Bound states in $\mathcal{PT}$-symmetric layers

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Joint work with David Krejčiřík

Outline of the talk

- Introduction
  - $\mathcal{PT}$-symmetric Quantum mechanics
  - Quantum waveguides
- $\mathcal{PT}$-symmetric waveguides
  - Model
  - Symmetries
  - Uniform waveguide
  - Perturbed waveguide
    - Essential spectrum
    - Weakly-coupled bound states
- Conclusions

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? Non-Hermitian observables in Quantum mechanics ?

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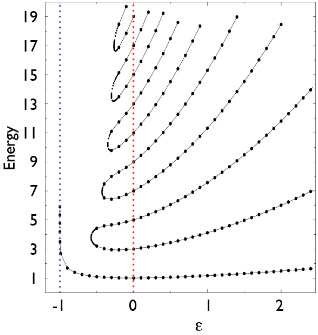
- Hamiltonian $-\Delta + \mathrm{i}x^3$ in $L^2(\mathbb{R})$ possess real spectrum
  [Bender, Boettcher 98]

- more generally: $-\Delta + x^2(\mathrm{i}x)^\varepsilon$ in $L^2(\mathbb{R})$ for $\varepsilon > 0$



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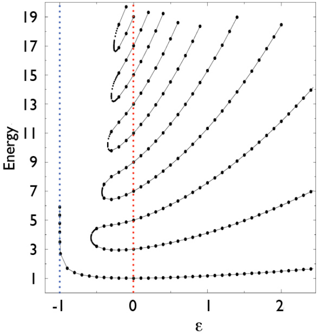
? Due to $\mathcal{PT}$-symmetry ?

$[H, \mathcal{PT}] = 0$ (in operator sense)

- Parity
  $(\mathcal{P}\psi)(x) = \psi(-x)$
- Time reversal
  $(\mathcal{T}\psi)(x) = \overline{\psi(x)}$



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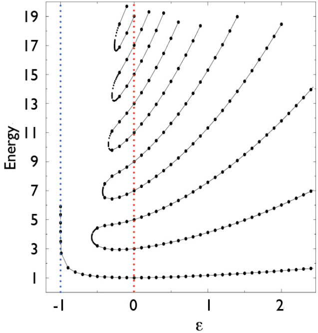

! Lack of techniques - no spectral theorem, no Min-max principle, ... !

Physical relevance

- Suggestions
  - nuclear physics [Scholtz, Geyer, Hahne 92], optics [Klaiman, Günther, Moiseyev 08], [Schomerus 10], solid state physics [Bendix, Fleischmann, Kottos, Shapiro 09], superconductivity [Rubinstein, Sternberg, Ma 07], electromagnetism [Ruschhaupt, Delgado, Muga 05], [Mostafazadeh 09], scattering [Hernandez-Coronado, Krejčiřík, Siegl 11]
- Experiments
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When metric operator $\Theta > 0, \|\Theta\| < +\infty, \|\Theta^{-1}\| < +\infty$ exists:
(H is then called quasi-Hermitian)

- H is Hermitian in Hilbert space $\left(L^2, \langle\cdot, \Theta\cdot\rangle\right)$
- $h = \Theta^{1/2} H \Theta^{-1/2}$ is Hermitian in $\left(L^2, \langle\cdot,\cdot\rangle\right)$

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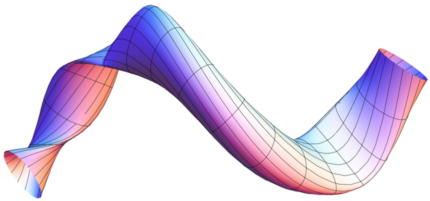
$\Rightarrow$ solves problem with reality of the spectrum, probability conservation, Stone's theorem…

Quantum waveguides

= microscopic structures of semiconductor material

- e.g. thin films, quantum wires, . . .

[Exner, Šeba 88], [Duclos, Exner 95]



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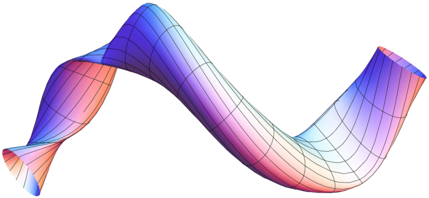
Mathematical description:

- unbounded tubular region
- free Laplacian
- boundary conditions



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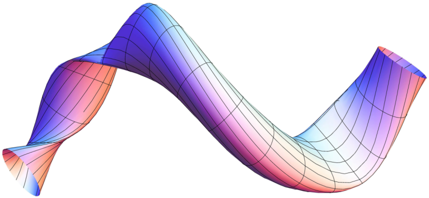


