Compactness properties of Schrödinger type operators

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A quote from Marx

Outside of a dog, a book is a man's best friend. Inside of a dog it's too dark to read.
Groucho Marx

Outside of a dog, compactness is an analyst's best friend. Inside of a dog it's relatively compact, anyway.
PS

Basic notions and results

Definition

In a topological space X a set K is called *compact*, if every open cover of K contains a finite subcover.

Theorem

In a metric space X for a set K the following conditions are equivalent:

(i) *K is compact.*

(ii) *K is precompact and complete.*

(iii) *K is sequentially compact.*

This contains as special cases the famous theorems of *Heine-Borel* and *Bolzano-Weierstrass*.

Basic notions and results

Compactness is at the heart of many existence results in analysis:

- On a compact set a real valued, continuous function attains its maximum and minimum.
- Peano's existence theorem for ODE is based on the Arzela-Ascoli theorem.
- Compactness is hidden in many important results from Hilbert space theory, because the unit ball of a Hilbert space is weakly compact.
- Compact operators allow a useful representation as an infinite matrix - the application of the corresponding techniques belong to the landmark results of Functional Analysis.
- Compact perturbations leave the essential spectrum of operators invariant.

Definition and Examples

Definition

A linear operator $T : E \to F$, E, F Banach spaces is called *compact*, if the image $T(B_E)$ of the unit ball of E is precompact. We write $\mathcal{K}(E, F)$ for the closed ideal (!) of compact operators from E to F.

- Finite rank operators are compact.
- Multiplication with a sequence in c_0 is a compact operator on any ℓ^p.
- Many integral operators are compact operators.
- The embedding of the Sobolev space $W_0^{1,2}(\Omega) \to L^2(\Omega)$ is compact for any open set Ω of finite measure in $\mathbb{R}^d$.

Absolutely p-summing operators

Throughout, E, F will denote Banach spaces, and $\|\cdot\|$ the respective norm.

Definition

A bounded operator T is called *absolutely p-summing*, if there exists a constant C s.t.

$$\left(\sum_n \|Tx_n\|^p\right)^{\frac{1}{p}} \leq C \sup_{x' \in E', \|x'\| \leq 1} \left(\sum_n |< x_n, x' >|^p\right)^{\frac{1}{p}}.$$

The smallest such constant, $\|T\|_{\mathcal{P}_p}$ is called the *p-summing norm*, and the ideal of these operators is denoted by $\mathcal{P}_p(E, F)$.

Hilbert Schmidt operators

If H_1, H_2 are Hilbert spaces, we have that

$$\begin{aligned}\mathcal{P}_2(H_1, H_2) &= \mathcal{HS}(H_1, H_2)\\ &:= \{T : H_1 \to H_2 \mid \sum_n \|Te_n\|^2 < \infty, \text{ for any ONB}\},\end{aligned}$$

moreover

$$\|T\|_{\mathcal{P}_2} = \|T\|_{\mathcal{HS}}$$

- An operator between L^2 spaces is Hilbert-Schmidt, iff it has an L^2-kernel.
- Important property: $\mathcal{HS}(H_1, H_2)$ is a Hilbert space.
- Hilbert-Schmidt norms are easy to calculate.

My favorite HS criterion

One of the deepest results in Functional Analysis is the celebrated GT, originally called *theorème fondamental de la théorie métrique des produits tensoriels topologiques* by Alexandre Grothendieck. A consequence is:

Theorem (Little GT)

An operator T between Hilbert spaces is Hilbert-Schmidt, provided it factors over L^∞, $C(K)$ or L^1. More precisely, if $T = BA$, where $B \in \mathcal{L}(L^\infty, H_2), A \in \mathcal{L}(H_1, L^\infty)$, then

$$\|T\|_{\mathcal{HS}} \leq K_G \|A\| \|B\|.$$

We'll see a baby version soon.

