The separation of two matrices and its application in eigenvalue perturbation theory

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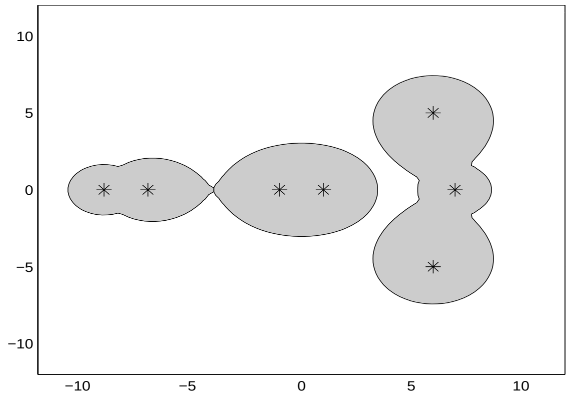
Outline.

- The 3 definitions of separation
- Inclusion theorems for pseudospectra of block triangular matrices
- Perturbation bounds for invariant subspaces

The definitions of separation



Pseudospectra

The pseudospectrum of $A \in \mathbb{C}^{n \times n}$ to the perturbation level $\epsilon > 0$ is

$$\Lambda_\epsilon(A) \;:=\; \text{set of all eigenvalues of all matrices of the form } A + E, \text{ where } E \in \mathbb{C}^{n\times n},\ \|E\| \le \epsilon.$$

$$= \text{ union of the spectra } \Lambda(A+E) \text{ where } E \in \mathbb{C}^{n\times n},\ \|E\| \le \epsilon$$

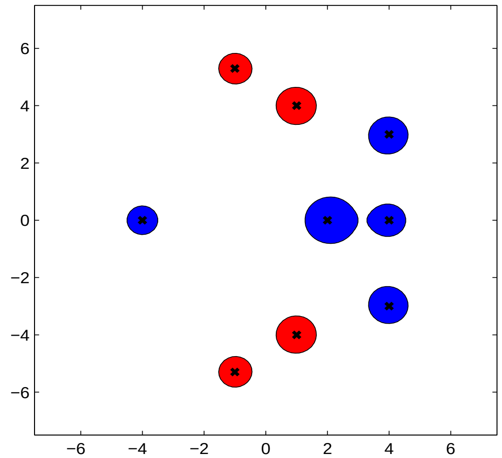
$$= \Lambda(A) \cup \{\, z \in \mathbb{C} \setminus \Lambda(A) \mid \quad \|(zI - A)^{-1}\|^{-1} \le \epsilon \,\}.$$

In this talk $\|\cdot\|$ denotes the spectral norm. Then

$$\Lambda_\epsilon(A) \;:=\; \{z \in \mathbb{C} \mid \sigma_{\min}(zI - A) \le \epsilon\,\}.$$

Separation of two matrices: Demmel's definition

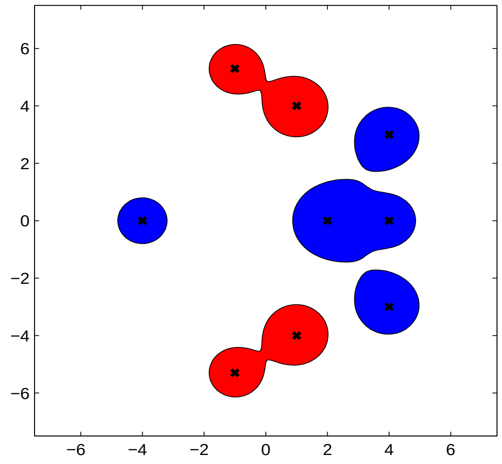
Pseudospectra of $L \in \mathbb{C}^{\ell \times \ell}$ (blue) and $M \in \mathbb{C}^{m \times m}$ (red):



$\epsilon = 0.50$

Separation of two matrices: Demmel's definition

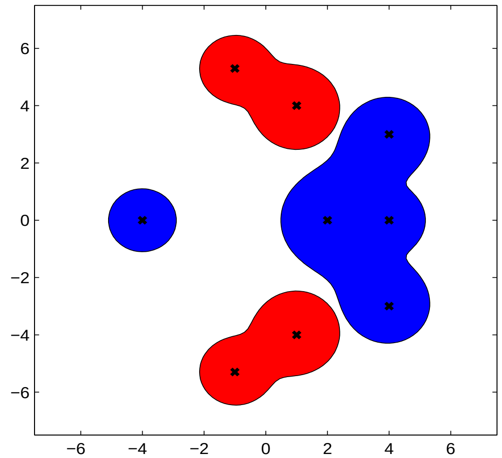
Pseudospectra of $L \in \mathbb{C}^{\ell\times\ell}$ (blue) and $M \in \mathbb{C}^{m\times m}$ (red):