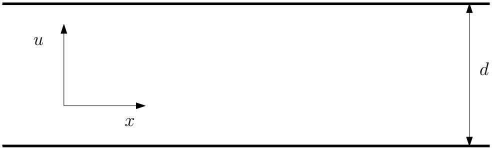
Straight waveguides have empty discrete spectrum

⇒ study of various perturbations

- small bumps [Bulla, Gesztezy, Renger, Simon 97]
- mixing of boundary conditions [Dittrich, Kříž 02]
- twisting and bending [Krejčiřík 08]

The model

- straight waveguide
  - 1 finite dimension (variable u)
  - n infinite dimensions (variable x)



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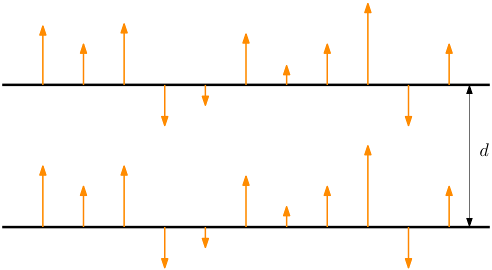

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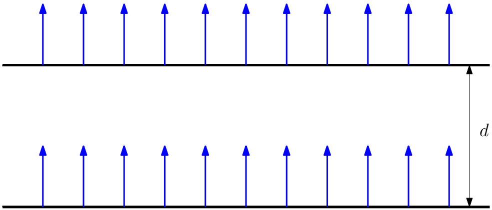

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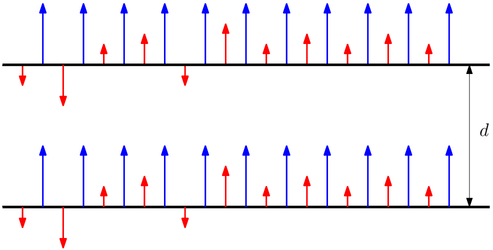
- influence of small perturbation in boundary conditions
  - existence of bound states?



Previous results

[Borisov, Krejčiřík 08]

Setup:

- planar waveguide $\Omega = \mathbb{R} \times I$
- Robin-type boundary conditions $\partial_u \Psi + \mathrm{i}(\alpha_0 + \varepsilon\beta)\Psi = 0$
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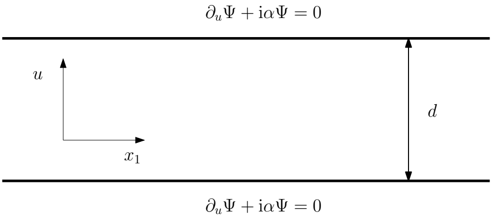

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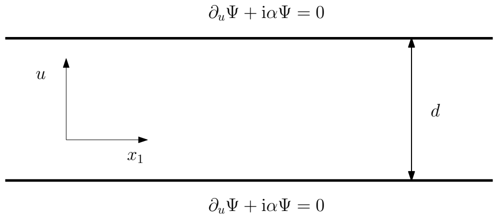
Results:

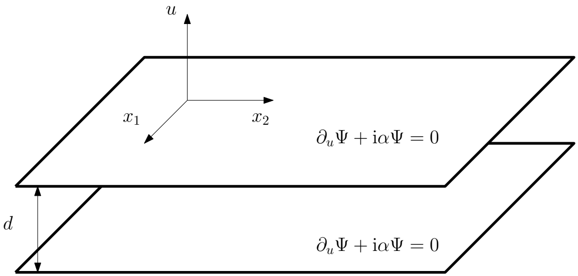
- conditions on existence and uniqueness of the bound state
- eigenvalue expansion up to order of ε^4
- wavefunction asymptotics



Definition of the Hamiltonian

The waveguide $\Omega := \mathbb{R}^n \times (0, d) = \mathbb{R}^n \times I$

$$H_\alpha \Psi := -\Delta \Psi,$$
$$\mathrm{Dom}(H_\alpha) := \left\{ \Psi \in W^{2,2}(\Omega) \middle| \, \partial_u \Psi + \mathrm{i}\alpha\Psi = 0 \quad \text{on} \quad \partial\Omega \right\}$$



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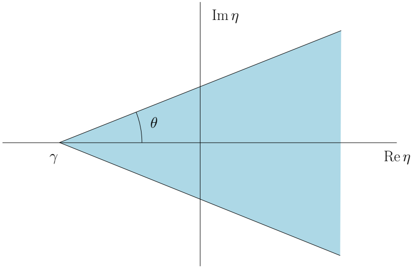

