Left Invertibility of Formal Hamiltonian Operators

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The formal Hamiltonian operator

Let X be an infinite dimensional Hilbert space. The formal Hamiltonian operator is given by the densely defined closed block operator matrix

$$H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : (\mathcal{D}(A) \cap \mathcal{D}(C)) \oplus (\mathcal{D}(B) \cap \mathcal{D}(A^*)) \to X \oplus X,$$

where A is a densely defined closed operator, B and C are symmetric operators in X.

- If B and C are self-adjoint, then H is Hamiltonian which naturally arise from linear infinite dimensional Hamiltonian systems. Due to the requirement in applications, one is also interested in the case of symmetric operators B and C.
- The formal Hamiltonian operator is of symplectic symmetry, i.e., $JH \subset (JH)^*$ with $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

Left invertibility

Let T be a closed operator between Banach spaces X and Y. Then, T is said to be left invertible if there exists a bounded operator S such that $ST = I|_{\mathcal{D}(T)}$.
It is well known that T is left invertible if and only if T is injective and its range $\mathcal{R}(T)$ is closed.

Completion problem

Partial operator matrix is operator matrix whose entries are specified only on a subset of its positions, while a completion of a partial operator matrix is the operator matrix resulting from filling in its unspecified entries. In operator matrix completion problems, one concerns the conditions under which a partial operator matrix has completions with some given properties.
Here, we are seeking for conditions on A and C such that the partial formal Hamiltonian operator

$$\begin{pmatrix} A & ? \\ C & -A^* \end{pmatrix}$$

has a left invertible completion with domain $\mathcal{D}(A) \cap \mathcal{D}(C) \oplus \mathcal{D}(A^*)$.
The operator matrix completion problem was shown to be very useful in operator theory, numerical analysis, optimal control, system theory and engineering problems.

Main result

Let A be a densely defined operator, and let C be a symmetric operator in X. Write $A_1 = A|_{\mathcal{D}(A)\cap\mathcal{D}(C)}$ and $C_1 = C|_{\mathcal{D}(A)\cap\mathcal{D}(C)}$. Then,

$$A_1 = (A_{11}\ A_{12}), \quad C_1 = (C_{11}, 0)$$

with respect to the domain space decomposition $X = \mathcal{N}(C_1)^\perp \oplus \mathcal{N}(\overline{C_1})$, and $A^* = \begin{pmatrix} (A^*)_1 \\ (A^*)_2 \end{pmatrix}$ with respect to the range space decomposition $X = \overline{\mathcal{R}(C_1)} \oplus \mathcal{R}(C_1)^\perp$, where $\overline{C_1}$ denotes the closure of C_1.

Theorem 1. Let A_{11} be a closable operator, and let A_{12}, $(A^*)_2$ and C_1 be closed operators with $(A^*)_1 \subset A_{11}^*$, $\mathcal{D}(C_{11}^*) \subset \mathcal{D}((A^*)_1^*)$ and $\mathcal{R}(C_1)$ being closed.
(i) Assume, in addition, codim $\mathcal{R}(A_{12}) < \infty$, then there exists a symmetric operator B with $\mathcal{D}(A^*) \subset \mathcal{D}(B)$ in X such that the formal Hamiltonian operator

$$H_B = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : (\mathcal{D}(A) \cap \mathcal{D}(C)) \oplus \mathcal{D}(A^*) \to X \oplus X$$

is left invertible if and only if A_{12} is left invertible and $\mathcal{R}((A^*)_2)$ is closed.
(ii) Assume, in addition, codim $\mathcal{R}(A_{12}) = \infty$, then there exists a symmetric operator B with $\mathcal{D}(A^*) \subset \mathcal{D}(B)$ in X such that the formal Hamiltonian operator

$$H_B = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : (\mathcal{D}(A) \cap \mathcal{D}(C)) \oplus \mathcal{D}(A^*) \to X \oplus X$$

is left invertible if and only if A_{12} is left invertible.

In particular, for the upper Triangular case, we have:

Theorem 2. There exists a symmetric operator B with $\mathcal{D}(A^*) \subset \mathcal{D}(B)$ in X such that the formal Hamiltonian operator

$$H_B = \begin{pmatrix} A & B \\ 0 & -A^* \end{pmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(A^*) \to X \oplus X$$

is left invertible if and only if A is left invertible.

Theorem 3. There exists a bounded self-adjoint operator B with $\mathcal{D}(A^*) \subset \mathcal{D}(B)$ in X such that the Hamiltonian operator

$$H_B = \begin{pmatrix} A & B \\ 0 & -A^* \end{pmatrix} : \mathcal{D}(A) \oplus \mathcal{D}(-A^*) \to X \oplus X$$

is left invertible if and only if A is left invertible.

References
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Personal Information

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- born October 10th, 1977, in Inner Mongolia, P.R. China
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- research interest is mainly focused on spectral properties of block operator matrices, especially the spectral theory of Hamiltonian operators and applications.