On a Multi-Physics Coupling Mechanism.

The 3rd Najman Conference on Spectral Problems for Operators and Matrices

Rainer Picard
Department of Mathematics
TU Dresden, Germany

Biograd 2013

The Shape of Evolutionary Equations.

General Form of Evolutionary Problems:

$$\partial_0 V + AU = f \text{ on } \mathbb{R},\ V = \mathscr{M} U.$$

Evolutionary Equation:

$$(\partial_0 \mathscr{M} + A)\, U = f.$$

Solution Theory: Does the operator

$$(\partial_0 \mathscr{M} + A)^{-1}$$

exist as a continuous linear mapping on a suitable Hilbert space? Which *"suitable"* Hilbert space?

The Shape of Evolutionary Equations.

General Form of Evolutionary Problems:

$$\partial_0 V + AU = f \text{ on } \mathbb{R},\ V = \mathscr{M} U.$$

Evolutionary Equation:

$$(\partial_0 \mathscr{M} + A)\, U = f.$$

Solution Theory: Does the operator

$$(\partial_0 \mathscr{M} + A)^{-1}$$

exist as a continuous linear mapping on a suitable Hilbert space? Which *"suitable"* Hilbert space?

The Time Derivative as a Normal Operator

Exponential weight function $t \mapsto \exp(-\rho t)$, $\rho \in \mathbb{R}$, generates a weighted L^2-space $H_{\rho,0}(\mathbb{R},\mathbb{C})$ by completion of the space $\mathring{C}_\infty(\mathbb{R},\mathbb{C})$ of smooth complex-valued functions with compact support w.r.t. $\langle \cdot | \cdot \rangle_{\rho,0}$ (norm: $|\cdot|_{\rho,0}$)

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \overline{\varphi(t)}\, \psi(t) \exp(-2\rho t)\, dt.$$

Time-differentiation ∂_0 as a closed operator in $H_{\rho,0}(\mathbb{R},\mathbb{C})$ induced by

$$\begin{aligned} \mathring{C}_\infty(\mathbb{R},\mathbb{C}) \subseteq H_{\rho,0}(\mathbb{R},\mathbb{C}) &\to H_{\rho,0}(\mathbb{R},\mathbb{C}), \\ \varphi &\mapsto \varphi'. \end{aligned}$$

The Time Derivative as a Normal Operator

Time-differentiation ∂_0 is a normal operator in $H_{\rho,0}(\mathbb{R},\mathbb{C})$

$$\partial_0 = \mathfrak{Re}\,\partial_0 + \mathrm{i}\,\mathfrak{Im}\,\partial_0 = \overline{\frac{1}{2}(\partial_0 + \partial_0^*)} + \mathrm{i}\,\overline{\frac{1}{2\mathrm{i}}(\partial_0 - \partial_0^*)}$$

with $\mathfrak{Re}\,\partial_0$, $\mathfrak{Im}\,\partial_0$ self-adjoint operators with commuting resolvents:

$$\mathfrak{Re}\,\partial_0 = \rho\,.$$

For $\rho \in \mathbb{R}\setminus\{0\}$: continuous invertibility of ∂_0, i.e. $0 \in \rho(\partial_0)$ (resolvent set):

$$\sigma(\partial_0) = \mathrm{i}\mathbb{R} + \rho \quad \text{(spectrum)}.$$

The Time Derivative as a Normal Operator

Time-differentiation ∂_0 is a normal operator in $H_{\rho,0}(\mathbb{R},\mathbb{C})$

$$\partial_0 = \mathfrak{Re}\,\partial_0 + \mathrm{i}\,\mathfrak{Im}\,\partial_0 = \overline{\frac{1}{2}\left(\partial_0+\partial_0^*\right)} + \mathrm{i}\,\overline{\frac{1}{2\mathrm{i}}\left(\partial_0-\partial_0^*\right)}$$

with $\mathfrak{Re}\,\partial_0$, $\mathfrak{Im}\,\partial_0$ self-adjoint operators with commuting resolvents:

$$\mathfrak{Re}\,\partial_0 = \rho.$$

For $\rho \in \mathbb{R}\setminus\{0\}$: continuous invertibility of ∂_0, i.e. $0 \in \rho\left(\partial_0\right)$ (resolvent set):

$$\sigma\left(\partial_0\right) = \mathrm{i}\mathbb{R} + \rho \quad \text{(spectrum)}.$$

