Najman Conference on Spectral Problems for Operators and Matrices

September 16-20, 2013

Spectral properties of discrete alloy-type models

Martin Tautenhahn

(joint work with A. Elgart, K. Leonhardt, N. Peyerimhoff and I. Veselić)

16.09.2013

Motivation

Discrete alloy-type model

Localization

Poisson statistics

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Operators, Schrödinger equation and long time behaviour

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$\mathcal{H} = L^2(\mathbb{R}^d), \ell^2(\mathbb{Z}^d)$

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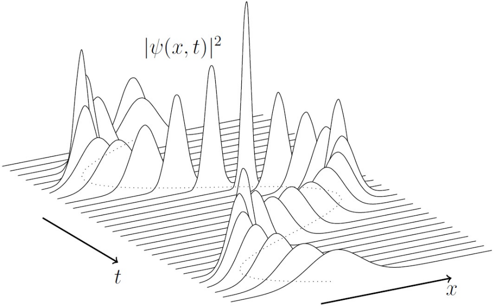

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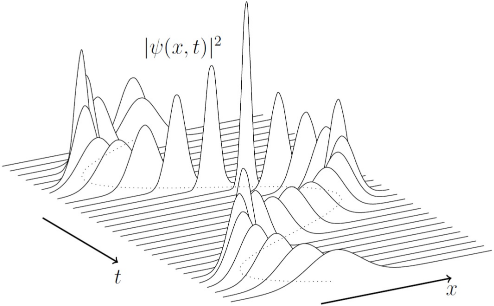

Long time behaviour with initial state $\psi_0 \in \mathcal{H} = \ell^2(\mathbb{Z}^d)$

$$\forall\ K \subset \mathbb{Z}^d \text{ finite: } \sum_{k \in K} \left|\psi(t,k)\right|^2 \to 0 \text{ if } t \to \infty$$

$$\forall\ \varepsilon > 0\ \exists\ K \subset \mathbb{Z}^d \text{ finite } \forall\, t\colon \sum_{k \in K} \left|\psi(t,k)\right|^2 \geq 1 - \varepsilon$$

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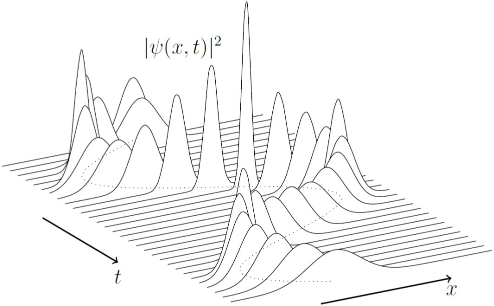

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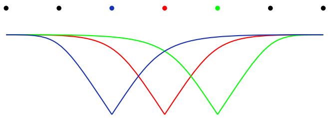
Long time behaviour depends on spectral properties of $H = -\Delta + V$!

Examples

- periodic potential, $H_{\text{per}} = -\Delta + V_{\text{per}}$, e.g.

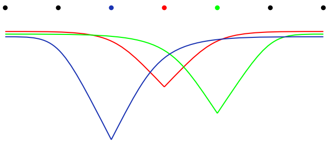


$$V_{\text{per}}(x) = \sum_{k \in \mathbb{Z}^d} u(x-k)$$

H_{per} has only absolutely continuous spectrum, i.e. $\mathcal{H} = \mathcal{H}_{\text{ac}}$

Examples

- family $(H_\omega)_\omega$, $\omega \in (\Omega, \mathcal{A}, \mathbb{P})$, $H_\omega = -\Delta + V_\omega$, e.g.



$$V_\omega(x) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x - k)$$

$\omega = (\omega_k)_{k \in \mathbb{Z}^d}$ sequence of i.i.d. random variables

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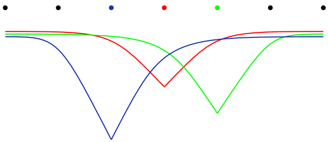

