Weyl theory and pseudospectral functions of canonical and Dirac systems

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The results are contained in:
A. Sakhnovich, L. Sakhnovich and I. Roitberg,
"Inverse problems and nonlinear evolution equations. Solutions, Darboux matrices and Weyl-Titchmarsh functions,"
De Gruyter, 2013.
See also
B. Fritzsche, B. Kirstein, I.Ya. Roitberg, and A.L. Sakhnovich,
Skew-Self-Adjoint Dirac System with a Rectangular Matrix Potential: Weyl Theory, Direct and Inverse Problems,
IEOT **74**:2 (2012), 163–187
and some other references in the book above.

The well-known canonical system has the form

$$\frac{dy(x,z)}{dx} = \mathrm{i}zJH(x)y(x,z), \tag{1}$$

$$J := \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \quad H(x) = H(x)^* \geq 0. \tag{2}$$

The $2p \times 2p$ matrix solution W of (1) normalized by the condition

$$W(0,z) = I_m, \quad m = 2p$$

is called the *fundamental solution*. H is called *Hamiltonian*.
$L^2(H,r)$ is the space of vector functions on $(0,\, r)$ with the product

$$(f, \widetilde{f})_H = \int_0^r \widetilde{f}(x)^* H(x) f(x) dx.$$

Introduce also the space $L^2_p(\tau)$ of vector functions with the product

$$\langle f, \widetilde{f} \rangle_\tau = \int_{-\infty}^{\infty} \widetilde{f}(t)^* d\tau(t) f(t) < \infty,$$

where τ is a nondecreasing $p \times p$ matrix function.

We set $$Uf := \begin{bmatrix} 0 & I_p \end{bmatrix} \int_0^r W(x,\overline{z})H(x)f(x)dx.$$

Definition. A nondecreasing $\tau(t)$ is called a *spectral* matrix function of (1) if U maps $L^2(H)$ isometrically into $L^2_p(d\tau)$.

Put $$\ker U = \left\{ f \in L^2(H) : \ Uf \equiv 0 \right\}, \quad L_1 = L^2(H) \ominus \ker U.$$

Definition. A nondecreasing matrix function $\tau(t)$ is called *pseudospectral* if U maps L_1 isometrically into $L^2_p(d\tau)$.

We describe spectral and pseudospectral functions in terms of Möbius transformations. The *de Branges space* B of functions

$$F = \int_0^r W(x,\overline{z})H(x)f(x)dx \quad (f \in H),$$

where $(F,\, F)_B := (f,\, f)_H$, is essential in our considerations.

Canonical system $dy(x,z)/dx = \mathrm{i}zJH(x)y(x,z)$ is important in itself and includes as particular cases Schrödinger and string equations and Dirac systems. The history of its study goes back to H. Poincare and A.M. Lyapunov. In the more recent period canonical system was studied, in particular, by L. de Branges, Gelfand-Lidskii, I. Gohberg, I. Kac, M.G. Krein and L. Sakhnovich. See more details in the Gohberg-Krein book on Volterra operators (ch. 6) and L. Sakhnovich book, OT: Adv. and Appl., vol. 107. See also quite recent papers by M. Langer, H. Woracek, H. Winkler and coauthers.

We note that in the case $\det H \neq 0$ (i.e., in the case of the strict inequality $H(x) > 0$) canonical system can be rewritten in the form $-\mathrm{i}H^{-1}J\frac{d}{dx}y = zy$ and U from the previous frame diagonalizes operators given by the diff.expr. $-\mathrm{i}H^{-1}J\frac{d}{dx}$ and boundary conditions $[I_p \quad 0]y(0) = 0, \quad y(r) = 0$:

$$U\left(-\mathrm{i}H^{-1}J\frac{d}{dx}\right)y = zUy.$$

The singular case, where $\det H$ may turn to zero and indivisible intervals appear, is more complicated but quite interesting.