The Time Derivative as a Normal Operator

Fourier-Laplace transform: unitary extension of

$$\mathring{C}_\infty(\mathbb{R},\mathbb{C}) \subseteq H_{\rho,0}(\mathbb{R},\mathbb{C}) \to H_{0,0}(\mathbb{R},\mathbb{C}) = L^2(\mathbb{R},\mathbb{C})$$

$$\varphi \mapsto \mathscr{L}_\rho \varphi$$

with $\mathscr{L}_\rho \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\mathrm{i}\, x\, t) \exp(-\rho\, t)\, \varphi(t)\, dt,\ x \in \mathbb{R}.$

is spectral representation for $\mathfrak{Im}\,\partial_0$:

$$\mathfrak{Im}\,\partial_0 = \mathscr{L}_\rho^{-1} \mathbf{m}_0 \mathscr{L}_\rho, \quad \partial_0 = \mathscr{L}_\rho^{-1}(\mathrm{i}\,\mathbf{m}_0 + \rho)\,\mathscr{L}_\rho.$$

Here $\mathbf{m}_0$ is the selfadjoint multiplication-by-argument operator in $L^2(\mathbb{R},\mathbb{C})$:

$$(\mathbf{m}_0 \varphi)(x) = x\varphi(x)$$

for $x \in \mathbb{R}$ and $\varphi \in \mathring{C}_\infty(\mathbb{R},\mathbb{C})$.

The Time Derivative as a Normal Operator

The canonical extension of ∂_0 to the X-valued case, X a Hilbert space, inherits the normality:

∂_0 is still a normal operator in $H_{\rho,0}(\mathbb{R},X)$

$$\rho = \mathfrak{Re}\,\partial_0 .$$

With the extended Fourier-Laplace transform

$$\mathscr{L}_\rho : H_{\rho,0}(\mathbb{R},X) \to L^2(\mathbb{R},X)$$

we still get

$$\partial_0 = \mathscr{L}_\rho^{-1}\left(\mathrm{i}\,\mathbf{m}_0 + \rho\right)\mathscr{L}_\rho .$$

The Time Derivative as a Normal Operator

The canonical extension of ∂_0 to the X-valued case, X a Hilbert space, inherits the normality:

∂_0 is still a normal operator in $H_{\rho,0}(\mathbb{R}, X)$

$$\rho = \mathfrak{Re}\,\partial_0.$$

With the extended Fourier-Laplace transform

$$\mathcal{L}_\rho : H_{\rho,0}(\mathbb{R}, X) \to L^2(\mathbb{R}, X)$$

we still get

$$\partial_0 = \mathcal{L}_\rho^{-1}(\mathrm{i}\,\mathbf{m}_0 + \rho)\,\mathcal{L}_\rho.$$

The Time Derivative as a Normal Operator

The canonical extension of ∂_0 to the X-valued case, X a Hilbert space, inherits the normality:

∂_0 is still a normal operator in $H_{\rho,0}(\mathbb{R}, X)$

$$\rho = \mathfrak{Re}\, \partial_0.$$

With the extended Fourier-Laplace transform

$$\mathcal{L}_\rho : H_{\rho,0}(\mathbb{R}, X) \to L^2(\mathbb{R}, X)$$

we still get

$$\partial_0 = \mathcal{L}_\rho^{-1} (\mathrm{i}\, \mathbf{m}_0 + \rho)\, \mathcal{L}_\rho.$$

Material Law Operators as Functions of the Time Derivative

- We also have that

$$\partial_0^{-1} = \mathscr{L}_\rho^{-1} \frac{1}{\mathrm{i}\,\mathbf{m}_0 + \rho} \mathscr{L}_\rho,$$

  and so

$$\sum_{k=0}^{N} M_k \partial_0^{-k} = \mathscr{L}_\rho^{-1} \sum_{k=0}^{N} M_k \frac{1}{(\mathrm{i}\,\mathbf{m}_0 + \rho)^k} \mathscr{L}_\rho$$

  with continuous linear operators M_k on X as coefficients, $k = 0, \ldots, N$.

