Markovian Extensions of Symmetric Second Order Elliptic Differential Operators

Andrea Posilicano
DiSAT, Università dell'Insubria, Como, Italy

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1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary Γ.
Let $A_{\min}$ be the symmetric operator

$$A_{\min} : \mathscr{C}_c^\infty(\Omega) \subset L^2(\Omega) \to L^2(\Omega)\,, \quad A_{\min} = \sum_{1\le i,j\le n} \partial_i(a_{ij}\partial_j)\,,$$

where $a_{ij} \in \mathscr{C}^\infty(\Omega)$, $a_{ij} = a_{ji}$, and

$$\forall x \in \Omega\,,\ \forall \underline{\xi} \in \mathbb{R}^n\,, \quad \mu_1\|\underline{\xi}\|^2 \le \sum_{1\le i,j\le n} a_{ij}(x)\xi_i\xi_j \le \mu_2\|\underline{\xi}\|^2\,,$$

$0 < \mu_1 < \mu_2 < +\infty$.

All self-adjoint extensions of $A_{\min}$ have been determined, and classified in term of (non-local) boundary conditions, by Grubb in 1968 (building on previous partial results by Višik).

Here we are interested in determine, and classify in terms of Wentzell-type boundary conditions, all Markovian extensions.

A not positive s.a. operator A is said Markovian whenever the semi-group e^{tA} is positivity-preserving and is contraction in $L^\infty(\Omega)$.

Nice features of Markovian operators:

• characterization in terms of Dirichlet forms;

• connections with Markov processes: $e^{tA}u(x) = E_x(u(X_t))$ (analytical properties of A correlate with path properties of X_t: recurrence, transience, irreducibility,....);

• equivalence between:

i) logaritmic Sobolev inequalites for quadratic forms,
ii) ultracontractivity properties of semi-groups,
iii) heat kernel estimates for semi-groups integral kernels.

2. Dirichlet Forms

A not negative, symmetric bilinear form (not necessarily densely defined) on the **real** Hilbert space $L^2(X)$, X a "nice" measure space,

$$F : \mathscr{D}(F) \times \mathscr{D}(F) \subseteq L^2(X) \times L^2(X) \to \mathbb{R}$$

is a *Markovian form* if

$$u \in \mathscr{D}(F) \implies \begin{cases} \Phi(u) \in \mathscr{D}(F)\,, \\ F(\Phi(u), \Phi(u)) \leq F(u,u) \end{cases}$$

for any normal contraction Φ, i.e. for any function

$$\Phi : \mathbb{R} \to \mathbb{R}\,, \quad \Phi(0) = 0 \quad \text{and} \quad |\Phi(t) - \Phi(s)| \leq |t - s|\,.$$

A closed Markovian form F is said to be a *Dirichlet form*.

A not positive s.a. operator A in $L^2(X)$ is said to be *Markovian* if

$$0 \le u \le 1\,,\ m\text{-a.e.} \quad \Longrightarrow \quad \forall t > 0\,, \quad 0 \le e^{tA} u \le 1\,,\ m\text{-a.e.}\,.$$

Let F_A be the not negative bilinear form associated with $-A$:

$$\forall u \in \mathscr{D}(F_A)\,, \forall v \in \mathscr{D}(A)\,, \quad F_A(u,v) = \langle u, -Av\rangle\,.$$

Theorem.(Beuling and Deny)
A is Markovian $\iff$ F_A is a Dirichlet form.

Theorem. (Fukushima)
(i) The closure of a closable Markovian form is a Dirichlet form.
(ii) A closed, not negative symmetric bilinear form F is a Dirichlet form if and only if

$$u \in \mathscr{D}(F) \Longrightarrow \begin{cases} u_\# \in \mathscr{D}(F)\,, \\ F(u_\#, u_\#) \le F(u,u)\,, \end{cases} \quad u_\# := \min\{1, u_+\}\,.$$

Let α, β and γ be continuous, monotone decreasing functions.