Trace class operators

An operator $T \in \mathcal{L}(E,F)$ is said to be of *trace class*, whenever there are $x_n' \in E'$, $y_n \in F$ such that

$$Tx = \sum_n < x, x_n' > y_n \text{ and } \sum_n \|x_n'\| \|y_n\| < \infty.$$

We write $\mathcal{N}(E,F)$ for the ideal of these operators and $\|\cdot\|_{tr}$ for its natural norm. There is a close analogy between trace ideals and sequence spaces via

$$\begin{aligned} \mathcal{N} &\leftrightsquigarrow \ell^1 \\ \mathcal{HS} &\leftrightsquigarrow \ell^2 \\ \mathcal{K} &\leftrightsquigarrow c_0 \\ \mathcal{L} &\leftrightsquigarrow \ell^\infty \end{aligned}$$

If $A, B \in \mathcal{HS}$ then $AB \in \mathcal{N}$, and the natural inner product on $\mathcal{HS}$ is given by $(A \mid B)_{\mathcal{HS}} = tr(B^*A)$.

A trace class criterion

We are concerned with operators on $L^2(X)$.

Theorem (Demuth, PS, Stolz, van Casteren)

Let $A \in \mathcal{L}(L^1, L^2), B \in \mathcal{L}(L^2, L^1)$ and assume that there exists $\Phi \in L^1$ s.t.

$$|Bf| \leq \Phi \text{ for every } \|f\| \leq 1.$$

Then

$$\|AB\|_{tr} \leq \|\Phi\|_1 \|A\|.$$

You can see the connection with little GT viz:

$$\begin{aligned} \Phi^{-1}B : L^2 \to L^\infty &\implies \Phi^{\frac{1}{2}}\Phi^{-1}B \in \mathcal{HS} \\ A\Phi^{\frac{1}{2}} : L^2 \to L^1 \to L^2 &\implies A\Phi^{\frac{1}{2}} \in \mathcal{HS} \\ \Rightarrow AB &= A\Phi^{\frac{1}{2}}\Phi^{\frac{1}{2}}\Phi^{-1}B \in \mathcal{N}. \end{aligned}$$

Application in perturbation theory

Consider $-\Delta$ on $\mathbb{R}^d$ (for definiteness)and suppose that two reasonable potentials V_0, V are given that differ only on a set B (you can think of as compact). Can we estimate

$$\|e^{-t(-\Delta+V)} - e^{-t(-\Delta+V_0)}\|_{\mathcal{HS},tr} \leq c(t,B)$$

irrespective of the size of the difference $V - V_0 \geq 0$ on B?
The answer is yes, and $c(t,B)$ is related to the capacity of B.
See PS, '94 and Demuth, PS, Stolz, van Casteren '95.
The above estimate has applications in Mathematical Physics, as trace class perturbations leave the absolutely continuous spectrum invariant. Moreover, in the latter paper, we find a natural HS operator with finite trace that is *not* trace class.

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Motivation

- *Friedrichs '34*: If $V(x) \to \infty$ as $|x| \to \infty$, then $e^{-t(-\Delta+V)} \in \mathcal{K}$ ($\Longrightarrow \sigma_{ess}(-\Delta + V) = \emptyset$)
- *Wang, Wu '08*: Generalization: $-\Delta \rightsquigarrow H_0$ $V(x) \to \infty \rightsquigarrow m\{V < E\} < \infty$ for all E. Very complicated proof :(
- *Simon '09*: little less general, much easier proof.
- *Lenz, PS, Wingert '10*: Even more general and even easier proof.

Question: Suppose $H_0 \geq 0$ in $L^2(X)$; what is the right condition to ensure that

$$\sigma_{ess}(H_0 + V) = \emptyset?$$

Answer: $1_{\{V<E\}}$ is $H_0 + V$*-relatively compact*!

Relative compactness

Fix a Hilbert space $\mathcal{H}$ and a selfadjoint operator H. $B \in \mathcal{L}(\mathcal{H})$ is called *H-relatively compact*, if $B 1_J(H) \in \mathcal{K}$ for any compact interval J. We use functional calculus!

Theorem

Let H be selfadjoint and $B \in \mathcal{L}$.