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- **Anderson localization [And58]:** There are intervals $I \subset \mathbb{R}$ containing only pure point spectrum for almost all ω (caught by disorder), i.e.

$$\sigma_c(H_\omega) \cap I = \emptyset \quad \text{a.s.}$$

- **Poisson statistics [Min94]:**
  - distribution of eigenvalues (if localization occurs)?
  - point process assiciated to rescaled eigenvalues converges to a Poisson process

Discrete alloy-type model

$$H_\omega = -\Delta + \lambda V_\omega \qquad \begin{cases} \text{on } \ell^2(\mathbb{Z}^d), \text{ where} \\ (\Delta\psi)(x) = \sum_{|e|_1=1} \psi(x+e), \\ (V_\omega\psi)(x) = V_\omega(x)\psi(x), \\ V_\omega(x) = \sum_{k\in\mathbb{Z}^d} \omega_k u(x-k). \end{cases}$$

- $\omega = (\omega_k)_{k\in\mathbb{Z}^d}$ sequence of i.i.d. random variables, $\omega_0 \sim \mu$, μ probability measure on $\mathbb{R}$ with compact support
- often: μ has density $\rho \in L^\infty(\mathbb{R}), W^{1,1}(\mathbb{R}), W^{2,1}(\mathbb{R})$
- single-site potential $u \in \ell^1(\mathbb{Z}^d;\mathbb{R})$
- $\lambda \geq 0$ strength of disorder

Discrete alloy-type model

Matrix representation for $d = 1$

$$H_\omega = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \ddots & \ddots & & & & \\ & \ddots & v_{x-1} & -1 & & & \\ & & -1 & v_x & -1 & & \\ & & & -1 & v_{x+1} & \ddots & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}, \quad v_x = \lambda \sum_{k \in \mathbb{Z}} \omega_k u(x-k)$$

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- $\Gamma \subset \mathbb{Z}^d$, $H_\Gamma : \ell^2(\Gamma) \to \ell^2(\Gamma)$ restriction of H_ω to Γ

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- $G_\Gamma(z) = (H_\Gamma - z)^{-1}$, $G_\Gamma(z; x, y) = \langle \delta_x, G_\Gamma(z)\delta_y \rangle$,

Discrete alloy-type model $V_\omega(x) = \sum \omega_k u(x-k)$

Important special cases

Anderson model: $u = \delta_0$, i.e. $V_\omega(x) = \omega_x$

monotone case: $u \geq 0$

non-monotone case: u changes sign

long range case: $\operatorname{supp} u$ unbounded

Abstract approach for correlated Anderson model [DK91,ASFH01]

- $V_\omega(x) = \eta_x(\omega)$, with random field $\eta = (\eta_x)_{x\in\mathbb{Z}^d}$
- random field η is assumed to be "conditional τ-Hölder continuous"
- **Question:** does the discrete alloy-type model satisfy this assumption?

Discrete alloy-type model $V_\omega(x) = \sum \omega_k u(x-k)$

Important special cases

Anderson model: $u = \delta_0$, i.e. $V_\omega(x) = \omega_x$ (loc,poi)

monotone case: $u \geq 0$ (loc)

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Abstract approach for correlated Anderson model [DK91,ASFH01] (loc,poi)

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Regularity properties I

Let $\eta : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ a random field. We say that this field is **conditional τ-Hölder continuous**, iff

Assumption (τ)

There are constants $\tau \in (0,1]$ and $C > 0$ such that

$$\sup_{x \in \mathbb{Z}^d} \sup_{a \in \mathbb{R}} \operatorname*{esssup}_{(\eta_k)_{k \neq x}} \mathbb{P}(\eta_x \in [a, a+\varepsilon] \mid (\eta_k)_{k \neq x}) \leq C\varepsilon^\tau .$$

Since this model is well studied [DK91,AM93] we ask

Question:

Does the discrete alloy-type model given by

$$\eta_x(\omega) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x-k)$$

satisfy this Assumption (τ)?

