Well-posedness for a class of non-autonomous differential inclusions.

Sascha Trostorff (joint work with Maria Wehowski)

Technische Universität Dresden

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An example: the non-autonomous heat equation

The setting

Well-posedness of non-autonomous problems

Classical approach

We consider

$$\begin{aligned} \partial_0 \theta(t) + \operatorname{div} q(t) &= f(t), \\ q(t) &= -k(t) \operatorname{grad} \theta(t), \end{aligned}$$

for $t \in \mathbb{R}$ in a domain $\Omega \subseteq \mathbb{R}^n$, where ∂_0 denotes the temporal derivative, $k : \mathbb{R} \to L_\infty(\Omega)$ suitable function and f is a given source term.

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Classically, one proves that $-A(t)$, subject to suitable boundary conditions, is a generator of an evolution family $(U(t,s))_{t\geq s\geq 0}$ and applies Duhamel's formula.

Our approach

We consider the system

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k(t)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\right) \begin{pmatrix} \theta(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}$$

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Thus, we end up with an equation of the form

$$(\partial_0 M_0(\mathrm{m}) + M_1(\mathrm{m}) + A)\, u = f,$$

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where A is skew-selfadjoint (b.c.), or more generally

$$(u, f) \in \partial_0 M_0(\mathrm{m}) + M_1(\mathrm{m}) + A,$$

where A is a maximal monotone relation.

The time derivative

Let H be a Hilbert space. For $\rho > 0$ denote by $H_\rho(\mathbb{R}; H)$ the space of H-valued square-integrable functions with respect to the measure $\exp(-2\rho t)\mathrm{d}t$.

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Define the derivative $\partial_{0,\rho}$ on $H_\rho(\mathbb{R}; H)$ as the closure of

$$\partial_{0,\rho}|_{C_c^\infty(\mathbb{R};H)} : C_c^\infty(\mathbb{R}; H) \subseteq H_\rho(\mathbb{R}; H) \to H_\rho(\mathbb{R}; H)$$
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In particular, $\partial_{0,\rho}^{-1}$ is causal.

The operators $M_0(\mathrm{m})$ and $M_1(\mathrm{m})$

Consider two strongly measurable, bounded functions $M_i : \mathbb{R} \to L(H)$ and denote by $M_i(\mathrm{m})$ the associated multiplication operator on $L_{2,\mathrm{loc}}(\mathbb{R}; H)$. Throughout, the following assumptions should hold:

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Hypotheses

(a) M_0 is Lipschitz-continuous and there exists a set $N \subseteq \mathbb{R}$ of measure zero, such that for all $x \in H$

$$\mathbb{R} \setminus N \ni t \mapsto M_0(t)x$$

is differentiable.

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(c) There exist $\rho_0 > 0$, $c > 0$ such that

$$\forall \rho \geq \rho_0, t \in \mathbb{R} \setminus N : \ \rho M_0(t) + \frac{1}{2}\dot{M}_0(t) + \Re M_1(t) \geq c.$$

Monotone relations

Let $A \subseteq H \oplus H$ be a binary relation. A is called *monotone* if for all pairs $(u, v), (x, y) \in A$ we have

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Remark

Let A, B be two monotone relations. Then

$$A + B = \{(x, y + z) \,|\, (x, y) \in A, (x, z) \in B\}$$

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Lemma

Assume that M_0, M_1 satisfy the hypotheses above. Then, there exist $\rho_0, c > 0$ such that for all $\rho \geq \rho_0$ the operator

$$\partial_{0,\rho} M_0(\mathrm{m}) + M_1(\mathrm{m}) - c$$

is monotone.

An example: the non-autonomous heat equation

The setting

Well-posedness of non-autonomous problems

Uniqueness and Continuous dependence

Let $A \subseteq H \oplus H$ be monotone and define

$$A_\rho := \{(u,v) \in H_\rho(\mathbb{R};H)^2 \,|\, (u(t),v(t)) \in A \text{ a.e.}\}$$

for $\rho > 0$. Then A_ρ is monotone.