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Idea of the proof:

$$\begin{aligned} h_\alpha[\Psi] := &\int_\Omega |\nabla\Psi(x,u)|^2 \, \mathrm{d}x \, \mathrm{d}u \\ &+ \mathrm{i} \int_{\mathbb{R}^n} \alpha(x) \, |\Psi(x,d)|^2 \, \mathrm{d}x - \mathrm{i} \int_{\mathbb{R}^n} \alpha(x) \, |\Psi(x,0)|^2 \, \mathrm{d}x \end{aligned}$$

- h_α is sectorial and closed
- First representation theorem

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Note that $H_\alpha^* = H_{-\alpha}$

Symmetries of H_α

The spatial reflection operator $\mathcal{P}$ and the time reversal operator $\mathcal{T}$:

$$(\mathcal{P}\Psi)(x,u) := \Psi(x, d-u),$$
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$\Rightarrow$ $\sigma_{\mathrm{r}}(H_\alpha) = \varnothing$

Transversal and longitudal operator

Transversal operator:

$$-\Delta^I_{\alpha_0}\psi := -\frac{\partial^2}{\partial u^2}\psi$$

$$\mathrm{Dom}(-\Delta^I_{\alpha_0}) := \left\{\psi \in W^{2,2}(I) \,|\, \psi' + \mathrm{i}\alpha_0\psi = 0 \quad \text{at} \quad 0, d\right\}$$

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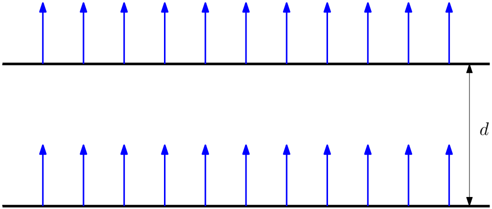
Uniform boundary conditions

Term in boundary conditions: $\alpha(x) = \alpha_0 \in \mathbb{R}$



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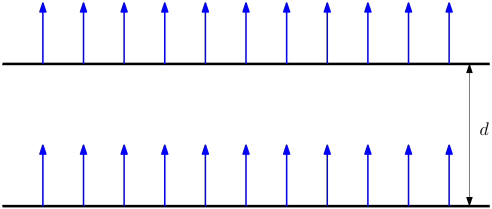
$$H_{\alpha_0} = \left(-\Delta' \otimes 1^I\right) + \left(1^{\mathbb{R}^n} \otimes -\Delta^I_{\alpha_0}\right),$$

because $\Psi(x,u) = \sum_{j=0}^{+\infty} \left(\phi_j, \Psi(x,\cdot)\right)_{L^2(I)} \psi_j(u)$ holds for every $\Psi \in L^2(\Omega)$



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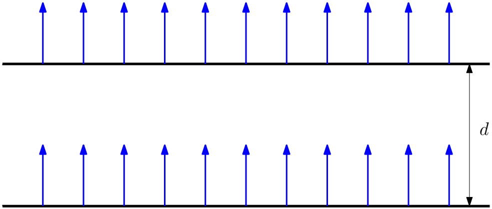

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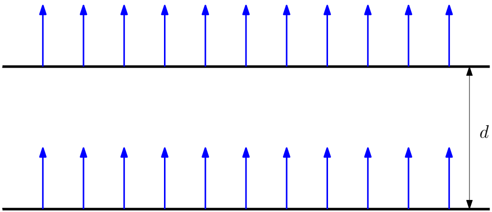
$\Rightarrow$ $\sigma(H_{\alpha_0}) = \sigma_{\text{ess}}(H_{\alpha_0}) = [\mu_0^2, +\infty)$



Essential spectrum of H_α

Theorem

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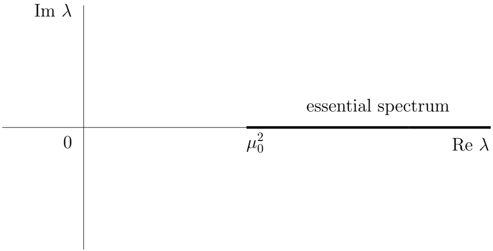
Corollary

$$\sigma_{\text{ess}}(H_\alpha) = \sigma_{\text{ess}}(H_{\alpha_0}) = [\mu_0^2, +\infty).$$



Weakly perturbed boundary conditions

$$\alpha(x) = \alpha_0 + \varepsilon\beta(x),$$

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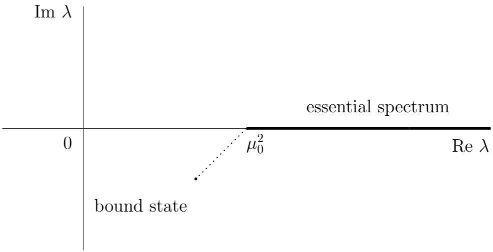

