Recent advances in oscillation theory of discrete symplectic systems

Roman Šimon Hilscher

$$z_{k+1} = \mathcal{S}_k(\lambda)\, z_k, \qquad \mathcal{S}_k^T(\lambda)\, \mathcal{J}\, \mathcal{S}_k(\lambda) = \mathcal{J}, \qquad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

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Abstract

Symplectic systems represent a discrete time analogue of linear Hamiltonian systems. They contain as special cases many important difference equations and systems, namely the Sturm–Liouville difference equations, symmetric three-term recurrence equations, Jacobi difference equations, and linear Hamiltonian difference systems. Following our recent work in Linear Algebra Appl. and SIAM J. Matrix Anal. Appl., we present a new theory of discrete symplectic systems, in which the dependence on the spectral parameter is nonlinear. This requires to develop new definitions of (finite) eigenvalues and (finite) eigenfunctions and their multiplicities for such systems. Our main results include the corresponding oscillation theorems, which relate the number of (finite) eigenvalues with the number of focal points of the principal solution in the given discrete interval, and comparison theorems for (finite) eigenvalues of two symplectic eigenvalue problems. The present theory generalizes several known results, which depend linearly on the spectral parameter. Our results are new even for the above mentioned special discrete symplectic systems.

Discrete symplectic systems

- **Traditional setting:**

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k - \lambda\, W_k x_{k+1}, \tag{$\mathrm{S}_\lambda^{\mathrm{lin}}$}$$

$$\mathcal{S}_k^T \mathcal{J}\, \mathcal{S}_k = \mathcal{J}, \quad \mathcal{S}_k := \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

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- **System ($\mathrm{S}_\lambda^{\mathrm{lin}}$) is symplectic:**

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  - ▹ linear Hamiltonian difference systems

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = (C_k - \lambda W_k)\, x_{k+1} - A_k^T u_k$$

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▷ $\lambda_0 \in \mathbb{C}$ is a finite eigenvalue if there exists a solution (x, u) of (E_0) with

$$W_k x_{k+1} \not\equiv 0 \quad \text{on } [0, N]_{\mathbb{Z}},$$

$z = (x, u)$ is then called a finite eigenfunction for λ_0, the dimension $\omega(\lambda_0)$ of functions $\{W_k x_{k+1}\}_{k=0}^{N}$, where (x, u) is a finite eigenfunction for λ_0, is called a geometric multiplicity of λ_0,

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▹ finite eigenvalues of (E_0) are real, isolated, and bounded from below,

▹ λ_0 is a finite eigenvalue of (E_0) if and only if

$$\theta(\lambda_0) := r - \operatorname{rank} \hat{X}_{N+1}(\lambda_0) \geq 1, \quad r := \max_{\nu \in \mathbb{R}} \operatorname{rank} \hat{X}_{N+1}(\nu),$$

and in this case $\theta(\lambda_0)$ is an algebraic multiplicity of λ_0, where $\big(\hat{X}(\lambda), \hat{U}(\lambda)\big)$ is the principal solution of $(\mathrm{S}_\lambda^{\mathrm{lin}})$, i.e., $\hat{X}_0(\lambda) = 0$ and $\hat{U}_0(\lambda) = I$,

- ▷ the oscillation theorem holds, i.e., (stated here for the scalar case) there exists $\ell \in \mathbb{N} \cup \{0\}$ such that m-th finite eigenfunction has precisely $m + \ell$ focal points in $(0, N + 1]$,

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- ▷ finite eigenfunctions corresponding to different finite eigenvalues are orthogonal with respect to the semi-inner product

$$\big\langle (x, u);\ (\tilde{x}, \tilde{u}) \big\rangle_W := \sum_{k=0}^{N} x_{k+1}^T W_k\, \tilde{x}_{k+1},$$

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- ▷ the Rayleigh principle, i.e., the finite eigenvalues have extremal properties with respect to the associated quadratic form (essential ingredient – the first equation in (S_λ^{lin}) does not depend on λ),

  ⊛ M. Bohner, O. Došlý, W. Kratz, Sturmian and spectral theory for discrete symplectic systems, *Trans. Amer. Math. Soc.* **361** (2009), no. 6, 3109–3123.