Regularity properties II

Theorem (T & Veselić 2013)

Let $d = 1$ and either

- $\operatorname{supp} u = \{0, \ldots, n-1\}$ and $\operatorname{supp} \mu$ bounded, **or**
- $\operatorname{supp} u = \{0, 1\}$, $u(0) = 1$, $u(1)^2 = 1$ and μ normal distribution.

Then the random field $\eta = (\eta_x)_{k \in \mathbb{Z}^d}$ defined by

$$\eta_x = \sum_{k \in \mathbb{Z}^d} \omega_k u(x - k)$$

does not satisfy Assumption (A).

Theorem (Kandler 2006, Ph.D. thesis)

Let η_k, $k \in \mathbb{Z}$ be a random field with compactly supported covariance function. Then there exists a discrete alloy type potential, with compactly supported $u : Z \to \mathbb{R}$, which has the same covariance function.

Motivation

Discrete alloy-type model

Localization

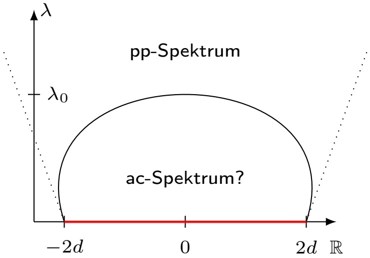
Poisson statistics

Expected spectral types for $H_\omega = -\Delta + \lambda V_\omega$

1. Let $\lambda > 0$. Then $\exists\ E_0 \in (2d, 2d+\lambda)$: $\sigma_c(H_\omega) \cap \{(-\infty, -E_0] \cup [E_0, \infty)\} = \emptyset$ a.s.

2. Let $d \geq 3$ and $\lambda > 0$ suff. small. Then $\exists\ E_m = E_m(\lambda) < E_0 : [-E_m, E_m] \cap \sigma_{ac} \neq \emptyset$ a.s. (conjecture)

3. $\exists\ \lambda_0 > 0 : \forall\ \lambda > \lambda_0\ :\ \sigma_c(H_\omega) = \emptyset$ a.s.



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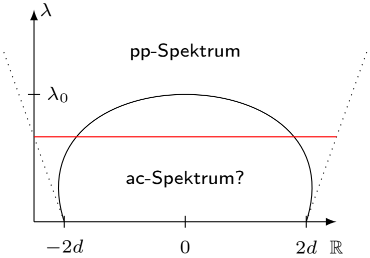

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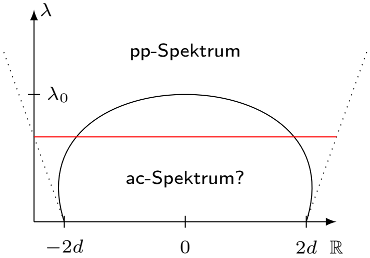

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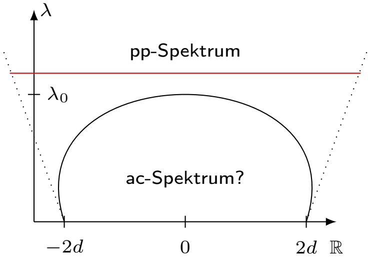

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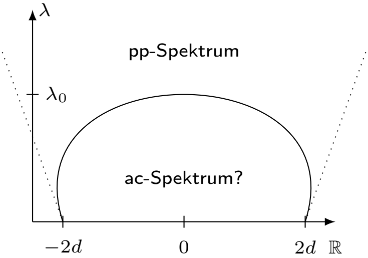

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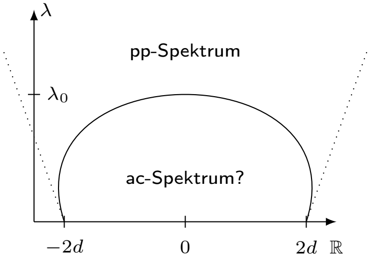