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Corollary

Assume that M_0, M_1 satisfy the hypotheses. Then there exists $\rho_0, c > 0$ such that $\partial_{0,\rho} M_0(\mathrm{m}) + M_1(\mathrm{m}) + A_\rho - c$ is monotone,

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Corollary

Assume that M_0, M_1 satisfy the hypotheses. Then there exists $\rho_0, c > 0$ such that $\partial_{0,\rho} M_0(\mathrm{m}) + M_1(\mathrm{m}) + A_\rho - c$ is monotone, i.e. for $(u,f),(v,g) \in \partial_{0,\rho} M_0(\mathrm{m}) + M_1(\mathrm{m}) + A_\rho$ we estimate

$$\Re\langle u-v|f-g\rangle_{H_\rho(\mathbb{R};H)} \geq c|u-v|^2_{H_\rho(\mathbb{R};H)},$$

yielding

$$|u-v|_{H_\rho(\mathbb{R};H)} \leq \frac{1}{c}|f-g|_{H_\rho(\mathbb{R};H)}.$$

Existence in the linear case

Assume that $A : D(A) \subseteq H \to H$ is skew-selfadjoint (in particular, A is monotone) and set

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GOAL: Find a dense subset $\mathcal{G} \subseteq H_\rho(\mathbb{R}; H)$, where we can show existence, i.e. for $f \in \mathcal{G}$ there exists $u \in D(\mathcal{B}_\rho)$ with $\mathcal{B}_\rho(u) = f$.

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Theorem (Picard,Waurick,Wehowski,T)

There exists $\rho_0 > 0$ such that for all $\rho \geq \rho_0$ the operator $\mathcal{B}_\rho^*$ is injective.

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Way out: Perturbation theory for maximal monotone relations.

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Corollary

Assume that M_0, M_1 satisfy the hypotheses. Then there exists $\rho_0 > 0, c > 0$ such that $\partial_{0,\rho} M_0(\mathrm{m}) + M_1(\mathrm{m}) - c$ is maximal monotone for all $\rho \geq \rho_0$.

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Proof.

This follows from the solution theory above with $A = 0$. □

Perturbation results

For A maximal monotone, $\lambda > 0$ we define

$$A_\lambda : H \to H \quad x \mapsto \lambda^{-1}(x - (1+\lambda A)^{-1}(x))$$

the *Yosida approximation*.

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Proposition

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Theorem

Let $A, B \subseteq H \oplus H$ be maximal monotone and $y \in H$. Then there exists $x \in H$ such that $(x, y) \in 1 + A + B$ if and only if

$$\sup_{\lambda > 0} |B_\lambda(x_\lambda)| < \infty,$$

where $(x_\lambda, y) \in 1 + A + B_\lambda$.

The operators M_0 and M_1 revised

We replace the hypothesis

(c) $\exists \rho_0, c > 0 \, \forall \rho \geq \rho_0, t \in \mathbb{R} \setminus N : \; \rho M_0(t) + \frac{1}{2}\dot{M}_0(t) + \Re M_1(t) \geq c.$

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(d') There exists $c > 0$ such that for all $t \in \mathbb{R}$ we have $M_0(t) \geq c$ on $N(M(0))^\perp = R(M_0(0))$ and $\Re M_1(t) \geq c$ on $N(M_0(0))$.

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For $\delta > 0$ we consider the following auxiliary problem:

$$(u, f) \in \partial_{0,\rho} M_0(\mathrm{m}) + \delta - \dot{M}_0(\mathrm{m}) + A_\rho.$$

Note that $M_1(\mathrm{m}) := \delta - \dot{M}_0(\mathrm{m})$ satisfies (d').

A regularity result

Proposition

Let $\rho > 0$ be large enough and $f \in D(\partial_{0,\rho}), \lambda > 0$. Moreover let $u_\lambda \in H_\rho(\mathbb{R}; H)$ such that

$$\partial_{0,\rho} M_0(\mathrm{m}) u_\lambda + \delta u_\lambda - \dot{M}_0(\mathrm{m}) u_\lambda + A_{\rho,\lambda}(u_\lambda) = f.$$

Then $u_\lambda \in D(\partial_{0,\rho})$.

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Then $u_\lambda \in D(\partial_{0,\rho})$.