$\epsilon = 0.80$

Separation of two matrices: Demmel's definition

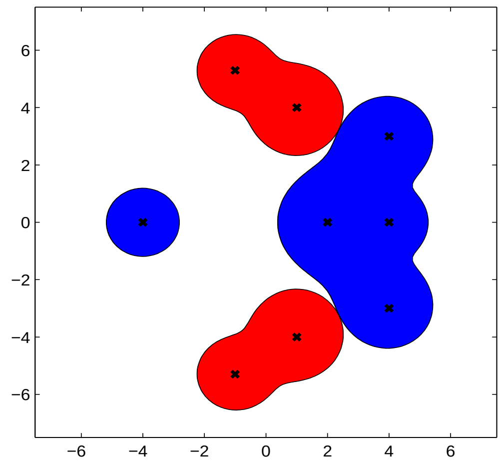
Pseudospectra of $L \in \mathbb{C}^{\ell \times \ell}$ (blue) and $M \in \mathbb{C}^{m \times m}$ (red):



$\epsilon = 1.19$

Separation of two matrices: Demmel's definition

Pseudospectra of $L \in \mathbb{C}^{\ell\times\ell}$ (blue) and $M \in \mathbb{C}^{m\times m}$ (red):

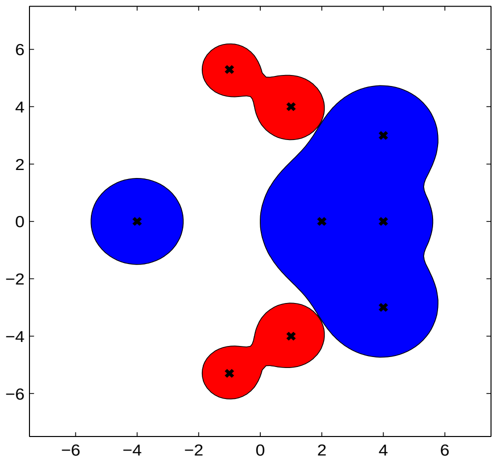


$$\epsilon = 1.19 = \text{sep}_\lambda^D(L, M)$$

$$\begin{aligned} \text{sep}_\lambda^D(L, M) &= \min\{\epsilon \mid \Lambda_\epsilon(L) \cap \Lambda_\epsilon(M) \neq \emptyset\} \\ &= \min_{z\in\mathbb{C}} \max\{\sigma_{\min}(zI - L), \sigma_{\min}(zI - M)\} \end{aligned}$$

Separation of two matrices: Varah's definition

Pseudospectra of $L \in \mathbb{C}^{\ell\times\ell}$ (blue) and $M \in \mathbb{C}^{m\times m}$ (red):



$$\epsilon_1 = 1.5$$

$$\epsilon_2 = 0.85$$

$$\begin{aligned}\mathrm{sep}_\lambda^V(L, M) &= \min\{\epsilon_1 + \epsilon_2 \mid \Lambda_{\epsilon_1}(L) \cap \Lambda_{\epsilon_2}(M) \neq \emptyset\ \} \\ &= \min_{z\in\mathbb{C}} \left[\sigma_{\min}(zI - L) + \sigma_{\min}(zI - M)\right]\end{aligned}$$

Separation of two matrices: Stewart's definition

Definition uses Sylvester-operator $Z \longmapsto T(Z) = MZ - ZL$:

$$\mathrm{sep}(L, M) = \min_{\|Z\|=1} \|MZ - ZL\| .$$

Facts:

- $\mathrm{sep}(L, M) \neq 0$ iff T nonsingular iff $\Lambda(L) \cap \Lambda(M) \neq \emptyset$
- $\mathrm{sep}(L, M) \leq \mathrm{sep}_\lambda^V(L, M)$ if $\|\cdot\|$ is unitarily invariant.

Proof:

$$\begin{aligned}
\Lambda(L + E_1) \cap \Lambda(M + E_2) \neq \emptyset \quad \Rightarrow \quad 0 &= \mathrm{sep}(L + E_1, M + E_2) \\
&= \min_{\|Z\|=1} \|(M + E_2)Z - Z(L + E_1)\| \\
&\geq \mathrm{sep}(L, M) - \|E_1\| - \|E_2\| \\
\Rightarrow \quad & \|E_1\| + \|E_2\| \geq \mathrm{sep}(L, M)
\end{aligned}$$

Comparison of the separations

Stewart's definition:

$$\text{sep}(L, M) = \min_{\|Z\|=1} \|MZ - ZL\|$$

Varah's definition:

$$\text{sep}_\lambda^V(L, M) = \min\{\epsilon_1 + \epsilon_2 \mid \Lambda_{\epsilon_1}(L) \cap \Lambda_{\epsilon_2}(M) \neq \emptyset\}$$

Demmel's definition:

$$\text{sep}_\lambda^D(L, M) = \min\{\epsilon \mid \Lambda_\epsilon(L) \cap \Lambda_\epsilon(M) \neq \emptyset\}$$

Computation of sep_λ^D in [Gu,Overton, 2006] . We have

$$\boxed{\text{sep}(L, M) \leq \text{sep}_\lambda^V(L, M) \leq 2\,\text{sep}_\lambda^D(L, M) \leq \text{dist}(\Lambda(\text{L}), \Lambda(\text{M}))}$$

Equality holds if L and M are both normal and $\|\cdot\|$ is the Frobenius norm.