- Weyl–Titchmarch theory – Weyl disks, $M(\lambda)$-function, square summable solutions, limit point and limit circle classification
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- Relative oscillation theory
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- **Discrete symplectic systems with nonlinear dependence on λ:**

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$$\Psi_k(\lambda) := \mathcal{J}\, \dot{\mathcal{S}}_k(\lambda)\, \mathcal{J}\, \mathcal{S}_k^T(\lambda)\, \mathcal{J} \quad \text{is symmetric and} \quad \Psi_k(\lambda) \geq 0.$$

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$$\mathcal{A}_k,\ \mathcal{B}_k,\ \mathcal{C}_k,\ \mathcal{D}_k : \mathbb{R} \to \mathbb{R}^{n \times n} \quad \text{are } \mathrm{C}^1_{\mathrm{p}} \text{ functions.}$$

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- **Matrix $\Psi_k(\lambda)$:** Lax 1997 – A differentiable function $\mathcal{S}(\lambda)$ is symplectic for all $\lambda \in \mathbb{R}$ if and only if $\mathcal{S}(0)$ is symplectic and $\dot{\mathcal{S}}(\lambda) = \mathcal{J}\,\Psi(\lambda)\,\mathcal{S}(\lambda)$ with a symmetric $\Psi(\lambda)$ for all $\lambda \in \mathbb{R}$.

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• **Monotonicity assumption** $\Psi_k(\lambda) \geq 0$**:** implies all the nice properties of solutions (conjoined bases) of system (S_λ), including the finite eigenvalues and finite eigenfunctions.

- **Special linear case – system (S_λ^{lin}):**

$$\Psi_k(\lambda) = \begin{pmatrix} W_k & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

- **Special linear case – system $(\mathrm{S}_\lambda^{\mathrm{lin}})$:**

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- **Conjoined bases $\big(X(\lambda), U(\lambda)\big)$ of (S_λ):**

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A conjoined basis $\big(X(\lambda), U(\lambda)\big)$ has a focal point in $(k, k+1]$ if

$$m_k(\lambda) := \operatorname{rank} M_k(\lambda) + \operatorname{ind} P_k(\lambda) \geq 1,$$

and then $m_k(\lambda)$ is its multiplicity, where

$$\begin{aligned} M_k(\lambda) &:= [I - X_{k+1}(\lambda)\, X_{k+1}^\dagger(\lambda)]\, \mathcal{B}_k(\lambda), \\ T_k(\lambda) &:= I - M_k^\dagger(\lambda)\, M_k(\lambda), \\ P_k(\lambda) &:= T_k(\lambda)\, X_k(\lambda)\, X_{k+1}^\dagger(\lambda)\, \mathcal{B}_k(\lambda)\, T_k(\lambda). \end{aligned}$$

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- $m_k(\lambda) \leq \operatorname{rank} \mathcal{B}_k(\lambda) \leq n$ and the matrix $P_k(\lambda)$ is always symmetric.

Lemma 1. *Under* $\Psi_k(\lambda) \geq 0$ *for* $k \in [0, N]_{\mathbb{Z}}$*, for every conjoined basis* $\big(X(\lambda), U(\lambda)\big)$ *of* (S_λ)*, whose initial conditions do not depend on* λ*, the one-sided limits*

$$\operatorname{rank} X_k(\lambda^{\pm}), \quad \operatorname{rank} M_k(\lambda^{\pm}), \quad \operatorname{ind} P_k(\lambda^{\pm}), \quad m_k(\lambda^{\pm})$$

exist finite for every $\lambda \in \mathbb{R}$ *and* $k \in [0, N+1]_{\mathbb{Z}}$*, resp.* $k \in [0, N]_{\mathbb{Z}}$.