Consider Möbius transformations

$$\varphi(z) = \mathrm{i}\big(a(z)\mathcal{P}_1(z) + b(z)\mathcal{P}_2(z)\big)\big(c(z)\mathcal{P}_1(z) + d(z)\mathcal{P}_2(z)\big)^{-1},$$

where $z \in \mathbb{C}_+$; a, b, c and d are $p \times p$ blocks of $W(r, \overline{z})^*$, and $\mathcal{P}_1$, $\mathcal{P}_2$ are nonsingular pairs with property-J, i.e., we require in $\mathbb{C}_+$:

$$\det\big(c(z)\mathcal{P}_1(z) + d(z)\mathcal{P}_2(z)\big) \neq 0, \quad W(r, \overline{z})^* =: \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix};$$

$$\mathcal{P}_1, \mathcal{P}_2 \quad \text{are meromorphic}, \quad \mathcal{P}_1(z)^*\mathcal{P}_1(z) + \mathcal{P}_2(z)^*\mathcal{P}_2(z) > 0,$$

$$\begin{bmatrix} \mathcal{P}_1(z)^* & \mathcal{P}_2(z)^* \end{bmatrix} J \begin{bmatrix} \mathcal{P}_1(z) \\ \mathcal{P}_2(z) \end{bmatrix} \geq 0.$$

Then, $\varphi(z)$ are well-defined and admit Herglotz representation

$$\varphi(z) = \mu z + \nu + \int_{-\infty}^{\infty} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\tau(t),$$

where $\mu \geq 0$, $\nu = \nu^*$, and τ is nondecreasing.

Denote the class of these φ by $\mathcal{N}(r)$.

Thus, we study operators U acting from $L^2(H)$ into $L^2_p(d\tau)$:

$$Uf = U(\tau)f := \begin{bmatrix} 0 & I_p \end{bmatrix} \int_0^r W(x,\bar{z})H(x)f(x)dx, \quad f \in L^2(H),$$

and functions $\varphi \in \mathcal{N}(r)$ with their Herglotz representations:

$$\varphi(z) = \mathrm{i}\big(a(z)\mathcal{P}_1(z) + b(z)\mathcal{P}_2(z)\big)\big(c(z)\mathcal{P}_1(z) + d(z)\mathcal{P}_2(z)\big)^{-1},$$

$$\varphi(z) = \mu z + \nu + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\tau(t). \qquad (3)$$

When $\mathcal{P}_1(z) = \mathcal{P}_2(z) \equiv I_p$ we add indices "0" into equalities above and write φ_0, μ_0, ν_0 and τ_0.

Theorem 1. If $\varphi \in \mathcal{N}(r)$, then $U(\tau)$ is contractive.

The description of pseudospectral matrix functions τ requires some additional conditions.

Recall that $\varphi_0(z) = \mathrm{i}\big(a(z) + b(z)\big)\big(c(z) + d(z)\big)^{-1}$, where a, b, c and d are the blocks of $W(r, \overline{z})^*$, and that μ_0, ν_0 and $\tau_0(t)$ are uniquely recovered from the Herglotz representation of φ_0.

Hypothesis I. If $c(z)h \equiv 0$ ($h \in \mathbb{C}^p$), then $h = 0$.

Theorem 2. (a) Let $\varphi \in \mathcal{N}(r)$ and let the condition

$$\lim_{\eta\to\infty} \eta^{-1}\big(c(-\mathrm{i}\eta)^* - d(-\mathrm{i}\eta)^*\big) \times \big(\varphi(\mathrm{i}\eta) - \varphi_0(\mathrm{i}\eta)\big)\big(c(\mathrm{i}\eta) - d(\mathrm{i}\eta)\big) = 0 \qquad (4)$$

hold. Then, the distribution function τ from the Herglotz representation of φ is pseudospectral.

(b) Let τ be pseudospectral, let $\mu_0 = 0$ and let Hypothesis I be valid. Then there exists $\varphi \in \mathcal{N}(r)$ with distribution function τ. For this function φ, equality (4) holds.

Positivity condition (see Gohberg-Krein book (1970), Ch. 6):

$\int_0^r H(x)dx > 0$.

If *positivity condition* holds, then Hypothesis I is valid and $\det\big(c(z)\mathcal{P}_1(z) + d(z)\mathcal{P}_2(z)\big) \neq 0$.

The positivity condition

$$H(x) \geq 0, \quad \int_0^r H(x)dx > 0 \tag{5}$$

is essentially weaker than the non-degeneracy condition that $H(x) > 0$ almost everywhere.

If $H(x) > 0$, then all the conditions of Theorem 2 are fulfilled automatically and we also have $\ker U = 0$.

Theorem 3. Let $H(x) > 0$ almost everywhere. Then the set of spectral functions of the canonical system coincides with the set of distribution functions from Herglotz representations of functions $\varphi \in \mathcal{N}(r)$.