(i) the densely defined Dirichlet form F_A satisfies a *logarithmic Sobolev inequality* with function α (α-LS for short) if

$\forall \epsilon > 0$, $\forall u \in \mathscr{D}(F_A) \cap L^1(X) \cap L^\infty(X)$, $u \geq 0$, $\|u\|_{L^2(X)} = 1$,

$$\int_X u^2 \log u \, dm \leq \epsilon F_A(u) + \alpha(\epsilon) \,,$$

(ii) the semigroup e^{tA} is *ultracontractive* with function β (β-UC for short) if

$$\forall t > 0 \,, \; \forall u \in L^2(X) \,, \quad \|e^{tA} u\|_{L^\infty(X)} \leq e^{\beta(t)} \|u\|_{L^2(X)} \,,$$

(iii) the integral kernel $\kappa_A(t, \cdot, \cdot)$ of e^{tA} satisfies a *heat kernel estimate* with function γ (γ-HK for short) if

$$\forall t > 0 \,, \; \text{for } m\text{-a.e. } x \text{ and } y, \quad \kappa_A(t, x, y) \leq e^{\gamma(t)} \,.$$

Theorem.(Gross, Davies and Simon, Davies,...)
For any Markovian operator one has LS $\iff$ UC $\iff$ HK:

$$\alpha\text{-LS} \quad \Longrightarrow \quad \beta\text{-UC}\,, \qquad \beta(t) = \frac{1}{t}\int_0^t \alpha(\epsilon)\,d\epsilon$$

$$\beta\text{-UC} \quad \Longrightarrow \quad \gamma\text{-HK}\,, \qquad \gamma(t) = 2\beta(t/2)\,.$$

$$\gamma\text{-HK} \quad \Longrightarrow \quad \beta\text{-UC}\,, \qquad \beta(t) = \gamma(t)/2$$

$$\beta\text{-UC} \quad \Longrightarrow \quad \alpha\text{-LS}\,, \qquad \alpha(t) = \beta(t)\,.$$

3. Markovian extensions of elliptic operators

Let $H^s(\Omega)$ and $H^s(\Gamma)$ be the usual Hilbert-Sobolev space of order s on the domain Ω and its boundary Γ. Let

$$\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)\,, \quad \gamma_1 : H^2(\Omega) \to H^{\frac{1}{2}}(\Gamma)\,,$$

the usual trace maps such that, for any $u \in \mathscr{C}^\infty(\bar{\Omega})$,

$$\gamma_0 u\,(x) = u(x)\,, \quad \gamma_1 u(x) = \nu(x)\cdot(a\nabla u)(x)\,, \quad x \in \Gamma\,,$$

where $\nu(x)$ is the (inward) normal at the boundary point x.
We pose

$$H^1_0(\Omega) := \{u \in H^1(\Omega)\ :\ \gamma_0 u = 0\}\,.$$

Both the bilinear forms F_D and F_N, defined by $\langle \nabla u, a\nabla v\rangle_{L^2(\Omega)}$ on the respective domains $\mathscr{D}(F_D) = H_0^1(\Omega)$ and $\mathscr{D}(F_N) = H^1(\Omega)$, are Dirichlet forms. Hence the corresponding self-adjoint operators A_D and A_N are Markovian. Both are self-adjoint extensions of $A_{\min}$: A_D is the Friedrichs extension and corresponds to Dirichlet b.c., A_N corresponds to Neumann b.c..

Given the dual operator $A'_{\min} : (\mathscr{C}_c^\infty(\Omega))' \to (\mathscr{C}_c^\infty(\Omega))'$, the Hilbert adjoint $A_{\max} := A^*_{\min}$ acts as $A'_{\min}$ on the domain $\mathscr{D}(A_{\max}) = \{u \in L^2(\Omega) : A'_{\min} u \in L^2(\Omega)\}$.