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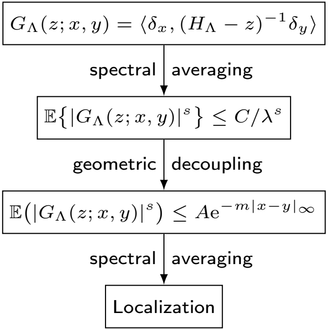
Aim:

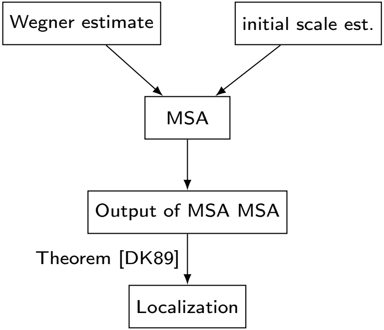
Study (1) and (3) in the non-monotone (long range) case!

Fractional moment method (FMM) vs. multiscale analysis (MSA)

FMM [AM93,AFSH01]



MSA [FS83,DK89]



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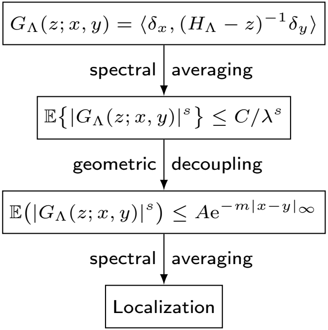

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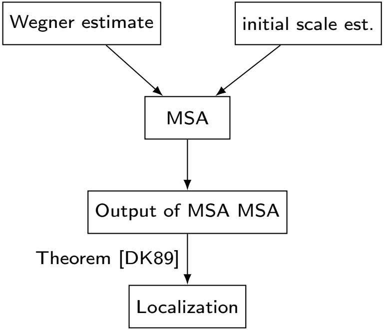
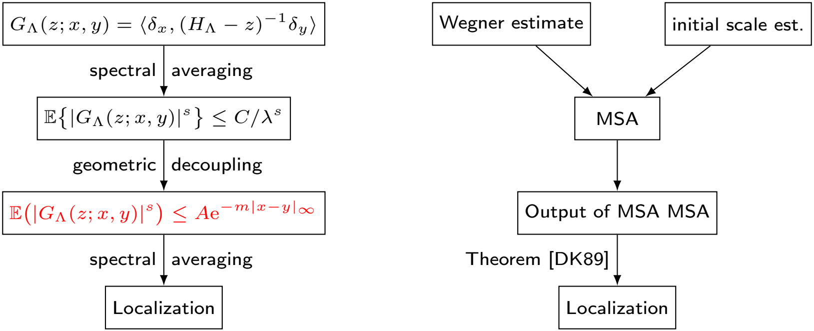


Both methods originally **strongly rely** on assumption $u \geq 0$.
If u changes sign:

- no results via FMM
- few results via MSA, e.g. [Klopp '95], [Veselić '02], [Krü12]

Fractional moment method (FMM) vs. multiscale analysis (MSA)

FMM [AM93,AFSH01] **MSA [FS83,DK89]**



New results for sign changing u:

- FMM [Elgart, T., Veselić '11] [Elgart, Shamis, Sodin 13]
- Wegner estimate [Peyerimhoff, T., Veselić '11]
- initial scale estimate and localization [Leonhardt, Peyerimhoff, T., Veselić '11]

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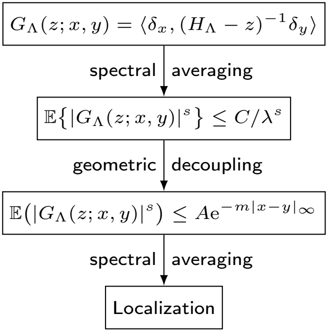