Lemma 1. *Under* $\Psi_k(\lambda) \geq 0$ *for* $k \in [0, N]_{\mathbb{Z}}$, *for every conjoined basis* $\big(X(\lambda), U(\lambda)\big)$ *of* (S_λ), *whose initial conditions do not depend on* λ, *the one-sided limits*

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exist finite for every $\lambda \in \mathbb{R}$ *and* $k \in [0, N+1]_{\mathbb{Z}}$, *resp.* $k \in [0, N]_{\mathbb{Z}}$. *Moreover,* $P_k(\lambda)$ *is nondecreasing in* λ *and* $Q_{k+1} := U_{k+1}(\lambda)\, X_{k+1}^{-1}(\lambda)$ *is nonincreasing in* λ *when* $X_{k+1}(\lambda)$ *is invertible.*

Lemma 1. *Under* $\Psi_k(\lambda) \geq 0$ *for* $k \in [0, N]_{\mathbb{Z}}$, *for every conjoined basis* $\big(X(\lambda), U(\lambda)\big)$ *of* (S_λ), *whose initial conditions do not depend on* λ, *the one-sided limits*

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- **Index theorem:** allows to compute the change of the index of a monotone matrix-valued function when its variable passes through a singularity:
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  This incorporates, in particular, Sturm–Liouville difference equations of arbitrary order.

Eigenvalue theory

- **Eigenvalue problem:**

$$(\mathrm{S}_\lambda), \quad \lambda \in \mathbb{R}, \quad x_0 = 0 = x_{N+1}. \tag{E$_0$}$$

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$$\theta(\lambda_0) := \operatorname{rank} \hat{X}_{N+1}(\lambda_0^-) - \operatorname{rank} \hat{X}_{N+1}(\lambda_0) \geq 1,$$

where $\big(\hat{X}(\lambda), \hat{U}(\lambda)\big)$ is the principal solution of (S_λ), i.e.,

$$\hat{X}_0(\lambda) = 0 \quad \text{and} \quad \hat{U}_0(\lambda) = I \quad \text{for all } \lambda \in \mathbb{R}.$$

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• **Regular case:** If $\hat{X}_{N+1}(\lambda)$ is invertible for all $\lambda \in \mathbb{R}$ except at isolated values of λ (e.g. under the Atkinson definiteness condition or when (S_λ) is controllable/normal), then $\operatorname{rank} \hat{X}_{N+1}(\lambda_0^-) = n$ and

$$\theta(\lambda_0) = n - \operatorname{rank} \hat{X}_{N+1}(\lambda_0) = \operatorname{def} \hat{X}_{N+1}(\lambda_0).$$

In this case λ_0 is a classical eigenvalue of (E_0).

Denote

$$n_1(\lambda) := \text{ the number of focal points of } \big(\hat{X}(\lambda), \hat{U}(\lambda)\big) \text{ in } (0, N+1],$$
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Then from this definition we have

$$
n_2(\lambda^+) = n_2(\lambda), \quad n_2(\lambda) - n_2(\lambda^-) = \theta(\lambda) \quad \text{for all } \lambda \in \mathbb{R},
$$

i.e., $n_2(\lambda)$ is right-continuous and the difference $n_2(\lambda) - n_2(\lambda^-)$ gives the number of finite eigenvalues at λ.

Theorem 2 (Global oscillation theorem)**.** *Assume* $\Psi_k(\lambda) \geq 0$ *and* $\operatorname{Im} \mathcal{B}_k(\lambda)$ *constant in* λ *on* $\mathbb{R}$ *for all* $k \in [0, N]_{\mathbb{Z}}$.

Theorem 2 (Global oscillation theorem). *Assume* $\Psi_k(\lambda) \geq 0$ *and* $\operatorname{Im} \mathcal{B}_k(\lambda)$ *constant in* λ *on* $\mathbb{R}$ *for all* $k \in [0, N]_{\mathbb{Z}}$. *Then* $n_1(\lambda)$ *is right-continuous,*

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Corollary 3. *Under* $\Psi_k(\lambda) \geq 0$ *and* $\operatorname{Im} \mathcal{B}_k(\lambda)$ *constant in* λ *on* $\mathbb{R}$ *for all* $k \in [0, N]_{\mathbb{Z}}$, *the finite eigenvalues of* (E_0) *are isolated and bounded from below.*