- Note that for $\rho \in]0,\infty[$

$$\left\|\partial_0^{-1}\right\| = \frac{1}{\rho} \text{ and } \left(\partial_0^{-1}\varphi\right)(x) = \int_{-\infty}^{x} \varphi(t)\, dt$$

  for all $\varphi \in \mathring{C}_\infty(\mathbb{R})$ and $x \in \mathbb{R}$.

Material Law Operators as Functions of the Time Derivative

- We also have that

$$\partial_0^{-1} = \mathscr{L}_\rho^{-1} \frac{1}{\mathrm{i}\,\mathbf{m}_0 + \rho} \mathscr{L}_\rho,$$

and so

$$\sum_{k=0}^{N} M_k \partial_0^{-k} = \mathscr{L}_\rho^{-1} \sum_{k=0}^{N} M_k \frac{1}{(\mathrm{i}\,\mathbf{m}_0 + \rho)^k} \mathscr{L}_\rho$$

with continuous linear operators M_k on X as coefficients, $k = 0, \ldots, N$.

- Note that for $\rho \in \,]0, \infty[$

$$\left\|\partial_0^{-1}\right\| = \frac{1}{\rho} \text{ and } \left(\partial_0^{-1}\varphi\right)(x) = \int_{-\infty}^{x} \varphi(t)\, dt$$

for all $\varphi \in \mathring{C}_\infty(\mathbb{R})$ and $x \in \mathbb{R}$.

Material Law Operators as Functions of the Time Derivative

Material Law Operator:

$$\mathscr{M} = M\left(\partial_0^{-1}\right).$$

It is $$M\left(\partial_0^{-1}\right) := \mathscr{L}_\rho^{-1}\, M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0+\rho}\right)\mathscr{L}_\rho,$$

where $$M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0+\rho}\right)\Phi := \left(\omega \mapsto M\left(\frac{1}{\mathrm{i}\,\omega+\rho}\right)\Phi(\omega)\right)$$

for $\Phi \in \mathring{C}_\infty(\mathbb{R}, X)$.

Here $(M(z))_{z\in B_{\mathbb{C}}(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in H with $r \geq \frac{1}{2\rho} > 0$. The operator $M\left(\partial_0^{-1}\right)$ will be referred to as the **material law operator**. The operator-valued function M will be referred to as the **material law function**.

Material Law Operators as Functions of the Time Derivative

Material Law Operator:

$$\mathscr{M} = M\left(\partial_0^{-1}\right).$$

It is $$M\left(\partial_0^{-1}\right) := \mathscr{L}_\rho^{-1} M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0 + \rho}\right) \mathscr{L}_\rho,$$

where $$M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0 + \rho}\right)\Phi := \left(\omega \mapsto M\left(\frac{1}{\mathrm{i}\,\omega + \rho}\right)\Phi(\omega)\right)$$

for $\Phi \in \mathring{C}_\infty(\mathbb{R}, X)$.

Here $(M(z))_{z \in B_{\mathbb{C}}(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in H with $r \geq \frac{1}{2\rho} > 0$. The operator $M\left(\partial_0^{-1}\right)$ will be referred to as the **material law operator**. The operator-valued function M will be referred to as the **material law function**.

Material Law Operators as Functions of the Time Derivative

Material Law Operator:

$$\mathscr{M} = M\left(\partial_0^{-1}\right).$$

It is $$M\left(\partial_0^{-1}\right) := \mathscr{L}_\rho^{-1}\, M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0 + \rho}\right) \mathscr{L}_\rho,$$

where $$M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0 + \rho}\right)\Phi := \left(\omega \mapsto M\left(\frac{1}{\mathrm{i}\,\omega + \rho}\right)\Phi(\omega)\right)$$

for $\Phi \in \mathring{C}_\infty(\mathbb{R}, X)$.

