$ and $A' \subseteq A^* =: A'_{\max}$
- there exist "boundary spaces" $\mathcal{H}$, $\mathcal{K}$ and "boundary operators",
  - $\Gamma_1 : D(A_{\max}) \to \mathcal{H}$ and $\Gamma_0 : D(A_{\max}) \to \mathcal{K}$,
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$$(A_{\max} u, v)_H - (u, A'_{\max} v)_H = (\Gamma_1 u, \Gamma'_0 v)_{\mathcal{H}} - (\Gamma_0 u, \Gamma'_1 v)_{\mathcal{K}}.$$

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$\{\mathcal{H} \oplus \mathcal{K}, (\Gamma_1, \Gamma_0), (\Gamma_1', \Gamma_0')\}$ is a boundary triple for the adjoint pair A, A'.

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$$\int_0^1 (-u''\overline{v} + u\,\overline{v}'')dx = \begin{pmatrix} -u'(1) \\ u'(0) \end{pmatrix} \cdot \overline{\begin{pmatrix} v(1) \\ v(0) \end{pmatrix}} - \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \cdot \overline{\begin{pmatrix} -v'(1) \\ v'(0) \end{pmatrix}}.$$

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$$(A_{\max} - \lambda)u = 0, \ (\Gamma_1 - B\Gamma_0)u = f,$$

- for $\lambda \in \rho(A_B)$ define the M-function via

$$M_B(\lambda) : \mathrm{Ran}\,(\Gamma_1 - B\Gamma_0) \to \mathcal{K}, \quad M_B(\lambda) f = \Gamma_0 S_{\lambda,B} f.$$

Relation to resolvent

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Theorem (Kreĭn-type formula)

- $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $\lambda \in \rho(A_B) \cap \rho(A_C)$. *Then*

$$(A_B - \lambda)^{-1} = (A_C - \lambda)^{-1} - S_{\lambda,C}(I + (B - C)M_B(\lambda))(C - B)\Gamma_0(A_C - \lambda)^{-1}$$

Results for poles

Theorem

Let $\mu \in \mathbb{C}$ be an isolated eigenvalue of finite algebraic multiplicity of the operator A_B. Assume unique continuation holds, i.e.

$$\ker(A_{\max}-\mu)\cap\ker(\Gamma_1)\cap\ker(\Gamma_0) = \ker(A'_{\max}-\bar{\mu})\cap\ker(\Gamma'_1)\cap\ker(\Gamma'_0) = \{0\}.$$

Then μ is a pole of finite multiplicity of $M_B(\cdot)$ and the order of the pole of $R(\cdot, A_B)$ at μ is the same as the order of the pole of $M_B(\cdot)$ at μ.

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Theorem

- *Let $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $\mu \in \mathbb{C}$.*
- *Assume there exists $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that $\mu \in \rho(A_C)$.*

Then μ is isolated eigenvalue of finite algebraic multiplicity of A_B iff μ is pole of finite multiplicity of $M_B(\cdot)$.
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A matrix differential operator I

The M-function M_B does not contain all spectral information on the operator A_B:

$$A_{\max} = \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & w(x) \\ w(x) & u(x) \end{pmatrix}, \quad A'_{\max} = \begin{pmatrix} -\frac{d^2}{dx^2} + \overline{q(x)} & \overline{w(x)} \\ \overline{w(x)} & \overline{u(x)} \end{pmatrix},$$

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$$D(A_{\max}) = D(A'_{\max}) = H^2(0,1) \times L^2(0,1),$$

$$\Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y'(1) \\ y'(0) \end{pmatrix}, \quad \Gamma_0 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y(1) \\ y(0) \end{pmatrix},$$

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$$A_B := A_{\max}|_{\ker(\Gamma_1 - B\Gamma_0)},$$

$$\sigma_{ess}(A_B) = \text{essran}(u) \quad \text{for any } B \in \mathbb{R}^{2\times 2}.$$

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so $z = \frac{wy}{\lambda - u}$ and

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Thus, if $w(x_0) = 0$, then $u(x_0)$ can be changed without affecting y or the M-function.

Results for $\begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & w(x) \\ w(x) & u(x) \end{pmatrix}$

Theorem

- *Let*
  - $\mathcal{W} = \{x \in [0,1] \,|\, w(x) \neq 0\}$,
  - $H_1 = \begin{pmatrix} L^2(0,1) \\ L^2(\mathcal{W}) \end{pmatrix}$,
  - $\lambda \notin \mathrm{essran}\,(\,u|_{\mathcal{W}})$.

  Then the bordered resolvent $P_{H_1}(A_B - \lambda I)^{-1} P_{H_1}$ *is analytic precisely where* $M_B(\lambda)$ *is analytic.*

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- *Assume*
  - q, u, w *analytic,*
  - $w(0), w(1) \neq 0$.