Proof

Decompose $u_\lambda = P_0 u_\lambda + P_1 u_\lambda \in N(M_0(0)) + R(M_0(0))$. Then $P_1 u_\lambda \in D(\partial_{0,\rho})$, since $M_0(\mathrm{m}) u_\lambda \in D(\partial_{0,\rho})$. Moreover,

$$\delta P_0 u_\lambda + P_0 A_{\rho,\lambda}(P_0 u_\lambda + P_1 u_\lambda) = P_0 f.$$

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An easy computation shows that B is monotone.

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&= \delta\tau_h P_0 u_\lambda + P_0 A_{\rho,\lambda}(\tau_h P_0 u_\lambda + P_1 u_\lambda) \\
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\end{aligned}$$

$$\delta P_0 u_\lambda + P_0 A_{\rho,\lambda}(P_0 u_\lambda + P_1 u_\lambda) = P_0 f$$

Define

$$B : H_\rho(\mathbb{R}; N(M_0(0))) \to H_\rho(\mathbb{R}; N(M_0(0))) : \quad v \mapsto P_0 A_{\rho,\lambda}(v + P_1 u_\lambda).$$

An easy computation shows that B is monotone. Moreover, using $\tau_h A_{\rho,\lambda} = A_{\rho,\lambda}\tau_h$

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The monotonicity of B yields

$$\Re\langle \underbrace{\delta \tau_h P_0 u_\lambda + B(\tau_h P_0 u_\lambda)}_{=g} - (\underbrace{\delta P_0 u_\lambda + B(P_0 u_\lambda)}_{=f}) | (\tau_h - 1) P_0 u_\lambda \rangle \geq \delta |(\tau_h - 1) P_0 u_\lambda|^2.$$

$$\tau_h P_0 f + P_0(A_{\rho,\lambda}(\tau_h P_0 u_\lambda + P_1 u_\lambda) - A_{\rho,\lambda}(\tau_h(P_0 u_\lambda + P_1 u_\lambda))) =: g$$

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Hence, by Cauchy-Schwarz

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$$\begin{aligned} |(\tau_h - 1)P_0 u_\lambda| &\leq \frac{1}{\delta}|f - g| \\ &\leq \frac{1}{\delta}(|(\tau_h - 1)f| + \frac{1}{\lambda}|(\tau_h - 1)P_1 u_\lambda|). \end{aligned}$$

Thus, $(\frac{1}{h}(\tau_h - 1)P_0 u_\lambda)_{h>0}$ is bounded, yielding the differentiability of $P_0 u_\lambda$. □

Existence for auxiliary problem

Proposition

Let $\rho > 0$ be large enough and $f \in D(\partial_{0,\rho})$. Then there exists $u \in H_\rho(\mathbb{R}; H)$ such that

$$(u, f) \in \partial_{0,\rho} M_0(\mathrm{m}) + \delta - \dot{M}_0(\mathrm{m}) + A_\rho.$$

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with maximal domain.Then $\partial_{0,\rho} u_\lambda \in D(B_\lambda)$ and

$$\langle B_\lambda(v) | v \rangle \geq 0 \quad (v \in D(B_\lambda)).$$

Moreover,

$$(\partial_{0,\rho} M_0(\mathrm{m}) + \delta + B_\lambda)(\partial_{0,\rho} u_\lambda) = \partial_{0,\rho} f,$$

yielding $|\partial_{0,\rho} u_\lambda| \leq C|\partial_{0,\rho} f|$.

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Since the right-hand side is uniformly bounded in λ we get the assertion. $\square$

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Corollary (Solution Theory for auxiliary problem)

$(\partial_{0,\rho} M_0(\mathrm{m}) + \delta - \dot{M}_0(\mathrm{m}) + A_\rho)^{-1}$ is a Lipschitz-continuous mapping on $H_\rho(\mathbb{R}; H)$ for sufficiently large $\rho > 0$.

Solution Theory

Theorem (Wehowski, T)

Let M_0, M_1 satisfy the hypotheses (a),(b),(c'),(d') and let $A \subseteq H \oplus H$ be maximal monotone with $(0,0) \in A$. Then, $(\overline{\partial_{0,\rho} M_0(\mathrm{m}) + M_1(\mathrm{m}) + A_\rho})^{-1}$ is a Lipschitz-continuous mapping on $H_\rho(\mathbb{R}; H)$ for sufficiently large $\rho > 0$.

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Proof.

The proof follows from the solution theory for the auxiliary problem and easy perturbation arguments. □

Thank you for your attention!

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