- **Quadratic functional:** For admissible $z = (x, u)$ we have

$$\begin{aligned}\mathcal{F}_0(z,\lambda) &= \sum_{k=0}^{N} \begin{pmatrix} x_k \\ u_k \end{pmatrix}^T \begin{pmatrix} \mathcal{C}_k^T(\lambda)\,\mathcal{A}_k(\lambda) & \mathcal{C}_k^T(\lambda)\,\mathcal{B}_k(\lambda) \\ \mathcal{B}_k^T(\lambda)\,\mathcal{C}_k(\lambda) & \mathcal{D}_k^T(\lambda)\,\mathcal{B}_k(\lambda) \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} \\ &= \sum_{k=0}^{N} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \mathcal{G}_k(\lambda) \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix},\end{aligned}$$

where

$$\mathcal{G}_k := \begin{pmatrix} \mathcal{A}_k^T \mathcal{E}_k \mathcal{A}_k - \mathcal{C}_k^T \mathcal{A}_k & \mathcal{C}_k^T - \mathcal{A}_k^T \mathcal{E}_k \\ \mathcal{C}_k - \mathcal{E}_k \mathcal{A}_k & \mathcal{E}_k \end{pmatrix}, \quad \mathcal{E}_k := \mathcal{B}_k \mathcal{B}_k^\dagger \mathcal{D}_k \mathcal{B}_k^\dagger.$$

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Theorem 4. *The number ℓ in Theorem 2 is zero, i.e.,*

$$n_1(\lambda) = n_2(\lambda) \quad \textit{for all } \lambda \in \mathbb{R},$$

if and only if the associated quadratic functional $\mathcal{F}_0(z, \lambda_0)$ is positive definite for some $\lambda_0 < 0$.

- **Recall:**

$$\mathcal{S}_k^T(\lambda)\,\mathcal{J}\,\mathcal{S}_k(\lambda)=\mathcal{J}, \quad \Psi_k(\lambda)\geq 0, \quad k\in[0,N]_{\mathbb{Z}}, \quad \lambda\in\mathbb{R},$$

$$\operatorname{Im}\mathcal{B}_k(\lambda) \text{ constant in } \lambda\in\mathbb{R} \text{ for each } k\in[0,N]_{\mathbb{Z}},$$

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• **Another symplectic system:** Together with (S_λ) we consider another symplectic system denoted by ($\underline{\mathrm{S}}_\lambda$), whose coefficients satisfy

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and the corresponding eigenvalue problem

$$(\underline{\mathrm{S}}_\lambda), \quad \lambda \in \mathbb{R}, \quad \underline{x}_0 = 0 = \underline{x}_{N+1}. \tag{$\underline{\mathrm{E}}_0$}$$

- **Comparison of finite eigenvalues:**

  ⊛ RŠH, Eigenvalue comparison for discrete symplectic systems, in: “Proceedings of the 18th International Conference on Difference Equations and Applications” (Barcelona, 2012), L. Alsedà, J. Cushing, S. Elaydi, and A. Pinto, editors, submitted (2012).

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$$\begin{aligned}
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n_2(\lambda) &:= \text{ the number of finite eigenvalues of } (\mathrm{E}_0) \text{ in } (-\infty, \lambda], \\
\underline{n}_1(\lambda) &:= \text{ the number of focal points of } \big(\underline{\hat{X}}(\lambda), \underline{\hat{U}}(\lambda)\big) \text{ in } (0, N+1], \\
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Theorem 5. *Assume that for all $\lambda \in \mathbb{R}$ we have*

$$\left.\begin{aligned} \mathcal{G}_k(\lambda) \geq \underline{\mathcal{G}}_k(\lambda), \\ \operatorname{Im}\big(\mathcal{A}_k(\lambda) - \underline{\mathcal{A}}_k(\lambda),\ \ \mathcal{B}_k(\lambda)\big) \subseteq \operatorname{Im} \underline{\mathcal{B}}_k(\lambda), \end{aligned}\right\} \quad k \in [0, N]_{\mathbb{Z}}.$$

