Weak Solutions for a Degenerate Elliptic Dirichlet Problem

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Branko Najman (1946–1996)

Picture taken by G.M. Bergmann at Oberwolfach in 1980.

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Closely Embedded Hilbert Spaces

Let $\mathcal{H}$ and $\mathcal{H}_+$ be two Hilbert spaces. The Hilbert space $\mathcal{H}_+$ is called closely embedded in $\mathcal{H}$ if:

(ce1) There exists a linear manifold $\mathcal{D} \subseteq \mathcal{H}_+ \cap \mathcal{H}$ that is dense in $\mathcal{H}_+$.

(ce2) The embedding operator j_+ with domain $\mathcal{D}$ is closed, as an operator $\mathcal{H}_+ \to \mathcal{H}$.

Axiom (ce1) means that on $\mathcal{D}$ the algebraic structures of $\mathcal{H}_+$ and $\mathcal{H}$ agree.

Axiom (ce2) means that the operator j_+ with $\mathrm{Dom}(j_+) = \mathcal{D} \subseteq \mathcal{H}_+$ defined by $j_+ x = x \in \mathcal{H}$, for all $x \in \mathcal{D}$, is closed.

The Kernel Operator

Let $\mathcal{H}_+$ be a Hilbert space that is closely embedded in $\mathcal{H}$, and let j_+ denote the corresponding closed embedding. Then $A = j_+ j_+^* \in \mathcal{C}(\mathcal{H})^+$ and

$$\langle j_+ h, k\rangle = \langle h, Ak\rangle_+, \quad h \in \mathrm{Dom}(j_+),\ k \in \mathrm{Dom}(A), \tag{2.1}$$

more precisely, A has the range in $\mathcal{H}_+$ and it can also be viewed as the adjoint of the embedding j_+. The operator A is called the *kernel operator* associated to the closed embedding of $\mathcal{H}_+$ in $\mathcal{H}$.

L. Schwartz — for continuous embeddings

The Space $\mathcal{R}(T)$

Let $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$ be a closed and densely defined linear operator, where $\mathcal{G}$ is another Hilbert space. On $\operatorname{Ran}(T)$ we consider a new inner product

$$\langle Tu, Tv \rangle_T = \langle u, v \rangle_{\mathcal{G}}, \tag{2.2}$$

where $u, v \in \operatorname{Dom}(T) \ominus \operatorname{Ker}(T)$. With respect to this new inner product $\operatorname{Ran}(T)$ can be completed to a Hilbert space that we denote by $\mathcal{R}(T)$, closely embedded in $\mathcal{H}$, and in such a way that $j_T \colon \mathcal{R}(T) \to \mathcal{H}$ has the property that $j_T j_T^* = TT^*$.

The Space $\mathcal{D}(T)$

Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{G})$ with $\operatorname{Ker}(T)$ a closed subspace of $\mathcal{H}$. Define the norm

$$|x|_T := \|Tx\|_{\mathcal{G}}, \quad x \in \operatorname{Dom}(T) \ominus \operatorname{Ker}(T), \tag{2.3}$$

and let $\mathcal{D}(T)$ be the Hilbert space completion of the pre-Hilbert space $\operatorname{Dom}(T) \ominus \operatorname{Ker}(T)$ with respect to the norm $|\cdot|_T$ associated the inner product $(\cdot, \cdot)_T$

$$(x, y)_T = \langle Tx, Ty \rangle_{\mathcal{G}}, \quad x, y \in \operatorname{Dom}(T) \ominus \operatorname{Ker}(T). \tag{2.4}$$

Define i_T from $\mathcal{D}(T)$ and valued in $\mathcal{H}$ by

$$i_T x := x, \quad x \in \operatorname{Dom}(i_T) = \operatorname{Dom}(T) \ominus \operatorname{Ker}(T). \tag{2.5}$$

The operator i_T is closed and $\mathcal{D}(T)$ is closely embedded in $\mathcal{H}$, with the underlying closed embedding i_T.
The operator Ti_T admits a unique isometric extension $\widehat{T} : \mathcal{D}(T) \to \mathcal{G}$.

Triplets of Closely Embedded Hilbert Spaces

By definition, $(\mathcal{H}_+;\mathcal{H};\mathcal{H}_-)$ is called a triplet of closely embedded Hilbert spaces if:

(th1) $\mathcal{H}_+$ is a Hilbert space closely embedded in the Hilbert space $\mathcal{H}$, with the closed embedding denoted by j_+, and such that $\operatorname{Ran}(j_+)$ is dense in $\mathcal{H}$.

(th2) $\mathcal{H}$ is closely embedded in the Hilbert space $\mathcal{H}_-$, with the closed embedding denoted by j_-, and such that $\operatorname{Ran}(j_-)$ is dense in $\mathcal{H}_-$.