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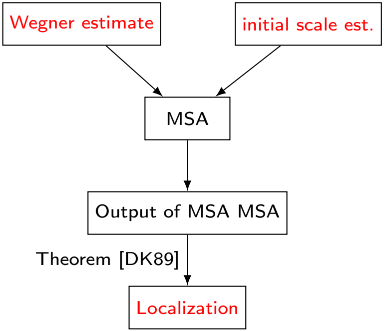

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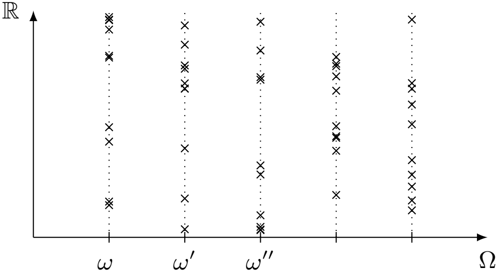
What is a Wegner estimate?

- consider box $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ and operator H_{Λ_L} on $\ell^2(\Lambda_L)$.



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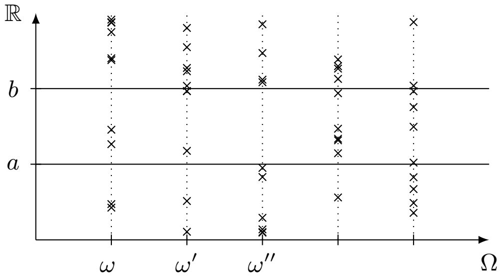

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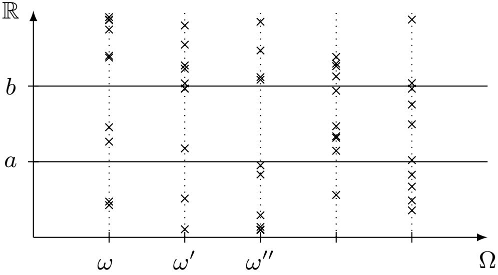

- Wegner estimate [Weg81] is upper bound on the expected number of eigenvalues of H_{Λ_L} in intervall $[a, b]$:

$$\forall\, L \in \mathbb{N} \text{ und } [a, b] \subset \mathbb{R} : \quad \mathbb{E}\big(\operatorname{Tr} \chi_{[a,b]}(H_{\Lambda_L})\big) \leq C_{\mathrm{W}}(b - a)|\Lambda_L|^m$$

with constants $C_{\mathrm{W}} > 0$ and $m \in [1, \infty)$.

Wegner estimate

Theorem 6 (Wegner estimate, [PTV11])

Let

- $|u(k)| \leq C\mathrm{e}^{-\alpha\|k\|_1}$ for all $k \in \mathbb{Z}^d$ and
- $\rho \in W^{1,1}(\mathbb{R})$.

Then there exist constants $C_u > 0$ und $m \in \mathbb{N}$, such that for all $L \in \mathbb{N}$ and all $[a,b] \subset \mathbb{R}$

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$$\mathbb{E}\big(\mathrm{Tr}\, \chi_{[a,b]}(H_{\Lambda_L})\big) \leq C_u \|\rho\|_{W^{1,1}} (b-a)(2L+1)^{2d+m}.$$

Remark

- allows long range interaction **and** non-monotonicity
- generalizes Wegner estimate of [Veselić 2010] where either
  - $d = 1$,
  - $\operatorname{supp} u$ finite, or
  - $\overline{u} := \sum_{k \in \mathbb{Z}^d} u(k) > 0$

  is assumed.

Localization

Wegner + Initial scale estimate $\Rightarrow$ Localization

strong disorder (λ large) Wegner implies initial scale estimate

weak disorder ($\lambda > 0$) u has small negative part

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Let u decay exponentially and $\rho \in W^{1,1}(\mathbb{R})$.

Theorem (Large disorder, [LPTV13])

Let λ be suff. large. Then, for almost all $\omega \in \Omega$, $\sigma_{\mathrm{c}}(H_\omega) = \emptyset$.