The finite eigenvalues of (E_0) *and* $(\underline{\mathrm{E}}_0)$ *are isolated, bounded from below, and*

$$n_2(\lambda) + \ell \leq \underline{n}_2(\lambda) + \underline{\ell} \quad \text{for all } \lambda \in \mathbb{R},$$

where

$$\ell := \lim_{\lambda \to -\infty} n_1(\lambda), \qquad \underline{\ell} := \lim_{\lambda \to -\infty} \underline{n}_1(\lambda).$$

• Comparison of finite eigenvalues:

Corollary 6. *Assume that the functional* $\underline{\mathcal{F}}_0(\underline{z}, \lambda_0)$ *is positive definite for some* $\lambda_0 < 0$ *and for all* $\lambda \in \mathbb{R}$ *we have*

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Then

$$n_2(\lambda) \leq \underline{n}_2(\lambda) \quad \text{for all } \lambda \in \mathbb{R}.$$

That is, if we denote by

$$-\infty < \lambda_1 \leq \cdots \leq \lambda_j \leq \dots \qquad \text{and} \qquad -\infty < \underline{\lambda}_1 \leq \cdots \leq \underline{\lambda}_j \leq \dots$$

the finite eigenvalues of (E_0) *and* ($\underline{E}_0$)*, respectively, then*

$$\underline{\lambda}_j \leq \lambda_j, \quad j = 1, 2, \dots,$$

whenever these finite eigenvalues exist.

Examples

- **Sturm–Liouville difference equations – second order:**

$$\Delta(r_k(\lambda)\,\Delta x_k) + q_k(\lambda)\,x_{k+1} = 0, \quad k \in [0, N-1]_{\mathbb{Z}},$$

where

$$r_k(\lambda) \neq 0, \quad \dot{r}_k(\lambda) \leq 0, \quad \dot{q}_k(\lambda) \geq 0.$$

In this case the matrices $\mathcal{S}_k(\lambda)$ and $\Psi_k(\lambda)$ have the form

$$\mathcal{S}_k(\lambda) = \begin{pmatrix} 1 & 1/r_k(\lambda) \\ -q_k(\lambda) & 1 - q_k(\lambda)/r_k(\lambda) \end{pmatrix},$$

$$\Psi_k(\lambda) = \frac{1}{r_k^2(\lambda)} \begin{pmatrix} r_k(\lambda) & q_k(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{q}_k(\lambda) & 0 \\ 0 & -\dot{r}_k(\lambda) \end{pmatrix} \begin{pmatrix} r_k(\lambda) & 0 \\ q_k(\lambda) & 1 \end{pmatrix}.$$

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- **Assumptions in the comparison theorem (Theorem 5 and Corollary 6):**

$$r_k(\lambda) \geq \underline{r}_k(\lambda), \quad q_k(\lambda) \geq \underline{q}_k(\lambda) \quad \text{for all } \lambda \in \mathbb{R}.$$

- **Sturm–Liouville difference equations – higher order:**

$$\sum_{j=0}^{n}(-1)^j \Delta^j \big(r_k^{[j]}(\lambda)\, \Delta^j y_{k+n-j}\big) = 0, \quad k \in [0, N-n]_{\mathbb{Z}},$$

where

$$r_k^{[n]}(\lambda) \neq 0, \quad \dot{r}_k^{[n]}(\lambda) \geq 0, \quad \dot{r}_k^{[i]}(\lambda) \leq 0 \quad \text{for all } i \in \{0, \dots, n-1\}.$$

In this case

$$\mathcal{B}_k(\lambda) = \frac{1}{r_k^{[n]}(\lambda)} \cdot \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix} \quad \text{has constant image.}$$

- **Matrix Sturm–Liouville difference equations:**

$$\Delta(R_k(\lambda)\,\Delta x_k) + Q_k(\lambda)\,x_{k+1} = 0, \quad k \in [0, N-1]_{\mathbb{Z}},$$

where $R_k(\lambda)$ and $Q_k(\lambda)$ are symmetric,

$$R_k(\lambda) \text{ is invertible}, \quad \dot{R}_k(\lambda) \le 0, \quad \dot{Q}_k(\lambda) \ge 0.$$