Further we consider canonical system on the semiaxis:

$$dy(x,z)/dx = \mathrm{i}zJH(x)y(x,z), \quad 0 \leq x < \infty. \tag{6}$$

Then $\mathcal{N}(r_1) \subseteq \mathcal{N}(r_2)$ for $r_1 > r_2$.

If (5) holds for some $\widehat{r} > 0$ (and so for all $r > \widehat{r}$), then $\bigcap_{r<\infty} \mathcal{N}(r) \neq \emptyset$.

Recall that we consider canonical system

$$dy(x,z)/dx = \mathrm{i}zJH(x)y(x,z),$$

where $0 \le x < \infty$ and $\int_0^{\widehat{r}} H(x)dx > 0$.

Then, there is $\varphi(z) \in \bigcap_{r<\infty} \mathcal{N}(r)$, see the previous frame. Such $\varphi(z)$ satisfy (for $z \in \mathbb{C}_+$) the inequality

$$\int_0^{\infty} \begin{bmatrix} I_p & \mathrm{i}\varphi(z)^* \end{bmatrix} W(x,z)^* H(x) W(x,z) \begin{bmatrix} I_p \\ -\mathrm{i}\varphi(z) \end{bmatrix} dx < \infty. \tag{7}$$

Definition. Holomorphic functions φ satisfying (7) are called Weyl functions of the canonical system on the semiaxis $0 \le x < \infty$.

Thus, positivity condition yields the existence of the Weyl function.
Uniqueness theorem. Let Hamiltonian H of the canonical system be locally summable on $[0,\,\infty)$ and satisfy positivity condition for some $\widehat{r} > 0$. Suppose additionally that $H(x) \ge -\delta J$ for some $\delta > 0$.

Then, there exists a unique Weyl function of the canonical system on $[0,\,\infty)$.

Consider canonical system on $[0, \infty)$ and assume that $\bigcap_{r<\infty} \mathcal{N}(r) = \varphi(z)$ (e.g., assume that the conditions of the Uniqueness theorem from the previous frame hold). Furthermore, assume that τ, which corresponds to φ, satisfies the Szegö cond.

$$\int_{-\infty}^{\infty} \left(1+t^2\right)^{-1} \ln\left(\det \tau'(t)\right) dt > -\infty.$$

Then (see Krein-Zasukhin theorem), τ' admits the factorization

$$\tau'(t) = \upsilon(t)^* \upsilon(t), \tag{8}$$

where the analytic (in $\mathbb{C}_+$) $p \times p$ matrix function $\upsilon(z)$, having the limit $\upsilon(t)$ on $\mathbb{R}$, belongs $\widetilde{D}_S$. That is, the entries of $\upsilon\big(z(\lambda)\big)^{\pm 1}$, with the mapping $z(\lambda)$ of the form $z(\lambda) = (\overline{\alpha}\lambda - \alpha)/(\lambda - 1)$, $\alpha \in \mathbb{C}_+$, belong to the Smirnov class of functions in the unit disc.

Theorem 4. If $c(r,z) \in \widetilde{D}_S$ (for all $r > r_0$ and some $r_0 > 0$), then we have the asymptotics

$$\lim_{r\to\infty} W(r,\overline{z})^* J W(r,\overline{\zeta}) = \frac{1}{2\pi} \begin{bmatrix} -\mathrm{i}\varphi(z) \\ I_p \end{bmatrix} \upsilon(z)^{-1} \times \left(\upsilon(\zeta)^*\right)^{-1} \begin{bmatrix} \mathrm{i}\varphi(\zeta)^* & I_p \end{bmatrix} \quad (z, \zeta \in \mathbb{C}_+).$$

Theorem 1 and Theorem 2 (b) are proved using an analog of the Potapov's Transformed Fundamental Matrix Inequality. Namely, assuming that $F, \widetilde{F} \in B$ and $\varphi \in \mathcal{N}(r)$ we use the inequality

$$(\widetilde{F}, \widetilde{F})_B + \Big(R_\omega F - G(z,\omega), \widetilde{F}\Big)_B + \Big(\widetilde{F}, R_\omega F - G(z,\omega)\Big)_B + (\overline{\omega} - \omega)^{-1}\Big(\Phi(\overline{\omega}) - \overline{\Phi(\overline{\omega})}\Big) \geq 0, \qquad \omega \in \mathbb{C}_-, \qquad (9)$$