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Theorem (weak disorder, small negative part, [LPTV13])

Let $\overline{u} = \sum u(k) > 0$ and $\operatorname{supp} \mu = [0, \omega_+]$. There is $\delta, \varepsilon > 0$, such that if

$$u = u_+ - \delta u_- \quad \text{with} \quad \|u_-\|_1 \leq 1,$$

then, for almost all $\omega \in \Omega$, $\sigma_{\mathrm{c}}(H_\omega) \cap [-\varepsilon, \varepsilon] = \emptyset$.

Motivation

Discrete alloy-type model

Localization

Poisson statistics

Minami's estimate

Poisson statistics

- Molchanov 1981 : (1-dim model in continuum)
- Minami 1996: i.i.d. Anderson model on $\mathbb{Z}^d$, i.e. $u = \delta_0$
- Germinet & Klopp 2011: abstract framework for $\mathbb{Z}^d$-models

Main ingredient for proving **Poisson statistics in the localized regime** is

Minami's estimate

There is $C_{\mathrm{Min}} > 0$ such that for all $x, y \in \Lambda \subset \mathbb{Z}^d$ and $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ there holds

$$\mathbb{E}\left(\det\left(\operatorname{Im}\begin{pmatrix} G_\Lambda(z;x,x) & G_\Lambda(z;x,y) \\ G_\Lambda(z;y,x) & G_\Lambda(z;y,y)\end{pmatrix}\right)\right) \leq C_{\mathrm{Min}}$$

No proof for $d > 1$ & $u \neq \delta_0$.

Minami's estimate

Assumption (A)

- $\operatorname{supp} u$ finite
- μ has density $\rho \in W^{2,1}(\mathbb{R})$
- Fourier transform $\hat{u} : [0, 2\pi)^d \to \mathbb{C}$

$$\hat{u}(\theta) = \sum_{k \in \mathbb{Z}^d} u(k) \mathrm{e}^{\mathrm{i} k \cdot \theta}$$

  does not vanish.

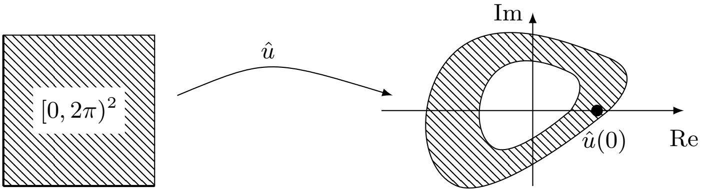
Theorem (T. & Veselić 2013)

Let Assumption (A) be satisfied. Then Minami's estimate holds.

Firs result on Minami's estimare with

- correlated potential values ($u \neq \delta_0$)
- non-monotone dependence on random parameters (u may change its sign)

$\hat{u} : [0, 2\pi)^d \to \mathbb{C}$, $\hat{u}(\theta) = \sum_{k \in \mathbb{Z}^d} u(k) \mathrm{e}^{i\langle k, \theta \rangle}$ does not vanish



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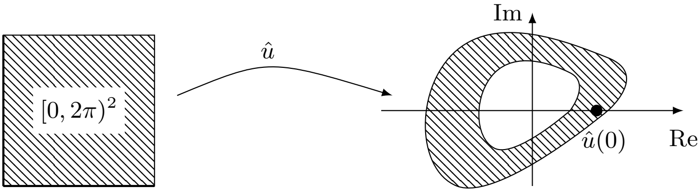

If $\overline{u} = \sum_{k \in \mathbb{Z}^d} u(k) = 0$, then $\hat{u}(0) = 0$ and Assumption (A) is not satisfied

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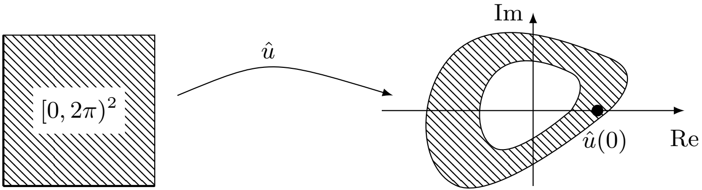

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Sufficient condition

$$|u(0)| > \sum_{k \neq 0} |u(k)| \quad \Rightarrow \quad \hat{u}(\theta) = u(0) + \sum_{k \neq 0} u(k) \mathrm{e}^{i\langle k, \theta \rangle} \neq 0 \quad \forall \theta$$

Corresponds to case $u = \delta_0 + \tilde{u}$ with $\sum_k |\tilde{u}(k)| < 1$.