(1) *The following assertions are equivalent:*
   - (i) *B is H-relatively compact.*
   - (ii) *There is some $\varphi \in C(\sigma(H))$ with $|\varphi| > 0$ s.t. $B\varphi(H) \in \mathcal{K}$.*
   - (iii) *For all $\varphi \in C_0(\sigma(H)) : B\varphi(H) \in \mathcal{K}$.*

(2) *Let $g \in C(\sigma(H), \mathbb{R})$ with $g(t) \to \infty$ as $|t| \to \infty$. Then B is $g(H)$- relatively compact whenever B is H-relatively compact.*

Relative compactness in terms of spaces

In case $H \geq 0$,

$$\mathcal{H}^p(H) = D((H+s)^{p/2}),$$

gives a scale of spaces (Sobolev spaces); $Q(H) = \mathcal{H}^1(H)$, $D(H) = \mathcal{H}^2(H)$

Theorem

The following are equivalent:

(i) *B is H-relatively compact,*

(ii) *$B(H-\lambda)^{-k} \in \mathcal{K}$ for some (all) $\lambda \in \rho(H)$, $k \in \mathbb{N}$,*

(iii) *$B : D(H) \to \mathcal{H}$ is compact.*

If $H \geq \gamma$ these conditions are in turn equivalent to each of the following:

(iv) *Be^{-tH} is compact for some (all) $t > 0$,*

(v) *$B : Q(H) \to \mathcal{H}$ is compact,*

(vi) *$B : \mathcal{H}^p(H) \to \mathcal{H}$ is compact for some (all) $p > 0$,*

(vii) *for any $C > 0$ and $(\psi_n) \subset Q(H)$ with $h[\psi_n] \leq C$ and $\psi_n \to 0$ weakly, it follows that $B\psi_n \to 0$ in norm.*

The main result

We now specialize to $\mathcal{H} = L^2(X)$; $H_0 \geq 0$, $V : X \to [0, \infty]$ measurable.
$H = H_0 + V$ defined as the form sum.

Theorem

If $1_{\{V<E\}}$ *is H-relatively compact for some* $E \in \mathbb{R}$, *then*

$$\sigma_{ess}(H_0 + V) \subset [E, \infty).$$

For the proof, let $\lambda \in \sigma_{ess}(H_0 + V)$. Then there exists a Weyl sequence, i.e.

$$\|f_n\| = 1, f_n \to 0 \text{ weakly }, \|(H - \lambda)f_n\| \to 0.$$

Assumption and equivalent reformulation: $\|1_{\{V<E\}} f_n\| \to 0$
$\Rightarrow \|1_{\{V \geq E\}} f_n\| \to 1$.

Proof of the main result

$$
\begin{aligned}
\lambda &= \lim_n (Hf_n \mid f_n) \\
&= \lim_n \left[(H_0 f_n \mid f_n) + (Vf_n \mid f_n)\right] \\
&\geq 0 + \liminf_n (Vf_n \mid f_n) \\
&= \liminf_n \left[(V1_{\{V<E\}} f_n \mid f_n) + (V1_{\{V\geq E\}} f_n \mid f_n)\right] \\
&\geq \liminf_n E(1_{\{V\geq E\}} f_n \mid f_n) \\
&= E
\end{aligned}
$$

☺

Application to the original question

$\mathcal{H} = L^2(X)$; $H_0 \geq 0$, $V : X \to [0, \infty]$ measurable.

Theorem

Assume $e^{-tH_0} : L^2 \to L^\infty$ *for some* $t > 0$ *and* $m\{V < E\} < \infty$ *for all* E. *Then*

$$\sigma_{ess}(H_0 + V) = \emptyset.$$

The proof is very short: by little GT:

$$1_{\{V<E\}} e^{-t(H_0+V)} : L^2 \to L^\infty \to L^2 \text{ is HS}$$

☺

Corollary

Assume that H_0 *is subelliptic on* $L^2(\mathbb{R}^d)$ *in the sense that* $Q(H_0) \subset W^{s,2}(\mathbb{R}^d)$ *continuously, for some* $s > 0$. *If* $m\{V < E\} < \infty$ *for all* E. *Then*

$$\sigma_{ess}(H_0 + V) = \emptyset.$$

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(In the latter paper more references can be found)