By Lions and Magenes both γ_0 and γ_1 have well-defined extensions $\hat\gamma_0$ and $\hat\gamma_1$ defined on $\mathscr{D}(A_{\max})$. Then for any $\lambda \geq 0$ one defines the Poisson operator $K_\lambda : H^{-\frac{1}{2}}(\Gamma) \to \mathscr{D}(A_{\max})$ which provides the unique solution of the adjoint Dirichlet problem for with boundary data in $H^{-\frac{1}{2}}(\Gamma)$:

$$\begin{cases} A^*_{\min} K_\lambda h = \lambda K_\lambda h\,, \\ \hat\gamma_0\, K_\lambda h = h\,. \end{cases}$$

By K_0 one defines P_0, be the Dirichlet-to-Neumann operator over Γ,

$$P_0 : H^s(\Gamma) \to H^{s-1}(\Gamma)\,, \quad s \ge -\frac{1}{2}\,, \quad P_0 := \hat{\gamma}_1\, K_0\,.$$

Any $u \in H^1(\Omega)$ admits the unique decomposition

$$u = u_0 + K_0 \gamma_0 u\,, \quad u_0 \in H^1_0(\Omega)$$

and the relation between F_N and F_D is given by

$$F_N(u, v) = F_D(u_0, v_0) - (P_0 \gamma_0 u, \gamma_0 v)_{-\frac{1}{2},\frac{1}{2}}\,.$$

Morover the bilinear form

$$f_\circ : H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \subseteq L^2(\Gamma) \times L^2(\Gamma) \to \mathbb{R}\,,$$

$$f_\circ(h_1, h_2) := -(P_0 h_1, h_2)_{-\frac{1}{2},\frac{1}{2}}$$

is a Dirichlet form.

Let

$$\mathsf{Ext}_0(A_{\min}) := \{A \supset A_{min} : A^* = A\,, -A \geq 0\}\,,$$

$$\begin{aligned}\mathsf{Ext}_{\mathrm{M}}(A_{\min}) :=& \{A \in \mathsf{Ext}_0(A_{\min}) : A \text{ is Markovian}\} \\ \equiv& \{A \in \mathsf{Ext}_0(A_{\min}) : F_A \text{ is Dirichlet }\}\end{aligned}$$

and define the semi-order

$$A_1 \preceq A_2 \iff \begin{cases} \mathscr{D}(F_{A_1}) \subseteq \mathscr{D}(F_{A_2})\,, \\ F_{A_1}(u,u) \geq F_{A_2}(u,u)\,. \end{cases}$$

Kreĭn proved that A_D is the minimum element of $\mathsf{Ext}_0(S)$ (hence of $\mathsf{Ext}_{\mathrm{M}}(A_{\min})$ also) and that there is a maximum element $A_K \in \mathsf{Ext}_0(A_{\min})$, the Kreĭn-von Neuman extension. However $A_K \notin \mathsf{Ext}_{\mathrm{M}}(A_{\min})$ (not necessarily true when Ω is unbounded) and

Theorem. (Watanabe, Fukushima).

A_N is the maximum element of $\mathsf{Ext}_{\mathrm{M}}(A_{\min})$.

Corollary. Let $A \in \mathrm{Ext_M}(A_{\min})$. Then

• $\mathscr{D}(F_A) \subseteq H^1(\Omega)$ and $F_N(u,u) \le F_A(u,u)$;

• F_A satisfies a logarithmic Sobolev inequality;

• the semigroup e^{tA} is ultracontractive;

• $\kappa_D \le \kappa_A \le \kappa_N$, where κ_D and κ_N denote the heat kernels of A_D and A_N respectively;

• the semigroup e^{tA} is irreducible (hence the ground state, if any, is unique and a.e. positive) and A is either recurrent or transient.