In this case

$$\mathcal{S}_k(\lambda) = \begin{pmatrix} I & R_k^{-1}(\lambda) \\ Q_k(\lambda) & I - Q_k(\lambda)\,R_k^{-1}(\lambda) \end{pmatrix},$$

$$\Psi_k(\lambda) = \begin{pmatrix} I & Q_k(\lambda)\,R_k^{-1}(\lambda) \\ 0 & R_k^{-1}(\lambda) \end{pmatrix} \begin{pmatrix} \dot{Q}_k(\lambda) & 0 \\ 0 & -\dot{R}_k(\lambda) \end{pmatrix} \begin{pmatrix} I & 0 \\ R_k^{-1}(\lambda)\,Q_k(\lambda) & R_k^{-1}(\lambda) \end{pmatrix}.$$

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In this case

$$\mathcal{S}_k(\lambda) = \begin{pmatrix} I & R_k^{-1}(\lambda) \\ Q_k(\lambda) & I - Q_k(\lambda)\,R_k^{-1}(\lambda) \end{pmatrix},$$

$$\Psi_k(\lambda) = \begin{pmatrix} I & Q_k(\lambda)\,R_k^{-1}(\lambda) \\ 0 & R_k^{-1}(\lambda) \end{pmatrix} \begin{pmatrix} \dot{Q}_k(\lambda) & 0 \\ 0 & -\dot{R}_k(\lambda) \end{pmatrix} \begin{pmatrix} I & 0 \\ R_k^{-1}(\lambda)\,Q_k(\lambda) & R_k^{-1}(\lambda) \end{pmatrix}.$$

- **Assumptions in the comparison theorem (Theorem 5 and Corollary 6):**

$$R_k(\lambda) \ge \underline{R}_k(\lambda), \quad Q_k(\lambda) \ge \underline{Q}_k(\lambda) \quad \text{for all } \lambda \in \mathbb{R}.$$

- **Matrix Sturm–Liouville difference equations:**

$$\Delta(R_k(\lambda)\,\Delta x_k) + Q_k(\lambda)\,x_{k+1} = 0, \quad k \in [0, N-1]_{\mathbb{Z}},$$

where $R_k(\lambda)$ and $Q_k(\lambda)$ are symmetric,

$$R_k(\lambda) \text{ is invertible}, \quad \dot{R}_k(\lambda) \le 0, \quad \dot{Q}_k(\lambda) \ge 0.$$

In this case

$$\mathcal{S}_k(\lambda) = \begin{pmatrix} I & R_k^{-1}(\lambda) \\ Q_k(\lambda) & I - Q_k(\lambda)\,R_k^{-1}(\lambda) \end{pmatrix},$$

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- **Relative oscillation theory:**

  ⊛ J. V. Elyseeva, ICDEA 2013, Muscat, Oman.

• **Symmetric three-term recurrence equations:**

$$V_{k+1}x_{k+2} - T_{k+1}(\lambda)\,x_{k+1} + V_k^T x_k = 0, \quad k \in [0, N-1]_{\mathbb{Z}},$$

where $T_k(\lambda)$ is symmetric and V_k is invertible, and

$$\dot{T}_k(\lambda) \leq 0.$$

In this case

$$\mathcal{S}_k(\lambda) = \begin{pmatrix} V_k^{-1}T_k(\lambda) & V_k^{-1} \\ -V_k^T & 0 \end{pmatrix},$$

$$\Psi_k(\lambda) = -\begin{pmatrix} 0 & 0 \\ 0 & V_k^{-1}\,\dot{T}_k(\lambda)\,V_k^{T-1} \end{pmatrix}.$$

- **Symmetric three-term recurrence equations:**

$$V_{k+1} x_{k+2} - T_{k+1}(\lambda)\, x_{k+1} + V_k^T x_k = 0, \quad k \in [0, N-1]_{\mathbb{Z}},$$