Here $(M(z))_{z \in B_{\mathbb{C}}(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in H with $r \geq \frac{1}{2\rho} > 0$. The operator $M\left(\partial_0^{-1}\right)$ will be referred to as the **material law operator**. The operator-valued function M will be referred to as the **material law function**.

Material Law Operators as Functions of the Time Derivative

Material Law Operator:

$$\mathscr{M} = M\left(\partial_0^{-1}\right).$$

It is
$$M\left(\partial_0^{-1}\right) := \mathscr{L}_\rho^{-1} M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0+\rho}\right)\mathscr{L}_\rho,$$

where
$$M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0+\rho}\right)\Phi := \left(\omega \mapsto M\left(\frac{1}{\mathrm{i}\,\omega+\rho}\right)\Phi(\omega)\right)$$

for $\Phi \in \mathring{C}_\infty(\mathbb{R}, X)$.

Here $(M(z))_{z\in B_{\mathbb{C}}(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in H with $r \geq \frac{1}{2\rho} > 0$. The operator $M\left(\partial_0^{-1}\right)$ will be referred to as the **material law operator**. The operator-valued function M will be referred to as the **material law function**.

Basic Solution Theory $H_{\rho,0}(\mathbb{R},H)$

Evolutionary Problem:

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)}U=F$$

When is $\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)$ (and its adjoint) strictly positive definite in $H_{\rho,0}(\mathbb{R},H)$ (for all sufficiently large $\rho\in\,]0,\infty[$)?

Assumptions (E):

- A **skew-selfadjoint** in H (lifted to $H_{\rho,0}(\mathbb{R},H)$),
- $M(z)=M_0+z\left(M_1+M^{(2)}(z)\right)$, $M^{(2)}$ a **causal material law function** (values in $L(H,H)$), e.g. analytic at 0,
- $\limsup_{\rho\to\infty}\left\|M^{(2)}(\mathrm{i}\cdot+\rho)\right\|=0$,
- $M_0\geq 0$ **selfadjoint, strictly positive definite on its range**,
- $\mathfrak{Re}\, M_1$ **strictly positive definite on the null space of** M_0.

Basic Solution Theory $H_{\rho,0}(\mathbb{R},H)$

Evolutionary Problem:

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)}U=F$$

When is $\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)$ (and its adjoint) strictly positive definite in $H_{\rho,0}(\mathbb{R},H)$ (for all sufficiently large $\rho\in\,]0,\infty[$)?

Assumptions (E):

- A **skew-selfadjoint** in H (lifted to $H_{\rho,0}(\mathbb{R},H)$),
- $M(z)=M_0+z\left(M_1+M^{(2)}(z)\right)$, $M^{(2)}$ a **causal material law function** (values in $L(H,H)$), e.g. analytic at 0,
- $\limsup_{\rho\to\infty}\left\|M^{(2)}(\mathrm{i}\,\cdot\,+\rho)\right\|=0$,
- $M_0\geq 0$ **selfadjoint**, **strictly positive definite on its range**,
- $\mathfrak{Re}\,M_1$ **strictly positive definite on the null space of** M_0.

The Basic Solution Theorem

Theorem

*Let M and A satisfy **Assumptions (E)**. Then we have for all sufficiently large $\rho \in]0, \infty[$ that for every $f \in H_{\rho,0}(\mathbb{R}, H)$ there is a unique solution $U \in H_{\rho,0}(\mathbb{R}, H)$ of the problem*

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)} U = f.$$

The solution operator $\left(\overline{\partial_0 M\left(\partial_0^{-1}\right) + A}\right)^{-1}$ is continuous and causal on $H_{\rho,0}(\mathbb{R}, H)$.

Causal? For every $a \in \mathbb{R}$ we have:

If $F \in H_{\rho,0}(\mathbb{R}, H)$ vanishes on the time interval $]-\infty, a]$, then so does $\left(\overline{\partial_0 M\left(\partial_0^{-1}\right) + A}\right)^{-1} F$.