Remark: For (scaled) Jordan blocks L, M:

$$\text{sep}(L, M) << \text{sep}_\lambda^D(L, M) << \text{dist}(\Lambda(\text{L}), \lambda(\text{M}))$$

Application:

Inclusion theorems for pseudospectra of **block triangular** matrices

The Problem

Let $A \in \mathbb{C}^{n\times n}$ be given in block Schur form:

$$A = U \begin{bmatrix} L & C \\ 0 & M \end{bmatrix} U^*, \qquad U \text{ unitary}, \qquad \Lambda(L) \cap \Lambda(M) = \emptyset.$$

We always have

$$\Lambda_\epsilon(L) \cup \Lambda_\epsilon(M) \subseteq \Lambda_\epsilon(A).$$

Problem: Find a tight function g of ϵ such that

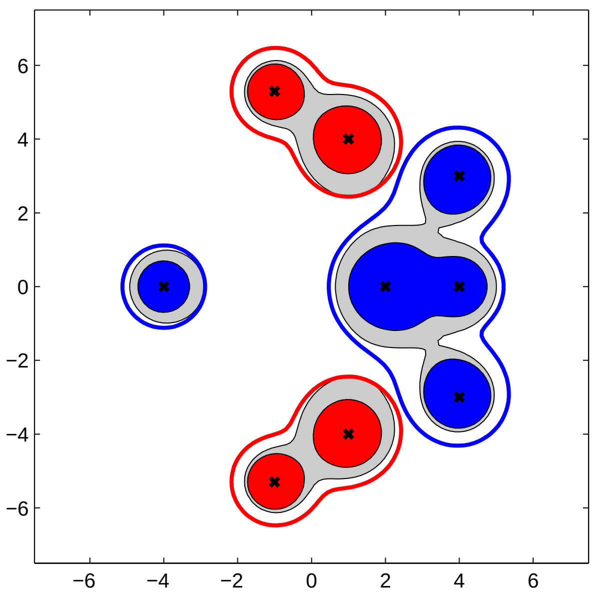
$$\Lambda_\epsilon(A) \subseteq \Lambda_{g(\epsilon)\epsilon}(L) \cup \Lambda_{g(\epsilon)\epsilon}(M). \qquad (*)$$

Relevance:
If $\|E\| = \epsilon$ and the union in $(*)$ is disjoint then precisely $\dim L$ eigenvalues of $A+E$ are contained in $\Lambda_{g(\epsilon)\epsilon}(L)$. The others are contained in $\Lambda_{g(\epsilon)\epsilon}(M)$.

Visualisation of the Problem

Problem again: Find a tight function g of ϵ such that

$$\Lambda_\epsilon\left(\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}\right) \subseteq \Lambda_{g(\epsilon)\epsilon}(L) \cup \Lambda_{g(\epsilon)\epsilon}(M).$$



grey region: $\Lambda_\epsilon\left(\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}\right)$

blue region: $\Lambda_\epsilon(L)$

red region: $\Lambda_\epsilon(M)$

blue curve: boundary of $\Lambda_{g(\epsilon)\epsilon}(L)$

red curve: boundary of $\Lambda_{g(\epsilon)\epsilon}(M)$

Upper bounds in terms of C

Let $A \in \mathbb{C}^{n\times n}$ be given in block Schur form:

$$A = U \begin{bmatrix} L & C \\ 0 & M \end{bmatrix} U^*, \qquad U \text{ unitary}, \qquad \Lambda(L) \cap \Lambda(M) = \emptyset.$$

Then

$$\Lambda_\epsilon(A) \subseteq \Lambda_{g(\epsilon)\epsilon}(L) \cup \Lambda_{g(\epsilon)\epsilon}(M)$$

for

$$g(\epsilon) = \sqrt{1 + \frac{\|C\|}{\epsilon}} \qquad (Grammont, Largillier, 2002)$$

and for

$$g(\epsilon) = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\|C\|}{\epsilon}} \qquad (Bora, 2001)$$

Good: Simple bounds which show that $\Lambda_\epsilon(A) \approx \Lambda_\epsilon(L) \cup \Lambda_\epsilon(M)$ for large ϵ.
Bad: $g(\epsilon) \to \infty$ as $\epsilon \to 0$.