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$$\mathcal{S}_k(\lambda) = \begin{pmatrix} V_k^{-1} T_k(\lambda) & V_k^{-1} \\ -V_k^T & 0 \end{pmatrix},$$

$$\Psi_k(\lambda) = -\begin{pmatrix} 0 & 0 \\ 0 & V_k^{-1}\, \dot{T}_k(\lambda)\, V_k^{T-1} \end{pmatrix}.$$

- **Assumptions in the comparison theorem (Theorem 5 and Corollary 6):**

$$\begin{pmatrix} T_k(\lambda) & -V_k \\ -V_k^T & 0 \end{pmatrix} \geq \begin{pmatrix} \underline{T}_k(\lambda) & -\underline{V}_k \\ -\underline{V}_k^T & 0 \end{pmatrix} \quad \text{for all } \lambda \in \mathbb{R}.$$

• **Linear Hamiltonian difference systems:**

$$\left.\begin{aligned} \Delta x_k &= A_k(\lambda)\, x_{k+1} + B_k(\lambda)\, u_k, \\ \Delta u_k &= C_k(\lambda)\, x_{k+1} - A_k^T(\lambda)\, u_k, \end{aligned}\right\} \quad k \in [0, N]_{\mathbb{Z}},$$

where $B_k(\lambda)$ and $C_k(\lambda)$ are symmetric, $I - A_k(\lambda)$ is invertible

$$\tilde{A}_k(\lambda) := [I - A_k(\lambda)]^{-1}, \quad \operatorname{Im}[\tilde{A}_k(\lambda)\, B_k(\lambda)] \text{ constant},$$

$$\dot{H}_k(\lambda) \geq 0, \quad H_k(\lambda) := \begin{pmatrix} -C_k(\lambda) & A_k^T(\lambda) \\ A_k(\lambda) & B_k(\lambda) \end{pmatrix}.$$

In this case

$$\mathcal{S}_k(\lambda) = \begin{pmatrix} \tilde{A}_k(\lambda) & \tilde{A}_k(\lambda)\, B_k(\lambda) \\ C_k(\lambda)\, \tilde{A}_k(\lambda) & C_k(\lambda)\, \tilde{A}_k(\lambda)\, B_k(\lambda) + I - A_k^T(\lambda) \end{pmatrix},$$

$$\Psi_k(\lambda) = \begin{pmatrix} I & -C_k(\lambda)\, \tilde{A}_k(\lambda) \\ 0 & \tilde{A}_k(\lambda) \end{pmatrix} \dot{H}_k(\lambda) \begin{pmatrix} I & 0 \\ -\tilde{A}_k^T(\lambda)\, C_k(\lambda) & \tilde{A}_k^T(\lambda) \end{pmatrix}.$$

Eigenvalue problems with separated endpoints

$$(\mathrm{S}_\lambda), \qquad \left.\begin{aligned} R_0^*(\lambda)\, x_0 + R_0(\lambda)\, u_0 = 0, \\ R_{N+1}^*(\lambda)\, x_{N+1} + R_{N+1}(\lambda)\, u_{N+1} = 0, \end{aligned}\right\}, \qquad \lambda \in \mathbb{R}, \qquad (\mathrm{E}_1)$$

$$(\underline{\mathrm{S}}_\lambda), \qquad \left.\begin{aligned} \underline{R}_0^*(\lambda)\, \underline{x}_0 + \underline{R}_0(\lambda)\, \underline{u}_0 = 0, \\ \underline{R}_{N+1}^*(\lambda)\, \underline{x}_{N+1} + \underline{R}_{N+1}(\lambda)\, \underline{u}_{N+1} = 0, \end{aligned}\right\}, \qquad \lambda \in \mathbb{R}, \qquad (\underline{\mathrm{E}}_1)$$

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- **Method:** Extend the interval $[0, N+1]_{\mathbb{Z}}$ by one point at each end to

  problem on $[-1, N+2]_{\mathbb{Z}}$ with Dirichlet endpoints $x_{-1} = 0 = x_{N+2}$,

Eigenvalue problems with separated endpoints

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Eigenvalue problems with separated endpoints

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  problem on $[-1, N+2]_{\mathbb{Z}}$ with Dirichlet endpoints $x_{-1} = 0 = x_{N+2}$,

and apply then the comparison theorems (Theorem 5 and Corollary 6) with Dirichlet endpoints.