where the functions Φ and G and operator R_ω are given by

$$\Phi(\overline{\omega}) := \big(F, R_\omega F - G(z,\omega)\big)_B,$$

$$G(z,\omega) := \frac{W(r,\overline{z})^* J W(r,\omega) - J}{z-\omega} \begin{bmatrix} I_p \\ -\mathrm{i}\varphi(\overline{\omega})^* \end{bmatrix} F_2(\omega),$$

$$R_\omega F = (z-\omega)^{-1}\big(F(z) - F(\omega)\big),$$

and $F_2(\omega) \in \mathbb{C}^p$ is the second block of $F(\omega)$.

We note that some restrictions of Theorems 1 and 2 for the case $p = 1$ were obtained earlier in a paper by L. Golinskii and I. Mikhailova (edited by V.P. Potapov) using J-theory and Potapov's Fundamental Matrix Inequality (nontransformed), see Oper. Theory: Adv. Appl., vol. 95 (1997), pp. 205-251.

The subcase of canonical systems

$$dy(x,z)/dx = \mathrm{i}zJH(x)y(x,z),$$

where $H = \gamma^*\gamma$ and the $p \times m$ $(m = 2p)$ matrix functions γ satisfy the identity $\gamma J \gamma^* = -I_p$, is equivalent to the self-adjoint Dirac system

$$\frac{d}{dx}u(x,z) = \mathrm{i}\Big(zj + jV(x)\Big)u(x,z), \quad j = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \tag{10}$$

$$V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}. \tag{11}$$

The more general case of the selfadjoint Dirac systems (10), where

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad m = m_1 + m_2, \tag{12}$$

and the potential v is an $m_1 \times m_2$ rectangular matrix function, is of interest.

Using the method of operator identities by L.A. Sakhnovich, we generalized the results from
A.L. Sakhnovich, Dirac type and canonical systems: spectral and Weyl-Titchmarsh fuctions, direct and inverse problems,
Inverse Problems **18** (2002), 331–348
for the the cases of *selfdjoint* and *skew-selfadjoint* Dirac systems with rectangular potentials.

Next, we consider the *skew-selfadjoint* Dirac system

$$\frac{d}{dx}u(x,z) = (\mathrm{i}zj + jV(x))u(x,z), \quad j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix} \tag{13}$$

on the interval $[0, r]$ and semiaxis $[0, \infty)$. The $m \times m$ fundamental solution u of this system is normalized by the condition

$$u(0,z) = I_m.$$

Direct problem. Weyl functions for Dirac system

$$du(x,z)/dx = (\mathrm{i}zj + jV(x))u(x,z) \tag{14}$$

are considered in some half-plane $\mathbb{C}_M = \{z : z \in \mathbb{C}, \quad \Im z > M\}$. These Weyl functions are constructed via Möbius transformations

$$\varphi(x,z,\mathcal{P}) = \Big(Y_{21}(x,z)\mathcal{P}_1(z) + Y_{22}(x,z)\mathcal{P}_2(z) \Big) \times \Big(Y_{11}(x,z)\mathcal{P}_1(z) + Y_{12}(x,z)\mathcal{P}_2(z) \Big)^{-1}, \tag{15}$$

where Y_{ik} are the blocks of u^{-1} and parameter matrix functions $\mathcal{P}_1$ and $\mathcal{P}_2$ are $m_1 \times m_2$ and $m_2 \times m_2$, respectively, matrix functions, which are meromorphic in $\mathbb{C}_M$ and satisfy inequalities

$$\mathcal{P}_1(z)^*\mathcal{P}_1(z) + \mathcal{P}_2(z)^*\mathcal{P}_2(z) > 0, \quad \mathcal{P}_1(z)^*\mathcal{P}_1(z) \geq \mathcal{P}_2(z)^*\mathcal{P}_2(z)$$

for $z \in \mathbb{C}_M$ (excluding, possibly, a discrete set of points).
NB. If $\|v(x)\| \leq M$ for all $x \leq r$, then formula (15) for $x \leq r$ is well-defined in $\mathbb{C}_M$.
Definition. Let system (14) be given on $[0, r]$ and let $\|v(x)\| \leq M$ for all $x \leq r$. Then Weyl functions are given by (15) with $x = r$.