The Basic Solution Theorem

Theorem

*Let M and A satisfy **Assumptions (E)**. Then we have for all sufficiently large $\rho \in]0,\infty[$ that for every $f \in H_{\rho,0}(\mathbb{R},H)$ there is a unique solution $U \in H_{\rho,0}(\mathbb{R},H)$ of the problem*

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)}U=f.$$

The solution operator $\left(\overline{\partial_0 M\left(\partial_0^{-1}\right)+A}\right)^{-1}$ is continuous and causal on $H_{\rho,0}(\mathbb{R},H)$.

Causal? For every $a \in \mathbb{R}$ we have:

If $F \in H_{\rho,0}(\mathbb{R},\, H)$ vanishes on the time interval $]-\infty,\, a]$, then so does $\left(\overline{\partial_0 M\left(\partial_0^{-1}\right)+A}\right)^{-1} F$.

Some Applications to a Particular Class of Problems

The structure of A as a block operator matrix is frequently of the form

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix} \tag{1}$$

with $G : D(G) \subseteq H_0 \to H_1$ a closed, densely defined linear operator between Hilbert spaces H_0 and H_1, and the material laws are often given simply as

$$M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1} M_1\,,$$

where M_0 is self-adjoint and strictly positive definite in $H := H_0 \oplus H_1$. The term $M^{(2)}$ can be treated as a perturbation.

Some Applications to a Particular Class of Problems

Maxwell's equations, acoustics equations, elasticity equations etc. are of this specific form if memory effects are not considered:

$$\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}.$$

M_0, M_1 block diagonal in simple cases.

Metamaterials and Other Complex Media

Complex materials: general material law operators

$$M\left(\partial_0^{-1}\right)$$

- not block-diagonal
- (linear) delay
- memory terms (such as temporal convolution operators or fractional derivatives).

New materials!

Metamaterials and Other Complex Media

Complex materials: general material law operators

$$M\left(\partial_0^{-1}\right)$$

- not block-diagonal
- (linear) delay
- memory terms (such as temporal convolution operators or fractional derivatives).

New materials!

Metamaterials and Other Complex Media

Complex materials: general material law operators

$$M\left(\partial_0^{-1}\right)$$

- not block-diagonal
- (linear) delay
- memory terms (such as temporal convolution operators or fractional derivatives).

New materials!

Metamaterials and Other Complex Media

Complex materials: general material law operators

$$M\left(\partial_0^{-1}\right)$$

- not block-diagonal
- (linear) delay
- memory terms (such as temporal convolution operators or fractional derivatives).

New materials!

Coupling of Different Physical Phenomena

Without coupling, block-diagonal operator matrix:

$$\partial_0 \begin{pmatrix} V_0 \\ \vdots \\ \vdots \\ V_n \end{pmatrix} + A \begin{pmatrix} U_0 \\ \vdots \\ \vdots \\ U_n \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ \vdots \\ f_n \end{pmatrix},$$

where

$$A = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix}$$

skew-selfadjoint in $H = \bigoplus_{k=0,\dots,n} H_k$, since diagonal block entries $A_k : D(A_k) \subseteq H_k \to H_k$, $k = 0, \dots, n$, are skew-self-adjoint.

Coupling of Different Physical Phenomena

The combined material laws now take the simple diagonal form

$$V = \begin{pmatrix} V_0 \\ \vdots \\ \vdots \\ V_n \end{pmatrix} = \begin{pmatrix} M_{00}\left(\partial_0^{-1}\right) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M_{nn}\left(\partial_0^{-1}\right) \end{pmatrix} \begin{pmatrix} U_0 \\ \vdots \\ \vdots \\ U_n \end{pmatrix}.$$

Proper coupling: M contains off-diagonal block entries

$$M\left(\partial_0^{-1}\right) := \begin{pmatrix} M_{00}\left(\partial_0^{-1}\right) & \cdots & \cdots & M_{0n}\left(\partial_0^{-1}\right) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ M_{n0}\left(\partial_0^{-1}\right) & \cdots & \cdots & M_{nn}\left(\partial_0^{-1}\right) \end{pmatrix}.$$

Coupling of Different Physical Phenomena

Canonical Form:

If
$$A_k = \begin{pmatrix} 0 & -G_k^* \\ G_k & 0 \end{pmatrix},$$

then, with the unitary permutation matrix

$$P = (e_0\, e_2 \cdots\, e_{2n} e_1\, e_3 \cdots e_{2n+1}),$$

based on $\begin{array}{rcl} \{0,\ldots,2n+1\} & \to & \{0,\ldots,2n+1\} \\ k & \mapsto & \frac{1-(-1)^k}{2}(n+1) + \lfloor \frac{k}{2} \rfloor \end{array}$, we obtain

$PAP^* = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}$ with

$$G = \begin{pmatrix} G_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & G_n \end{pmatrix}.$$

Example: Plasma Field Equations

Plasma field equations, [Felsen-Marcuvitz-1973]: Maxwell equation and acoustic equation coupled (average electron velocity v, electron pressure p).

$$(\partial_0 M_0 + M_1 + A)\begin{pmatrix}\begin{pmatrix} p \\ E \end{pmatrix} \\ \begin{pmatrix} v \\ H \end{pmatrix}\end{pmatrix} = F$$

$$M_0 = \begin{pmatrix} \begin{pmatrix} \frac{1}{\gamma p_0} & 0 \\ 0 & \varepsilon_0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} n_0 m & 0 \\ 0 & \mu_0 \end{pmatrix} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -n_0 q & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & n_0 q \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -n_0 m \omega_c b_0 \times & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}, \quad G = \begin{pmatrix} \mathring{\operatorname{grad}} & 0 \\ 0 & \mathring{\operatorname{curl}} \end{pmatrix}$$

M_0 strictly positive definite, M_1 skew-selfadjoint.

Example: Thermo-Piezo-Electro-Magnetism in Micromorphic Media

We base our consideration on a model suggested by R.D. Mindlin, 1974. We are led to the system

$$\left(\partial_0 M_0 + M_1 + A\right)\begin{pmatrix} \dot{u} \\ \dot{\psi} \\ E \\ \theta \\ \tau + \sigma \\ \mu \\ \iota_{\mathrm{sym}}^{*}\sigma \\ H \\ Q \end{pmatrix} = \begin{pmatrix} f \\ h \\ -J \\ g \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Example: Thermo-Piezo-Electro-Magnetism in Micromorphic Media

$$AU := \begin{pmatrix} 0 & 0 & 0 & 0 & -\nabla\cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\nabla\cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\operatorname{curl} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nabla\cdot \\ -\mathring{\nabla} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathring{\nabla} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathring{\operatorname{curl}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathring{\nabla} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{\psi} \\ E \\ \theta \\ \tau + \sigma \\ \mu \\ \iota_{\mathrm{sym}}^{*}\sigma \\ H \\ Q \end{pmatrix}$$

$$H = L^{2,1}(\Omega) \oplus L^{2,2}(\Omega) \oplus L^{2,1}(\Omega) \oplus L^{2,0}(\Omega) \oplus L^{2,2}(\Omega) \oplus L^{2,3}(\Omega) \oplus \mathrm{sym}\left[L^{2,2}(\Omega)\right] \oplus \mathrm{skew}\left[L^{2,2}(\Omega)\right] \oplus L^{2,1}(\Omega).$$

$$\tau \in \mathrm{sym}\left[L^{2,2}(\Omega)\right]$$

Example: Thermo-Piezo-Electro-Magnetism in Micromorphic Media

M_0 is continuous, selfadjoint, M_1 continuous, such that

$$\rho M_0 + \mathfrak{Re}\, M_1 \geq c_0 > 0$$

for all sufficiently large $\rho \in]0,\infty[$. Here

$$M_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{skew} & 0 & -\imath_{\mathrm{sym}} & 0 & 0 \\ 0 & 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{skew} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \imath_{\mathrm{sym}}^{*} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa \end{pmatrix}.$$

Summary

- The key to well-posedness of evolutionary problems is strict positive definiteness.
- Causality is a characterizing property for evolutionary equations.
- The framework provides for an abundance of applications in particular for coupled phenomena with a single highly unified approach.

Literature

L. B. Felsen and N. Marcuvitz.
Radiation and Scattering of Waves (IEEE Press Series on Electromagnetic Wave Theory).
Wiley-IEEE Press, January 1994.