Theorem. Let F be a closed bilinear form on $L^2(\Omega)$. Then $F = F_A$, $A \in \mathrm{Ext}_{\mathrm{M}}(A_{min})$ if and only if there exists a Markovian form f_b on $L^2(\Gamma)$, such that

$$F_A(u, v) = F_N(u, v) + f_b(\gamma_0 u, \gamma_0 v)\,,$$

$$\mathscr{D}(F_A) = \{u \in H^1(\Omega) : \gamma_0 u \in \mathscr{D}(f_b)\}\,.$$

A is recurrent (equivalently conservative) if and only if $1 \in \mathscr{D}(f_b)$ and $f_b(1, 1) = 0$.

4. Wentzell-type boundary conditions

Theorem. Let $A \in \mathrm{Ext_M}(A_{\min})$. Then $u \in \mathscr{D}(A)$ if and only if $u \in \mathscr{D}(A_{\max}) \cap H^1(\Omega)$ and there exist a Markovian form f_b on $L^2(\Gamma)$ such that $\gamma_0 u \in \mathscr{D}(f_b)$ and

$$\forall h \in \mathscr{D}(f_b) \cap H^{\frac{1}{2}}(\Gamma)\,, \quad f_b(\gamma_0 u, h) = (\hat{\gamma}_1 u, h)_{-\frac{1}{2},\frac{1}{2}}\,.$$

Notice that f_b is not necessarily densely defined: $\overline{\mathscr{D}(f_b)}$ can be a strict subspace of $L^2(\Gamma)$.

When f_b is a densely defined, closed (i.e. Dirichlet) and regular (i.e. $\mathscr{C}(\Gamma) \cap \mathscr{D}(f_b)$ is dense in $\mathscr{C}(\Gamma)$ w.r.t. the L^∞ norm and is a form-core), by the famed Buerling-Deny decomposition one gets Wentzell-type b.c.:

$$f_b^{(c)}(\gamma_0 u, h) + \int_{\Gamma\times\Gamma} (\gamma_0 u(x) - \gamma_0 u(y))(h(x) - h(y))\, dJ(x,y)$$

$$+ \int_\Gamma \gamma_0 u(x) h(x)\, d\kappa(x) = (\hat{\gamma}_1 u, h)_{-\frac{1}{2},\frac{1}{2}}\,,$$

$$f_b^{(c)}(h_1, h_2) = \sum_{1\le i,j\le n-1} \int_\Gamma \nabla_i h_1(x) \nabla_j h_2(x)\, d\nu_{ij}(x)\,.$$

When B, $f_B = f_b$, is bounded on $H^s(\Gamma)$ to $H^{s-\alpha}(\Gamma)$ for all $s \ge -\frac{1}{2}$ and for some $\alpha \in [0,2]$, and, in case $\alpha \ge 1$, is elliptic pseudo-differential, one has

$$\mathscr{D}(A) = \{u \in \mathscr{D}(A_{\max}) \cap H^{2\wedge\alpha}(\Omega) : B\gamma_0 u = \hat{\gamma}_1 u\}\,.$$

Thus, when $\alpha = 2$,

$$\mathscr{D}(A) = \{u \in H^2(\Omega) : B\gamma_0 u = \gamma_1 u\}\,.$$

Theorem.

i) (variation on Grubb's theory) $A \in Ext(A_{\min})$ if and only if there exist an orthogonal projector Π on $L^2(\Gamma)$ and a self-adjoint operator Θ on the subspace associated with Π such that

$$\mathscr{D}(A) = \{u \in \mathscr{D}(A_{\max}) \,:\, \Sigma\hat{\gamma}_0 u \in \mathscr{D}(\Theta)\,,\; \Pi\Lambda(\hat{\gamma}_1 u - P_0\hat{\gamma}_0 u) = \Theta\Sigma\hat{\gamma}_0 u\}\,.$$

