A Carleman estimate for elliptic second order PDE

Christian Rose (TU Chemnitz)

in part joint with I. Nakić, M. Tautenhahn

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Theorem [Bourgain, Kenig 05]

There exist $C(d), D(d), E(d) > 0$ and a function $w\colon \mathbb{R}^d \to \mathbb{R}$ such that for all $f \in \mathcal{C}_c^\infty(B_1(0) \setminus \{0\})$ and all $\alpha > D$

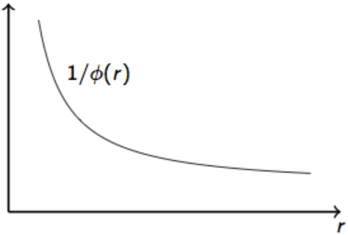
$$\alpha^3 \int w^{-1-2\alpha} f^2 \leq E \int w^{2-2\alpha} (\Delta f)^2.$$

weight function

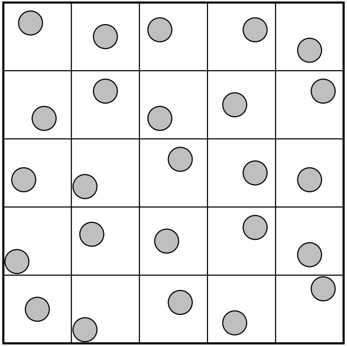
$$\phi\colon \mathbb{R}_+ \to \mathbb{R}_+,$$

$$\phi(r) = r \exp\left(\int_0^\infty \frac{\mathrm{e}^{-\mathrm{t}} - 1}{\mathrm{t}} \mathrm{dt} \right),$$

$$w\colon \mathbb{R}^d \to \mathbb{R},\ w(x) = \phi(|x|).$$



Application: Scale-free unique continuation principle



- $\Lambda_L = (-L/2, L/2)^d$
- $S_L = \Lambda_L \cap \left(\bigcup_{j \in \mathbb{Z}^d} B(x_j, \delta) \right)$
- $H_L = (-\Delta + V)_{\Lambda_L}$ with Dirichlet or periodic b.c. at $\partial\Lambda_L$

Theorem [Rojas-Molina & Ves. 13]

Let $\delta, K > 0$. Then $\exists\, C \in (0, \infty)$, s.t. $\forall$

- $V : \mathbb{R}^d \to [-K, K]$, $L \in \mathbb{N}$
- $\psi \in W^{2,2}(\Lambda_L; \mathbb{R})$, $H_L \psi = 0$
- $(x_j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}^d$, $B(x_j, \delta) \subset \Lambda_1 + j$

$$\Longrightarrow \int_{S_L} \psi^2 \geq C \int_{\Lambda_L} \psi^2$$

Generalisation to elliptic second order partial differential operators

Assumptions

L second order partial differential operator $L = \sum_{i,j=1}^d \partial_i(a^{ij}\partial_j)$ satisfies

- symmetric coefficients, $a^{ij} = a^{ji}\,\forall i,j \in \{1,\dots,d\}$,
- ellipticity, $\exists \xi\colon \forall x \in B_1(0)\colon 1/\xi|x|^2 \le \sum_{i,j=1}^d a^{ij}(x)x_i x_j \le \xi|x|^2$,
- Lipschitz continuity, $\exists \theta\colon \forall x,y \in B_1(0)\colon \sum_{i,j=1}^d |a^{i,j}(x) - a^{i,j}(y)| \le \theta|x-y|$.

Theorem [ongoing work]

There exist $C, D, E > 0$ and a function $w\colon \mathbb{R}^d \to \mathbb{R}$ depending on Lipschitz and ellipticity constants, such that for all $f \in \mathcal{C}_c(B_1(0) \setminus \{0\})$ and all $\alpha > D$

$$\int \alpha w^{1-2\alpha}|\nabla f|^2 + \alpha^3 w^{-1-2\alpha} f^2 \le E \int w^{2-2\alpha}(Lf)^2.$$

- Similar weight function, depends on constant μ adapted to the operator.
- Open question: How does μ scale with Lipschitz constant? Conjecture is linear.