P. Neff.
The Cosserat couple modulus for continuous solids is zero viz the linearized Cauchy-stress tensor is symmetric.
ZAMM, Z. Angew. Math. Mech., 86(11):892–912, 2006.

Patrizio Neff and Krzysztof Chełmiński.
Infinitesimal elastic-plastic Cosserat micropolar theory. Modelling and global existence in the rate-independent case.
Proc. R. Soc. Edinb., Sect. A, Math., 135(5):1017–1039, 2005.

R. Picard and D. McGhee.
Partial differential equations. A unified Hilbert space approach.
Berlin: de Gruyter, 2011.

A Side Note: The "Mother" of "All" Evolutionary PDE

The "Mother":

$$A = \begin{pmatrix} 0 & -\nabla^* \\ \nabla & 0 \end{pmatrix} \tag{2}$$

with a suitable domain making A skew-selfadjoint in the Hilbert space

$$H = \left(\bigoplus_{k \in \mathbb{N}} L_k^2(\Omega) \right) \oplus \left(\bigoplus_{k \in \mathbb{N}} L_k^2(\Omega) \right).$$

$L_k^2(\Omega)$ tensors of order k with $L^2(\Omega)$-coefficients.

∇ co-variant derivative and $-\nabla^*$ its skew-adjoint (tensorial divergence).

The "Mother" of "All" Evolutionary PDE

Dirichlet boundary condition $G = \mathring{\nabla}$:

$$A := \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}$$

Initial boundary value problems of classical mathematical physics can be produced from this particular "mother" operator A by choosing suitable projections for constructing "descendants".

The "Mother" of "All" Evolutionary PDE

Theorem

Let $C : D(C) \subseteq H_0 \to H_1$ be a closed densely defined linear operator, H_k, $k = 0, 1$, Hilbert spaces. If $B_k : H_k \to X_k$ are continuous linear mappings, X_k Hilbert space, $k = 0, 1$, such that

- *$C^* B_1^*$ densely defined and B_0 is a bijection*
  or
- *$C B_0^*$ densely defined and B_1 is a bijection.*

Then $\overline{\begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}} \begin{pmatrix} B_0^* & 0 \\ 0 & B_1^* \end{pmatrix}$ *is skew-selfadjoint.*

"Mother" and "descendant".

The "Mother" of "All" Evolutionary PDE

Examples:

- tensor order (or degree; "Stufe")
- symmetric/alternating

3-dimensional		
order 0, 1	———	acoustics
order 1, 2	symmetric	elastics
order 1, 2	alternating	electrodynamics

- descend in space dimension
- vanishing trace condition (divergence-free; incompressible Stokes equation)

The “Mother” of “All” Evolutionary PDE

Examples:

- tensor order (or degree; “Stufe”)
- symmetric/alternating

3-dimensional		
order 0, 1	————	acoustics
order 1, 2	symmetric	elastics
order 1, 2	alternating	electrodynamics

- descend in space dimension
- vanishing trace condition (divergence-free; incompressible Stokes equation)

The "Mother" of "All" Evolutionary PDE

Examples:

- tensor order (or degree; "Stufe")
- symmetric/alternating

3-dimensional		
order 0, 1	———	acoustics
order 1, 2	symmetric	elastics
order 1, 2	alternating	electrodynamics

- descend in space dimension
- vanishing trace condition (divergence-free; incompressible Stokes equation)

The “Mother” of “All” Evolutionary PDE

Examples:

- tensor order (or degree; “Stufe”)
- symmetric/alternating

3-dimensional		
order $0, 1$	————	acoustics
order $1, 2$	symmetric	elastics
order $1, 2$	alternating	electrodynamics

- descend in space dimension
- vanishing trace condition (divergence-free; incompressible Stokes equation)

The “Mother” of “All” Evolutionary PDE

Examples:

- tensor order (or degree; “Stufe”)
- symmetric/alternating

3-dimensional		
order $0, 1$	————	acoustics
order $1, 2$	symmetric	elastics
order $1, 2$	alternating	electrodynamics

- descend in space dimension
- vanishing trace condition (divergence-free; incompressible Stokes equation)