Here $\Lambda := (-\Delta_{LB} + 1)^{\frac{1}{4}} : H^s(\Gamma) \to H^{s-\frac{1}{2}}(\Gamma)\,, \quad \Sigma := \Lambda^{-1}\,.$

ii) (Kreĭn-type formula)

$$R^A_\lambda = R^D_\lambda + K_\lambda\Lambda\Pi(\Theta + \lambda\Pi\Lambda\gamma_1 R^D_\lambda K_\lambda\Lambda\Pi)^{-1}\Pi(K_\lambda\Lambda)^*\,.$$

iii) Let f^Λ_Θ be the closed, not negative bilinear form on $\overline{\Lambda\mathscr{D}(f_\Theta)}$ defined by

$$f^\Lambda_\Theta(h_1, h_2) := f_\Theta(\Sigma h_1, \Sigma h_2)\,.$$

Then $A \in \mathrm{Ext}_{\mathrm{M}}(A_{\min})$ if and only if $\mathscr{D}(f_\Theta) \subseteq H^1(\Gamma)$ and f^Λ_Θ is a Dirichlet form such that

$$f_b := f^\Lambda_\Theta - f_\circ \quad \text{is Markovian}\,.$$

5. Some examples

Example 1. One gets a Markovian extension by taking

$$B = -b_1\,\Delta_{LB} + b_s(-\Delta_{LB})^s + b_0\,, \quad 0 < s < 1\,,\ b_1, b_s, b_0 \geq 0\,.$$

To such an extension corresponds the Wentzell-type b.c.

$$b_1\,\Delta_{LB}\,\gamma_0 u - b_s(-\Delta_{LB})^s\gamma_0 u - b_0\gamma_0 u + \gamma_1 u = 0\,.$$

For the corresponding Markovian extension A_B one has

$$\mathscr{D}(A_B) \subseteq H^2(\Omega) \quad \text{whenever } b_1 \neq 0\,,$$

$$\mathscr{D}(A_B) \subseteq H^{2s}(\Omega) \quad \text{whenever } b_1 = 0,\ b_s \neq 0 \text{ and } 1/2 \leq s < 1\,,$$

$$\mathscr{D}(A_B) \subseteq H^{\frac{3}{2}-s}(\Omega) \quad \text{whenever } b_1 = 0,\ b_s \neq 0 \text{ and } 0 < s \leq 1/2\,.$$

F_{A_B} is a regular Dirichlet form with Beurling-Deny decomposition

$$F_{A_B} = F^{(c)}_{A_B} + F^{(j)}_{A_B} + F^{(k)}_{A_B}\,,$$

$$F^{(c)}_{A_B}(u,v) = F_N(u,v) + b_1(-\Delta_{LB}\gamma_0 u, \gamma_0 v)_{-1,1}\,,$$

$$F^{(j)}_{A_B}(u,v) = b_s((-\Delta_{LB})^s\gamma_0 u, \gamma_0 v)_{-s,s}\,,$$

$$F^{(k)}_{A_B}(u,v) = b_0\langle\gamma_0 u, \gamma_0 v\rangle_{L^2(\Gamma)}\,.$$

Hence F_{A_B} has the local property (and so the corresponding Markov process is a Diffusion) if and only if $b_s = 0$ and is strongly local if and only if $b_s = b_0 = 0$. The corresponding Markovian extension A_B is recurrent if $b_0 = 0$ and transient otherwise.

Example 2. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open and bounded and such that Γ is a smooth, compact, $n-1$ dimensional Lie group (for example this is true if Ω is a solid torus, or a plane disc with $N \geq 0$ circular holes, or a four-dimensional ball). Let $L_1, \ldots, L_{n-1}$ be a basis of left-invariant vector field in the corresponding Lie algebra. By Hunt's theorem any symmetric convolution semigroup of measures in Γ has a generators given by a Markovian self-adjoint operator B on $L^2(\Gamma)$ such that $\mathscr{C}^2(\Gamma) \subset \mathscr{D}(B)$ and, for any $h \in \mathscr{C}^2(\Gamma)$,

$$Bh(x) = \sum_{i,j=1}^{n-1} c_{ij} L_i L_j h(x) + \frac{1}{2} \int_\Gamma \left(h(xy) - 2h(x) + h(xy^{-1}) \right) d\nu(y) \,,$$

where $c \equiv (c_{ij})$ is a constant, real-valued, not-negative-definite matrix and ν is a symmetric Lévy measure.