  Then q, u, w can be recovered from the M-function.

Symmetric case

Definition

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Let A be symmetric, A_B self-adjoint and invertible. If A is simple, then $M_B(\lambda) - M_B(0)$ determines A_B up to unitary equivalence.

Lemma

A is simple iff

$$H = \overline{\mathrm{Span}_{\delta \notin \sigma(A_B)} (A_B - \delta I)^{-1} Ran(S_{0,B})}.$$

Non-symmetric case

For $\mu_0 \notin \sigma(A_B)$, define the spaces

$$\mathcal{S} = \mathrm{Span}_{\delta \notin \sigma(A_B)} (A_B - \delta I)^{-1} \mathrm{Ran}(S_{\mu_0,B}), \tag{1}$$

$$\mathcal{T} = \mathrm{Span}_{\mu \notin \sigma(A_B)} \mathrm{Ran}(S_{\mu,B}), \tag{2}$$

where $S_{\mu,B} = \left((\Gamma_1 - B\Gamma_0)|_{\ker(A_{\max} - \mu I)} \right)^{-1}$.

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Under some extra assumptions on the spectrum of extensions of A, we have that $\overline{\mathcal{S}}$ is independent both of μ_0 and B. Moreover, $\overline{\mathcal{S}} = \overline{\mathcal{T}}$.

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Lemma

Under some extra assumptions on the spectrum of extensions of A, we have that $\overline{\mathcal{S}}$ is independent both of μ_0 and B. Moreover, $\overline{\mathcal{S}} = \overline{\mathcal{T}}$.

Lemma

The space $\overline{\mathcal{S}}$ is a regular invariant subspace for the resolvent of A_B, i.e. $\overline{(A_B - \mu I)^{-1}\overline{\mathcal{S}}} = \overline{\mathcal{S}}$ for all $\mu \in \rho(A_B)$.

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Theorem

- λ_0 *point of analyticity of* M_B
- $\lambda_0 \in \overline{\rho(A_B)}$
- $P_{n,\mathcal{S}}$ *and* $P_{m,\mathcal{S}'}$ *projections onto any finite-dimensional subspaces of* $\mathcal{S}$ *and* $\mathcal{S}'$

Then $P_{m,\mathcal{S}'}(A_B - \lambda I)^{-1} P_{n,\mathcal{S}}$ *analytic at* $\lambda = \lambda_0$.

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Theorem

- λ_0 *point of analyticity of* M_B
- λ_0 *at worst an isolated singularity of* $(A_B - \lambda I)^{-1}$
- $\rho(A'_{B^*})$ *has finitely many connected components*

Then $P_{\overline{\mathcal{S}'}}(A_B - \lambda I)^{-1}P_{\overline{\mathcal{S}}}$ *is analytic at* $\lambda = \lambda_0$.

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Theorem

Assume we know

- $M_B(\lambda)$ *for all* $\lambda \in \rho(A_B)$,
- $\mathrm{Ran}\,(S_{\mu,B})$ *for some* $\mu \in \rho(A_B)$,
- $\mathrm{Ran}\,(S'_{\mu',B^*})$ *for some* $\mu' \in \rho(A_B^*)$.

Then we can reconstruct $P_{\overline{\mathcal{S}'}}(A_B - \lambda)^{-1} P_{\overline{\mathcal{S}}}$ *for all* $\lambda \in \rho(A_B)$.

Result for $\begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & \widetilde{w}(x) \\ w(x) & u(x) \end{pmatrix}$

Theorem

Assume $w\widetilde{w} = 0$*, that* $\theta(x,\lambda), \phi(x,\lambda)$ *solve* $-y'' + (q - \lambda)y = 0$ *and*

$$E_{u,w} := \mathrm{Span}_{n\in\mathbb{N}} w(x)\theta(x, u(x))u^n(x) + \mathrm{Span}_{n\in\mathbb{N}} w(x)\phi(x, u(x))u^n(x).$$

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Then

$$S^{\perp} = \left\{ \begin{pmatrix} h \\ g \end{pmatrix} : g \perp E_{u,w}, \right.$$
$$\left. h(x) = \int_0^x (wg)(t)[\phi(t, u(t))\theta(x, u(t)) - \theta(t, u(t))\phi(x, u(t))]dt \right\}$$

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In particular,

$$\overline{\mathcal{S}} = \begin{pmatrix} L^2(0,1) \\ \chi_{\{w\neq 0\}}L^2(0,1) \end{pmatrix} = H_1.$$

iff $E_{u,w} = \chi_{\{w\neq 0\}}L^2(0,1)$.

Thank you
for your attention!