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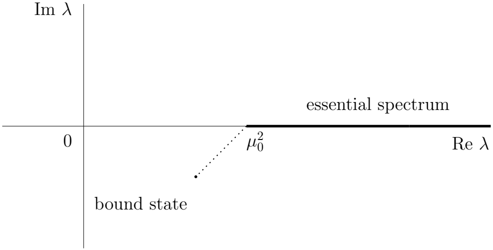

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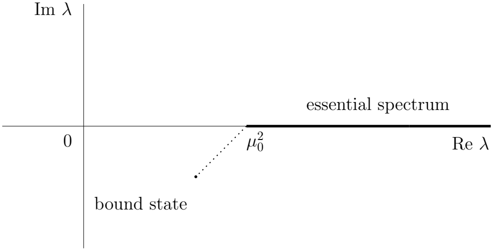
- singular perturbation theory - perturbation of the threshold of the essential spectrum



The bound state

Theorem

Let us assume $\beta \in W^{2,\infty}(\mathbb{R}^n)$ and that β and its first and second derivations go in the infinity to 0 faster than $x^{-4-\delta}$ for some $\delta > 0$

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If $\varepsilon > 0$ is sufficiently small, $|\alpha_0| < \pi/d$ and $\alpha_0\langle\beta\rangle < 0$, then H_α possesses the real and unique eigenvalue $\lambda = \lambda(\varepsilon) \in (-\infty, \mu_0^2)$.

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The asymptotic expansion

$$\lambda(\varepsilon) = \begin{cases} \mu_0^2 - \varepsilon^2\alpha_0^2\langle\beta\rangle^2 + \mathcal{O}(\varepsilon^3) & (\text{if } n = 1), \\ \mu_0^2 - \mathrm{e}^{2/w(\varepsilon)}, & (\text{if } n = 2), \end{cases}$$

where $w(\varepsilon) = \frac{\varepsilon}{\pi}\langle\beta\rangle\alpha_0 + \mathcal{O}(\varepsilon^2)$, holds as $\varepsilon \to 0$.

Ingredients of the proof

Unitary transformation:

$$\begin{aligned} U_\varepsilon^{-1} H_\alpha U_\varepsilon &= H_{\alpha_0} + \varepsilon Z_\varepsilon, \\ \mathrm{Dom}(U_\varepsilon^{-1} H_\alpha U_\varepsilon) &= \mathrm{Dom}(H_{\alpha_0}), \end{aligned}$$

where

$\dots (U_\varepsilon \Psi)(x,u) := \mathrm{e}^{-\mathrm{i}\varepsilon\beta(x)u}\Psi(x,u)$ for any $\Psi \in L^2(\Omega)$

$\dots Z_\varepsilon := 2\mathrm{i}u\nabla'\beta(x)\cdot\nabla' + 2\mathrm{i}\beta(x)\frac{\partial}{\partial u} + \left(\varepsilon\beta^2(x) - \mathrm{i}\Delta'\beta(x)u - \varepsilon u^2|\nabla'\beta|^2\right)$

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$$U_\varepsilon^{-1} H_\alpha U_\varepsilon = H_{\alpha_0} + \varepsilon Z_\varepsilon = H_{\alpha_0} + \varepsilon C_\varepsilon^* D,$$
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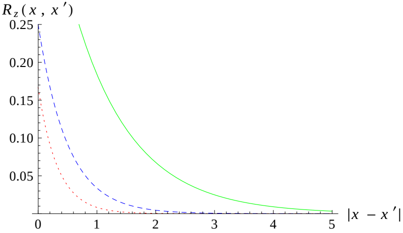
Birman-Schwinger principle:

$$\lambda \in \sigma_{\mathrm{p}}(H_\alpha) \quad \Leftrightarrow \quad -1 \in \sigma_{\mathrm{p}}\left(\varepsilon D(H_{\alpha_0} - \lambda)^{-1} C_\varepsilon^*\right).$$