- **Assumptions:** The extended system must be symplectic, i.e.,

$$\mathcal{S}_{-1}(\lambda) := \begin{pmatrix} R_0^{*T}(\lambda) & -R_0^T(\lambda) \\ R_0^T(\lambda) & R_0^{*T}(\lambda) \end{pmatrix}, \qquad \Psi_{-1}(\lambda) := \mathcal{J}\,\dot{\mathcal{S}}_{-1}(\lambda)\,\mathcal{J}\,\mathcal{S}_{-1}^T(\lambda)\,\mathcal{J} \geq 0,$$

$$\mathcal{S}_{N+1}(\lambda) := \dots, \qquad \Psi_{N+1}(\lambda) := \mathcal{J}\,\dot{\mathcal{S}}_{N+1}(\lambda)\,\mathcal{J}\,\mathcal{S}_{N+1}^T(\lambda)\,\mathcal{J} \geq 0.$$

Eigenvalue problems with joint endpoints

$$(\mathrm{S}_\lambda), \quad R^*(\lambda)\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix} + R(\lambda)\begin{pmatrix}u_0\\ u_{N+1}\end{pmatrix} = 0, \quad \lambda \in \mathbb{R}, \qquad (\mathrm{E}_2)$$

$$(\underline{\mathrm{S}}_\lambda), \quad \underline{R}^*(\lambda)\begin{pmatrix}-\underline{x}_0\\ \underline{x}_{N+1}\end{pmatrix} + \underline{R}(\lambda)\begin{pmatrix}\underline{u}_0\\ \underline{u}_{N+1}\end{pmatrix} = 0, \quad \lambda \in \mathbb{R}, \qquad (\underline{\mathrm{E}}_2)$$

Eigenvalue problems with joint endpoints

$$(\mathrm{S}_\lambda), \quad R^*(\lambda)\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + R(\lambda)\begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = 0, \quad \lambda \in \mathbb{R}, \qquad (\mathrm{E}_2)$$

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- **Method:** Augment the system and the boundary conditions into double dimension $2n$ to

problem on $[0, N+1]_{\mathbb{Z}}$ with separated endpoints in dimension $2n$,

Eigenvalue problems with joint endpoints

$$(\mathrm{S}_\lambda), \quad R^*(\lambda)\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + R(\lambda)\begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = 0, \quad \lambda \in \mathbb{R}, \qquad (\mathrm{E}_2)$$

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Eigenvalue problems with joint endpoints

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- **Assumptions:** The augmented system must be symplectic.

Eigenvalue problems with joint endpoints

$$(\mathrm{S}_\lambda), \quad R^*(\lambda)\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + R(\lambda)\begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = 0, \quad \lambda \in \mathbb{R}, \qquad (\mathrm{E}_2)$$

$$(\underline{\mathrm{S}}_\lambda), \quad \underline{R}^*(\lambda)\begin{pmatrix} -\underline{x}_0 \\ \underline{x}_{N+1} \end{pmatrix} + \underline{R}(\lambda)\begin{pmatrix} \underline{u}_0 \\ \underline{u}_{N+1} \end{pmatrix} = 0, \quad \lambda \in \mathbb{R}, \qquad (\underline{\mathrm{E}}_2)$$

• **Method:** Augment the system and the boundary conditions into double dimension $2n$ to

problem on $[0, N+1]_{\mathbb{Z}}$ with separated endpoints in dimension $2n$,

and apply then the comparison theorems with separated endpoints.

• **Assumptions:** The augmented system must be symplectic.

• **Periodic endpoints:** $x_0 = x_{N+1}$ and $u_0 = u_{N+1}$,

$$R(\lambda) = \underline{R}(\lambda) \equiv \begin{pmatrix} I & -I \\ 0 & 0 \end{pmatrix}, \quad R^*(\lambda) = \underline{R}^*(\lambda) \equiv \begin{pmatrix} 0 & 0 \\ I & I \end{pmatrix}.$$

Go symplectic!

and

Thank you for your attention!