Proof of the Grammont-Largillier-bound

Let

$$a_z := \max\{\|(z\,I - L)^{-1}\|, \|(z\,I - M)^{-1}\|\}.$$

Then we have the following chain of inclusions and inequalities.

$$\begin{aligned}
z \in \Lambda_\epsilon(A) \quad \Rightarrow \quad \epsilon^{-1} \quad &\leq \quad \|(z\,I - A)^{-1}\| \\
&= \quad \left\| \begin{bmatrix} (z\,I - L)^{-1} & -(z\,I - L)^{-1}\,C\,(z\,I - M)^{-1} \\ 0 & (z\,I - M)^{-1} \end{bmatrix} \right\| \\
&\leq \quad \left\| \begin{bmatrix} a_z & {a_z}^2\,\|C\| \\ 0 & a_z \end{bmatrix} \right\|_2 \\
&= \quad a_z \frac{a_z\|C\| + \sqrt{(a_z\|C\|)^2 + 4}}{2}
\end{aligned}$$

$$\Rightarrow \quad 2(\epsilon a_z)^{-1} - a_z\|C\| \quad \leq \quad \sqrt{(a_z\|C\|)^2 + 4}$$

$$\Rightarrow \quad (\epsilon\sqrt{1 + \|C\|/\epsilon})^{-1} \quad \leq \quad a_z$$

$$\Rightarrow$$

$$z \in \Lambda_{\epsilon\sqrt{1+\|C\|/\epsilon}}(L) \,\cup\, \Lambda_{\epsilon\sqrt{1+\|C\|/\epsilon}}(M).$$

Demmel's bound (1983)

Let T be such that
$$T^{-1}\begin{bmatrix} L & C \\ 0 & M \end{bmatrix} T = \begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}.$$

Then the Bauer-Fike-Theorem yields

$$\Lambda_\epsilon\left(\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}\right) \subseteq \Lambda_{\|T\|\,\|T^{-1}\|\,\epsilon}(L) \cup \Lambda_{\|T\|\,\|T^{-1}\|\,\epsilon}(M)$$

Problem: Find such T with smallest condition number $\|T\|\,\|T^{-1}\|$.

Solution: Let R be such that $\boxed{RM - LR = C}$. Then

$$T = \begin{bmatrix} I & R/p \\ 0 & I/p \end{bmatrix}, \qquad p = \sqrt{1 + \|R\|^2}$$

has smallest possible condition number

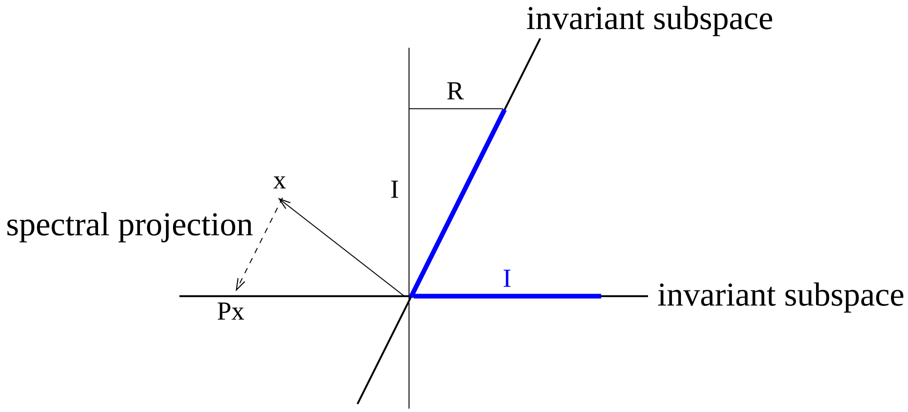
$$\kappa := \|T\|\,\|T^{-1}\| = p + \|R\| = p + \sqrt{p^2 - 1} \leq 2p.$$

Note: $\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}$ has invariant subspaces range $\begin{bmatrix} I \\ 0 \end{bmatrix}$, range $\begin{bmatrix} R \\ I \end{bmatrix}$

and p is the norm of the associated spectral projector.

Illustration: invariant subspaces of

$$A = \begin{bmatrix} L & C \\ 0 & M \end{bmatrix} = \begin{bmatrix} L & RM - LR \\ 0 & M \end{bmatrix}, \quad \Lambda(L) \cap \Lambda(M) = \emptyset.$$



Invariant subspaces: $\text{range} \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \text{range} \begin{bmatrix} R \\ I \end{bmatrix}$

Spectral projector: $P = \begin{bmatrix} I & -R \\ 0 & 0 \end{bmatrix}, \quad p := \|P\| = \sqrt{1 + \|R\|^2}.$

Demmel's result and the separation.