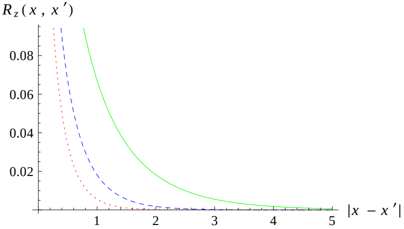
Ingredients of the proof

Separation of the resolvent singularity

$\Rightarrow$ implicit equation for the eigenvalues



$n = 1$



$n = 2$

Ingredients of the proof

Separation of the resolvent singularity

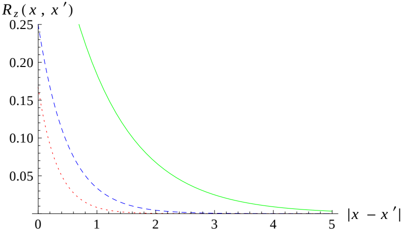
$\Rightarrow$ implicit equation for the eigenvalues

$$-1 = \varepsilon\, F_n\left(\sqrt{\mu_0^2 - \lambda}\right) \int_\Omega \psi_0(u) \left(C_\varepsilon^* (I + M_\varepsilon^\lambda)^{-1} D\overline{\phi_0}\right)(x,u)\, \mathrm{d}x\, \mathrm{d}u,$$

where

... F_n is remainder of the singular part

... M_ε^λ is remainder of the regular part



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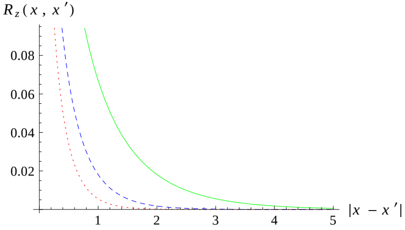

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Banach fixed point theorem

- existence and uniqueness of the solution
- alternatively Rouché theorem

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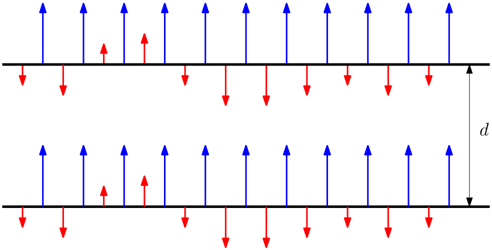
$\mathcal{PT}$-symmetry

- reality of the bound state

Discussion of results

- $\alpha_0 \langle \beta \rangle < 0$
  - perturbation acts against uniform boundary conditions



Discussion of results

- $\alpha_0 \langle \beta \rangle < 0$
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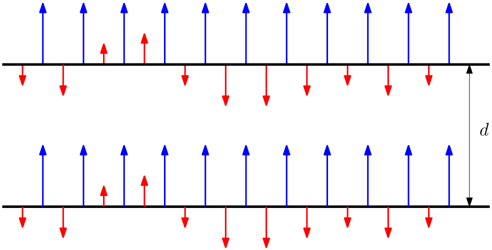
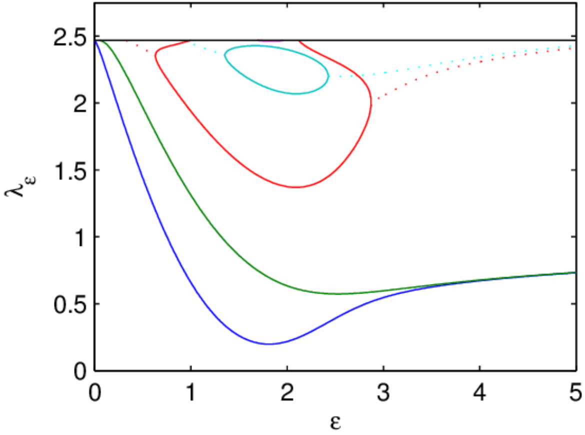
- the ground state energy is real
  - other eigenvalues can be complex!



Numerical results

- the case $n = 1$ [Krejčiřík, Tater 08]



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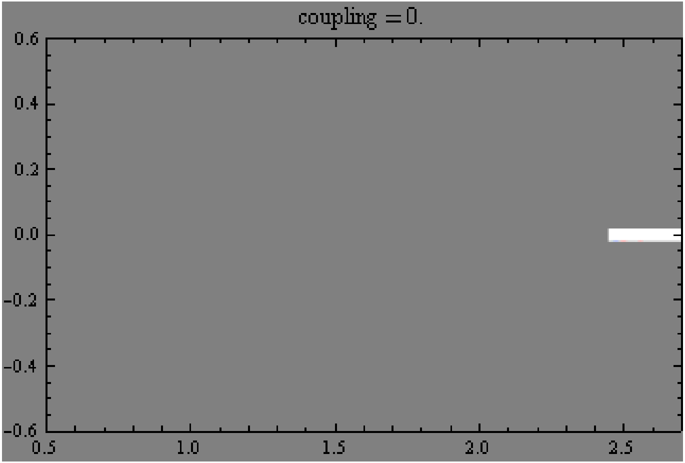

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? Existence of the metric operator

Thank you for your attention!