Consider Dirac system on the semiaxis $[0, \infty)$.
The set of values of our Möbius transformations

$$\varphi(x,z,\mathcal{P}) = \Big(Y_{21}(x,z)\mathcal{P}_1(z) + Y_{22}(x,z)\mathcal{P}_2(z)\Big) \times \Big(Y_{11}(x,z)\mathcal{P}_1(z) + Y_{12}(x,z)\mathcal{P}_2(z)\Big)^{-1},$$

at the fixed points x and z ($z \in \mathbb{C}_M$) is denoted by $\mathcal{N}(x,z)$.

Definition. Let $\|v(x)\| \leq M$ for all $x < \infty$. Then a Weyl function is a function s.t.

$$\int_0^\infty \begin{bmatrix} I_{m_1} & \varphi(z)^* \end{bmatrix} u(x,z)^* u(x,z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} dx < \infty, \quad z \in \mathbb{C}_M. \tag{16}$$

Theorem 5. There is a unique Weyl function of the skew-selfadjoint Dirac system on the semiaxis. It is analytic and contractive in $\mathbb{C}_M$ and is given by the relation

$$\varphi(z) = \bigcap_{x<\infty} \mathcal{N}(x,z).$$

Inverse problem. Theorem 6. Let the potential v of the Dirac system on $[0,\, r]$ be bounded. Then v can be uniquely recovered from the Weyl function φ in the following way.
First, we construct the $m_2 \times m_1$ matrix function Φ_1:

$$\Phi_1\Big(\frac{x}{2}\Big) = \frac{1}{\pi}\mathrm{e}^{x\eta}\mathrm{l.i.m.}_{a\to\infty}\int_{-a}^{a}\mathrm{e}^{-\mathrm{i}x\xi}\frac{\varphi(\xi+\mathrm{i}\eta)}{2\mathrm{i}(\xi+\mathrm{i}\eta)}d\xi, \quad \eta > M.$$

Next, we construct the bounded in $L^2_{m_2}(0,r)$, invertible and strictly positive operator S:

$$S = I + \int_0^r s(x,t)\cdot dt, \;\; s(x,t) = \int_0^{\min(x,t)} \Phi_1'(x-\zeta)\Phi_1'(t-\zeta)^* d\zeta.$$

Finally, we construct an $m_1 \times m$ matrix function β:

$$\beta(x) = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} - \int_0^x \Big(S_x^{-1}\Phi_1'\Big)(t)^* \begin{bmatrix} \Phi_1(t) & I_{m_2} \end{bmatrix} dt,$$

where S_x is the block of S, which maps the subspace $L^2_{m_2}(0,x)$ onto $L^2_{m_2}(0,x)$. We obtain the potential v via the formula

$$v(x) = \beta'(x)\gamma(x)^*,$$

where the $m_2 \times m$ matrix function γ is uniquely recovered from β.

Recall our last formula $v = \beta' \gamma^*$. Here β and γ are the block rows of the fundamental solution $u(x, z)$ at $z = 0$:

$$\beta(x) = \begin{bmatrix} I_{m_1} & 0 \end{bmatrix} u(x, 0), \quad \gamma(x) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix} u(x, 0).$$

The matrix function γ is easily recovered from β using equalities

$$\gamma(0) = \begin{bmatrix} 0 & I_{m_2} \end{bmatrix}, \quad \gamma' \gamma^* \equiv 0, \quad \beta \gamma^* \equiv 0.$$

Inverse problem on the semiaxis.

Since the Weyl function φ of the Dirac system on the semiaxis is given by $\varphi(z) = \bigcap_{x<\infty} \mathcal{N}(x, z)$ (see Theorem 5), this Weyl function is a Weyl function of the same Dirac system on all the intervals $[0, r]$. Thus, our procedure also grants the solution of the inverse problem on the semiaxis.

Weyl functions for Dirac systems on the semiaxis satisfy for all $r < \infty$ the inequalities

$$\sup_{x \leq r, z \in \mathbb{C}_M} \left\| \mathrm{e}^{-\mathrm{i}zx} u(x, z) \begin{bmatrix} I_{m_1} \\ \varphi(z) \end{bmatrix} \right\| < \infty. \tag{17}$$

Inequalities (17) are used as the definiton of the generalized Weyl functions. In terms of the generalized Weyl functions, we deal with the inverse problem for the case of locally bounded potentials. Easy sufficient condition, under which φ is a generalized Weyl function, are also given.