Let $A \in \mathbb{C}^{n\times n}$ be given in block Schur form:

$$A = U\begin{bmatrix} L & C \\ 0 & M \end{bmatrix} U^* = U\begin{bmatrix} L & RM - LR \\ 0 & M \end{bmatrix} U^*, \quad U \text{ unitary}, \quad \Lambda(L) \cap \Lambda(M) = \emptyset.$$

Let

$$\kappa = \|R\| + \sqrt{\|R\|^2 + 1} = \sqrt{p^2 - 1} + p.$$

Then for all $\epsilon \geq 0$,

$$\Lambda_\epsilon(A) \quad \subseteq \quad \Lambda_{\kappa\epsilon}(L) \cup \Lambda_{\kappa\epsilon}(M),$$

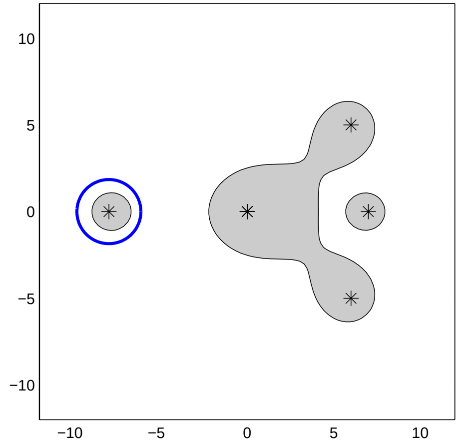
Moreover, if $\epsilon < \text{sep}_\lambda^D(L, M)/\kappa$ then

$$\Lambda_{\kappa\epsilon}(L) \cap \Lambda_{\kappa\epsilon}(M) = \emptyset.$$

Corollary to Demmel's result.

If $L = \lambda I$ (i.e. λ is a semisimple eigenvalue of A) then

$$\begin{aligned} \Lambda_\epsilon(A) &\subseteq \Lambda_{\kappa\,\epsilon}(L) \cup \Lambda_{\kappa\,\epsilon}(M) \\ &= \underbrace{D_{\kappa\,\epsilon}(\lambda)}_{\text{Disk of radius } \kappa\epsilon} \cup \Lambda_{\kappa\,\epsilon}(M), \end{aligned}$$



where $\kappa = \|R\| + p = \sqrt{p^2 - 1} + p \approx 2p$

and $p = \sqrt{1 + \|R\|^2}$ is the norm of the spectral projector.

Furthermore, if ϵ is small enough then $D_{\kappa\,\epsilon}(\lambda)$ contains only one connected component $\mathcal{C}_\epsilon(\lambda)$ of $\Lambda_\epsilon(A)$. But we know that for small ϵ

$$\mathcal{C}_\epsilon(\lambda) \approx D_{p\,\epsilon}(\lambda)$$

since p is the condition number of λ.

Question: Is Demmel's bound to large (factor ≈ 2)?

Inclusion bound for small ϵ: Demmel's separation

Let $A \in \mathbb{C}^{n\times n}$ be given in block Schur form:

$$A = U\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}U^* = U\begin{bmatrix} L & RM - LR \\ 0 & M \end{bmatrix}U^*, \quad U \text{ unitary}, \quad \Lambda(L)\cap\Lambda(M) = \emptyset.$$

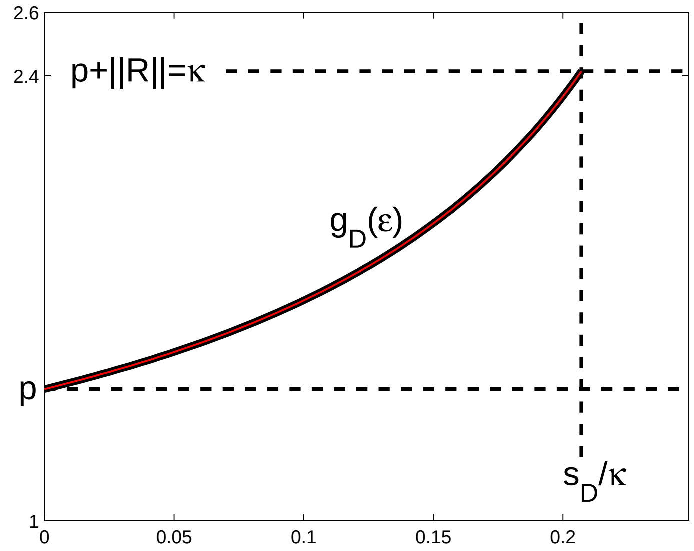
Let $s_D = \text{sep}_\lambda^D(L,M)$, $\kappa = \|R\| + \sqrt{\|R\|^2+1} = \sqrt{p^2-1}+p$.

Then for $\epsilon \le s_D/\kappa$,

$$\Lambda_\epsilon(A) \quad\subseteq\quad \Lambda_{g_D(\epsilon)\,\epsilon}(L) \cup \Lambda_{g_D(\epsilon)\,\epsilon}(M)$$

where

$$g_D(\epsilon) = p + \frac{\|R\|^2\,\epsilon}{s_D - p\,\epsilon}.$$



Inclusion bound for small ϵ: Varah's separation

Let $A \in \mathbb{C}^{n\times n}$ be given in block Schur form:

$$A = U\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}U^* = U\begin{bmatrix} L & RM - LR \\ 0 & M \end{bmatrix}U^*, \quad U \text{ unitary}, \quad \Lambda(L)\cap\Lambda(M) = \emptyset.$$