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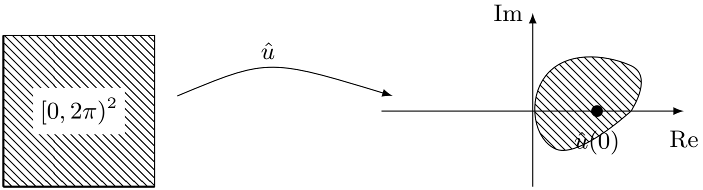

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Idea of proof

Schur complement formula gives

$$\begin{pmatrix} G_\Lambda(z;x,x) & G_\Lambda(z;x,y) \\ G_\Lambda(z;y,x) & G_\Lambda(z;y,y) \end{pmatrix} = \left(M + \begin{pmatrix} V_\omega(x) & 0 \\ 0 & V_\omega(y) \end{pmatrix} \right)^{-1} .$$

M depends on $V_\omega(k)$, $k \in \Lambda \setminus \{x, y\}$.

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Lemma [Min94,GV06]

Let $M = (m_{i,j})_{i,j=1}^2$ with $\operatorname{Im} M < 0$. Then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \det \left(\operatorname{Im} \left[M + \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \right]^{-1} \right) \mathrm{d}v_1 \mathrm{d}v_2 \leq \pi .$$

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Linear transformation $A : \Lambda_+ \to \Lambda_+$

- $A_{\Lambda_+}(x,y) = u(x-k)$ $x \in \Lambda$, $y \in \Lambda_+$
- A_{Λ_+} invertible

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linear transformation

$$\eta_{\Lambda_+} = A_{\Lambda_+} \omega_{\Lambda_+}$$

$$E = \int_{\mathbb{R}^n} \det \operatorname{Im} \left(M + \begin{pmatrix} \eta_x & 0 \\ 0 & \eta_y \end{pmatrix} \right)^{-1} k(\eta_{\Lambda_+}) \prod_{k \in \Lambda_+} \mathrm{d}\eta_k .$$

Since $\eta_k = V_\omega(k)$ for all $k \in \Lambda$, M is (analytically) independent of η_x, η_y.

$$E = \int_{\mathbb{R}^n} \left(M + \begin{pmatrix} \eta_x & 0 \\ 0 & \eta_y \end{pmatrix} \right)^{-1} k(\eta_{\Lambda_+}) \prod_{k \in \Lambda_+} \mathrm{d}\eta_k .$$

$$\leq \int_{\mathbb{R}^{n-2}} \sup_{\eta_x, \eta_y} k(\eta_{\Lambda_+}) \int_{\mathbb{R}^2} \det \operatorname{Im} \left(M + \begin{pmatrix} \eta_x & 0 \\ 0 & \eta_y \end{pmatrix} \right)^{-1} \mathrm{d}\eta_x \mathrm{d}\eta_y \prod_{k \in \Lambda_+ \setminus \{x,y\}} \mathrm{d}\eta_k$$

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It remains to show that Assumption (A) implies

- A_{Λ_+} is invertible
- there is $C_u > 0$ such that for all $\Lambda_+ \subset \mathbb{Z}^d$

$$\int_{\mathbb{R}^{n-2}} \sup_{\eta_x, \eta_y} k(\eta_{\Lambda_+}) \prod_{k \in \Lambda_+ \setminus \{x,y\}} \mathrm{d}\eta_k \leq C_u$$