Hence the boundary conditions $B\gamma_0 u = \hat{\gamma}_1 u$ produce a Markovian extension A_B which, since $f_B(1) = 0$, is recurrent.
By the known Beurling-Deny decomposition of f_B the Beurling-Deny decomposition of the Dirichlet form F_{A_B} is

$$F^{(c)}_{A_B}(u,v) = F_N(u,v) + \sum_{i,j=1}^{n-1} c_{ij} \int_\Gamma L_i\gamma_0 u(x) L_j\gamma_0 v(x)\, d\sigma(x)\,,$$

$$F^{(j)}_{A_B}(u,v) = \int_{\Gamma\times\Gamma} (\gamma_0 u(x) - \gamma_0 u(y))\,(\gamma_0 v(x) - \gamma_0 v(y))\, dJ(x,y)\,,$$

$$F^{(k)}_{A_B}(u,v) = 0\,,$$

where the measure J is defined by

$$J(E_1 \times E_2) := \int_{E_2} \nu(y^{-1}E_1)\, d\sigma(y)\,.$$

Example 3. Given the decomposition $\Gamma = \Gamma_0 \cup \Gamma_1$, Γ_0 open, let $\Pi : L^2(\Gamma) \to L^2(\Gamma)$ be the orthogonal projector onto $L^2(\Gamma_0)$. Then one gets the self-adjoint extension A_{DN} with domain

$$\mathscr{D}(A_{DN}) = \{u \in \mathscr{D}(A_{\max}) \cap H^1(\Omega) : \text{supp}(\gamma_0 u) \subseteq \bar{\Gamma}_0, \, \text{supp}(\hat{\gamma}_1 u) \subseteq \Gamma_1\}\,.$$

The conditions appearing in $\mathscr{D}(A_{DN})$ are a weak form of the mixed Dirichlet-Neumann boundary conditions

$$\gamma_0 u = 0 \text{ on } \Gamma_1\,, \quad \gamma_1 u = 0 \text{ on } \Gamma_0\,.$$

Such boundary conditions are Wentzell-type and define a (transient) Markovian extension with associated Dirichlet form

$$\mathscr{D}(F_{DN}) = \{u \in H^1(\Omega) \,:\, \text{supp}(\gamma_0 u) \subseteq \bar{\Gamma}_0\}\,, \quad F_{DN}(u, v) = F_N(u, v)\,.$$

Example 4. Let $\Pi : L^2(\Gamma) \to L^2(\Gamma)$ be the rank-one orthogonal projection $\Pi = (|\Gamma|^{-\frac{1}{2}}1) \otimes (|\Gamma|^{-\frac{1}{2}}1)$, where $|\Gamma|$ denotes the volume of the boundary Γ, and let $b \geq 0$. Then, denoting by $\langle h \rangle$ the mean of h over Γ, i.e. $\langle h \rangle := |\Gamma|^{-1} \int_\Gamma h d\sigma$, one gets the Markovian extension A with domain

$$\mathscr{D}(A) = \{u \in H^2(\Omega) : b\,\gamma_0 u = \langle \gamma_1 u \rangle\}\,.$$

with corresponding Dirichlet form

$$\mathscr{D}(F_A) = \{u \in H^1(\Omega) : \gamma_0 u = \text{const.}\}\,.$$

$$F_A(u,v) = F_N(u,v) + b\,\langle \gamma_0 u \rangle \langle \gamma_0 v \rangle\,.$$

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