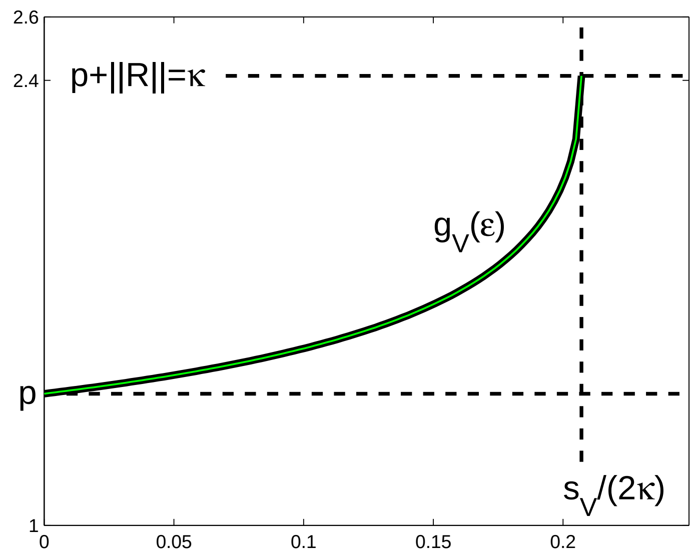
Let $s_V = \mathrm{sep}_\lambda^V(L,M)$, $\kappa = \|R\| + \sqrt{\|R\|^2+1} = \sqrt{p^2-1}+p$.

Then for $\epsilon \le s_V/(2\kappa)$,

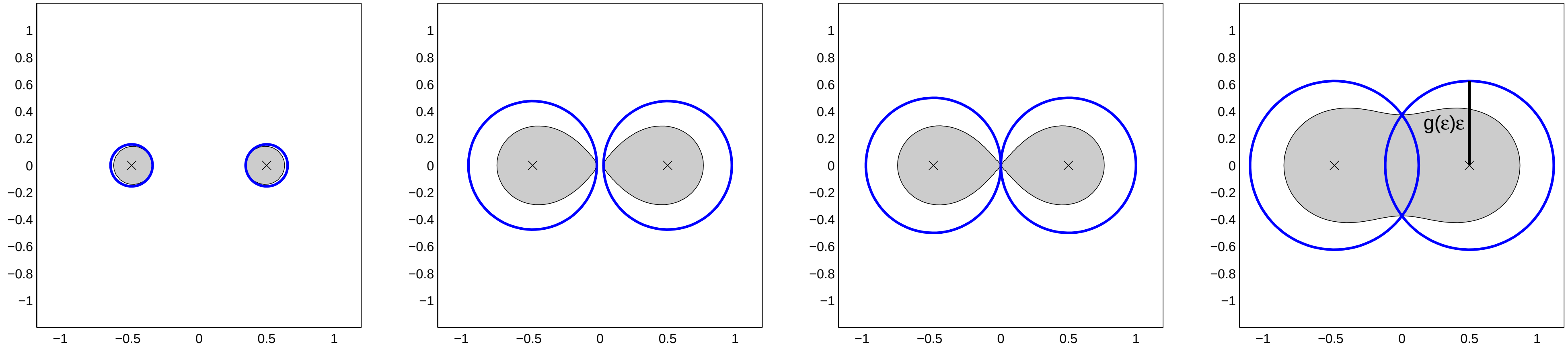
$$\Lambda_\epsilon(A) \quad \subseteq \quad \Lambda_{g_V(\epsilon)\,\epsilon}(L) \cup \Lambda_{g_V(\epsilon)\,\epsilon}(M),$$

where

$$g_V(\epsilon) = \frac{p - \epsilon/s_V}{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\epsilon}{s_V}\left(p - \frac{\epsilon}{s_V}\right)}}.$$



The 2×2 case



Let $A = \begin{bmatrix} \frac{s}{2} & c \\ 0 & -\frac{s}{2} \end{bmatrix} = \begin{bmatrix} \frac{s}{2} & s\,r \\ 0 & -\frac{s}{2} \end{bmatrix}$, $s > 0$, $c, r \geq 0$. Then

$$s = \mathrm{sep}^V_\lambda(-s/2, s/2) = 2\,\mathrm{sep}^D_\lambda(-s/2, s/2) = \mathrm{sep}(L, M).$$

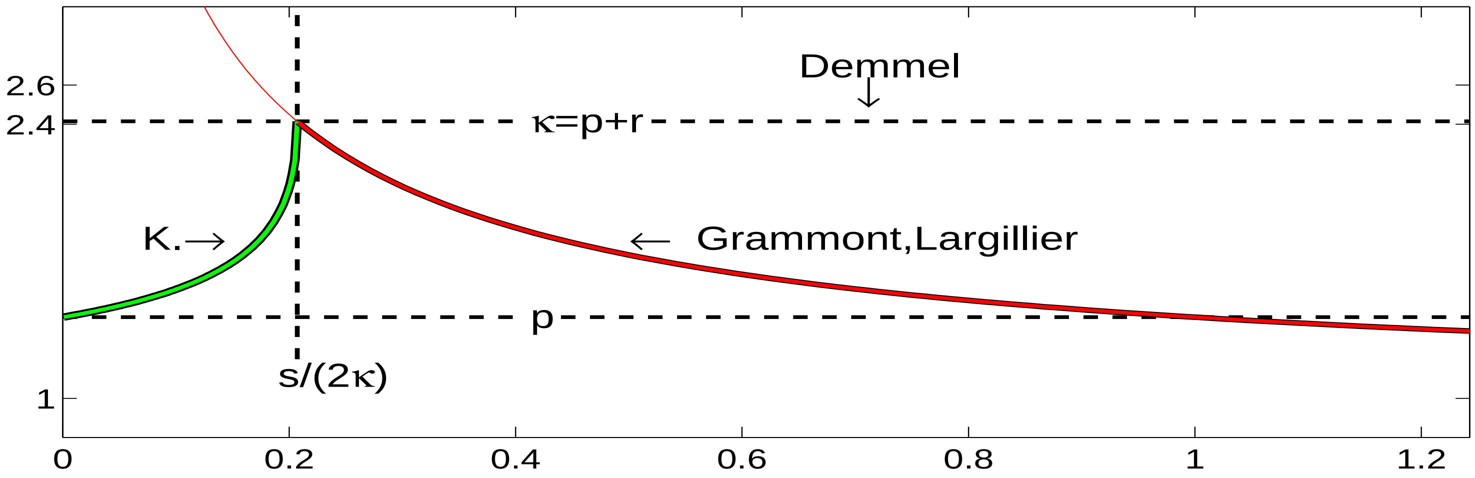
The 1×1 pseudospectra of the eigenvalues $\pm s/2$ are disks:

$$\Lambda_\epsilon(\pm s/2) = D_\epsilon(\pm s/2) = \{z \in \mathbb{C} \mid |z \mp s/2| \leq \epsilon\}.$$

We are looking for

$$g(\epsilon) = \min\{\, g \geq 0 \mid \ \Lambda_\epsilon(A) \subseteq D_{g\,\epsilon}(-s/2) \cup D_{g\,\epsilon}(s/2) \ \}.$$

Bounds are exact in the 2×2 case



Let $A = \begin{bmatrix} \frac{s}{2} & c \\ 0 & -\frac{s}{2} \end{bmatrix} = \begin{bmatrix} \frac{s}{2} & s\,r \\ 0 & -\frac{s}{2} \end{bmatrix}$, $s > 0$, $c, r \geq 0$, and let

$$g(\epsilon) = \min\{\, g \geq 0 \mid \Lambda_\epsilon(A) \subseteq D_{g\,\epsilon}(-s/2) \cup D_{g\,\epsilon}(s/2) \,\}.$$

Then we have ($p = \sqrt{1 + r^2}$, $\kappa = p + r$):

$$g(\epsilon) = \begin{cases} \dfrac{p - \epsilon/s}{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\epsilon}{s}\left(p - \frac{\epsilon}{s}\right)}} & \text{if } \epsilon \leq s/(2\kappa), \qquad \text{(K.)} \\ \sqrt{1 + c/\epsilon} & \text{if } \epsilon \geq s/(2\kappa) \qquad \text{(Grammont, Largillier)} \end{cases}$$

Literature:

1. J.M. Varah: On the separation of two matrices, SIAM J. Numer. Anal. 16, No. 2, 1979

2. On ϵ-spectra and stability radii, J. Comp. Appl. Math. 147, 2002

3. J. W. Demmel: Computing Stable Eigendecompositions of Matrices, Lin. Alg. Appl. 79, 1986.

4. J. W. Demmel: The Condition Number of Equivalence Transformations that Block Diagonalize Matrix Pencils, SIAM J. Numer. Anal. 20, No. 3, 1983.

Application of Stewart's separation: perturbation bounds for invariant subspaces

Joint work with Daniel Kressner

Recall: $$\mathrm{sep}(L, M) = \min_{\|Z\|=1} \| \underbrace{MZ - ZL}_{T(Z)} \|$$

Invariant subspaces and Riccati equations

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{C}^{(\ell+m)\times(\ell+m)}$, $Z \in \mathbb{C}^{m\times\ell}$

Basic fact:

range $\begin{bmatrix} I \\ Z \end{bmatrix}$ is an ℓ-dimensional invariant subspace of A iff

Z satisfies the (nonsymmetric) Riccati equation

$$\underbrace{A_{21} + A_{22}Z - ZA_{11} - ZA_{12}Z}_{=:\mathcal{R}(A,Z)} = 0$$

since then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I \\ Z \end{bmatrix} = \begin{bmatrix} I \\ Z \end{bmatrix} (A_{11} + A_{12}Z).$$

On the following slides:

- $$A = \underbrace{\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}}_{E}, \qquad \Lambda(L) \cap \Lambda(M) = \emptyset.$$

- E is perturbation of A_0.
- The invariant subspace range $\begin{bmatrix} I \\ Z \end{bmatrix}$ of $A_0 + E$

  is perturbation of

  the invariant subspace range $\begin{bmatrix} I \\ 0 \end{bmatrix}$ of A_0, where

$$\mathcal{R}(A_0 + E, Z) = 0.$$

Problem: Bound for $\|Z\|$ (with E as large as possible)

Stewart's bound for invariant subspace of

$$A = \underbrace{\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}}_{E} = \begin{bmatrix} L + E_{11} & E_{12} + C \\ E_{21} & M + E_{22} \end{bmatrix}.$$

Let $s_E = \mathrm{sep}(L + E_{11}, M + E_{22})$ w.r.t $\|\cdot\|$ and suppose

$$\|E_{21}\|\,\|E_{12} + C\| < \frac{{s_E}^2}{4}$$

Then $\mathcal{R}(A_0 + E, Z) = 0$ has a unique solution Z, and

$$\boxed{\|Z\| \;\leq\; \frac{2\,\|E_{21}\|}{s_E + \sqrt{s_E^2 - 4\|E_{21}\|\,\|E_{12} + C\|}} \;\leq\; \frac{2\,\|E_{21}\|}{s_E}.}$$

Proof: Write Riccati equation in fixed point form,

$$Z = T_E^{-1}(E_{21} - ZE_{12}Z), \qquad T_E(Z) = (M + E_{22})Z - Z(L + E_{11}),$$

and apply the contraction mapping theorem. We have $s_E = \|T_E^{-1}\|^{-1}$.

New bound for invariant subspace of

$$A = \underbrace{\begin{bmatrix} L & C \\ 0 & M \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}}_{E}.$$

Let $s = \mathrm{sep}(L, M)$ w.r.t. $\|\cdot\|$ and suppose

$$\|E\| \left(\|E\| + \|C\|\right) < \frac{s^2}{4}$$

Then $\mathcal{R}(A_0 + E, Z) = 0$ has a unique solution Z, and

$$\boxed{\|Z\| \;\leq\; \frac{2\,\|E\|}{s + \sqrt{s^2 - 4\|E\|\left(\|E\| + \|C\|\right)}} \;\leq\; \frac{2\,\|E\|}{s}.}$$

Proof: Write Riccati equation in fixed point form,

$$Z = T^{-1}([-Z \;\; I]E[I \;\; Z^\top]^\top), \qquad T(Z) = MZ - ZL,$$

and apply Brouwer's fixed point theorem. We have $s = \|T^{-1}\|^{-1}$.

Block diagonal case

$$A = \underbrace{\begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}}_{E}.$$

Let $s = \mathrm{sep}(L, M)$ w.r.t. $\|\cdot\|$ and suppose

$$\|E\| < \frac{s}{2} \qquad (*)$$

Then $\mathcal{R}(A_0 + E, Z) = 0$ has a unique solution Z, and

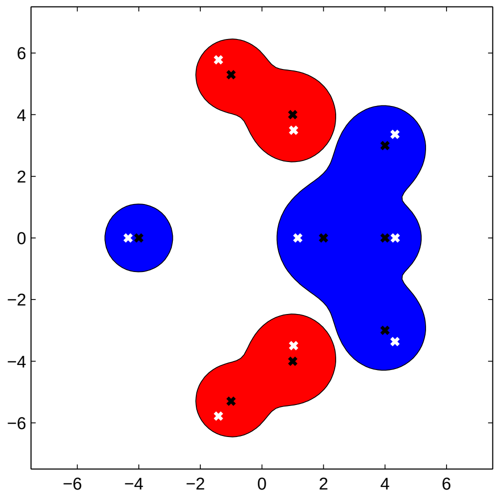
$$\|Z\| \ \le \ \frac{2\,\|E\|}{s + \sqrt{s^2 - 4\|E\|^2}} \ \le \ \frac{2\,\|E\|}{s}.$$

Open problem: Can condition $(*)$ be replaced by

$$\|E\| < \mathrm{sep}_\lambda^D(L, M) \qquad ?$$

Open question extended

Let $$A = \underbrace{\begin{bmatrix} L & 0 \\ 0 & M \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}}_{E}.$$ Then $\Lambda_\epsilon(A_0) = \Lambda_\epsilon(L) \cup \Lambda_\epsilon(M)$.



If $\|E\| = \epsilon < \mathrm{sep}_\lambda^D(L, M)$ then precisely $\dim L$ eigenvalues of $A_0 + E$ (white crosses) are contained in $\Lambda_\epsilon(L)$ (blue region).

Is the associated invariant subspaces always of the form

$$\mathrm{range} \begin{bmatrix} I \\ Z \end{bmatrix} \qquad \text{(graph subspace)} \quad ?$$

Thanks for listening