(th3) $\operatorname{Dom}(j_+^*) \subseteq \operatorname{Dom}(j_-)$ and for every vector $y \in \operatorname{Dom}(j_-) \subseteq \mathcal{H}$ we have

$$\|y\|_- = \sup\{\frac{|\langle x, y\rangle_{\mathcal{H}}|}{\|x\|_+} \mid x \in \operatorname{Dom}(j_+),\ x \neq 0\}.$$

The kernel operator $A = j_+ j_+^*$ is a positive selfadjoint operator in $\mathcal{H}$ that is one-to-one. Then, $H = A^{-1}$ is a positive selfadjoint operator in $\mathcal{H}$ and it is called the Hamiltonian of the triplet. Note that, as a consequence of (th3), we actually have $\operatorname{Dom}(j_+^*) = \operatorname{Dom}(j_-)$.

Generation of Triplets of Hilbert Spaces: Factoring the Hamiltonian

Theorem

Let H be a positive selfadjoint operator in the Hilbert space $\mathcal{H}$, that admits an inverse $A = H^{-1}$, possibly unbounded. Then there exists $T \in \mathcal{C}(\mathcal{H}, \mathcal{G})$, with $\operatorname{Ran}(T)$ dense in $\mathcal{G}$ and $H = T^*T$. In addition, let $S = T^{-1} \in \mathcal{C}(\mathcal{G}, \mathcal{H})$. Then:

- (i) *The Hilbert space $\mathcal{H}_+ := \mathcal{D}(T) := \mathcal{R}(S)$ is closely embedded in $\mathcal{H}$ with its embedding i_T having range dense in $\mathcal{H}$, and its kernel operator $A = i_T i_T^*$ coincides with H^{-1}.*
- (ii) *$\mathcal{H}$ is closely embedded in the Hilbert space $\mathcal{H}_- = \mathcal{R}(T^*)$ with its embedding $j_{T^*}^{-1}$ having range dense in $\mathcal{R}(T^*)$. The kernel operator $B = j_{T^*}^{-1} j_{T^*}^{-1*}$ of this embedding is unitary equivalent with $A = H^{-1}$.*

Generation of Triplets of Hilbert Spaces: Weak Solutions

Theorem (continued)

(iii) *The operator* $V = i_T^* | \operatorname{Ran}(T^*)$, *that is,*

$$\langle i_T x, y \rangle_{\mathcal{H}} = (x, Vy)_T, \quad x \in \operatorname{Dom}(T),\ y \in \operatorname{Ran}(T^*), \tag{2.6}$$

extends uniquely to a unitary operator $\widetilde{V}$ *between the Hilbert spaces* $\mathcal{R}(T^*)$ *and* $\mathcal{D}(T)$.

(iv) *The operator* H, *when viewed as a linear operator with domain dense in* $\mathcal{D}(T)$ *and range in* $\mathcal{R}(T^*)$, *extends uniquely to a unitary operator* $\widetilde{H}\colon \mathcal{D}(T) \to \mathcal{R}(T^*)$, *and* $\widetilde{H} = \widetilde{V}^{-1}$.

Generation of Triplets of Hilbert Spaces: Dual Space

Theorem (continued)

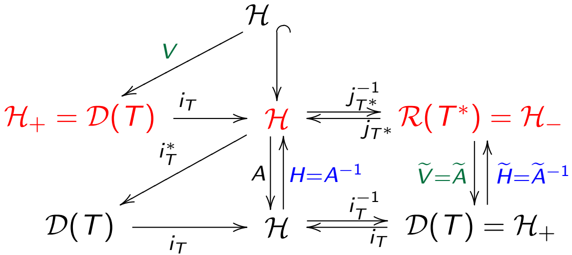
(v) *The operator* $\Theta\colon \mathcal{R}(T^*) \to \mathcal{D}(T)^*$ *defined by*

$$(\Theta\alpha)(x) := (\widetilde{V}\alpha, x)_T, \quad \alpha \in \mathcal{R}(T^*),\ x \in \mathcal{D}(T), \tag{2.7}$$

provides a canonical and unitary identification of the Hilbert space $\mathcal{R}(T^*)$ *with the conjugate space* $\mathcal{D}(T)^*$*, in particular, for all* $y \in \operatorname{Dom}(T^*)$

$$\|y\|_{T^*} = \sup\Big\{\frac{|\langle y, x\rangle_{\mathcal{H}}|}{|x|_T} \mid x \in \operatorname{Dom}(T) \setminus \{0\}\Big\}. \tag{2.8}$$

Generation of Triplets of Hilbert Spaces: The General Picture



H	Hamiltonian	Berezansky — continuous embeddings
$A = H^{-1}$	Kernel Operator	
$H = T^*T$	Factor Operator	
$A = SS^*$	Factor Operator	

The Gradient

Let Ω be an open (nonempty) set of the $\mathbb{R}^N$. Let $D_j = \mathrm{i}\frac{\partial}{\partial x_j}$, ($j = 1, \ldots, N$) be the operators of differentiation with respect to the coordinates of points $x = (x_1, \ldots, x_N)$ in $\mathbb{R}^N$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$, let $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$. $\nabla_l = (D^\alpha)_{|\alpha|=l}$ denotes the gradient of order l, where l is a fixed nonnegative integer. Letting $m = m(N, l)$ denote the number of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_N = l$, ∇_l can be viewed as an operator acting from $L_2(\Omega)$ into $L_2(\Omega; \mathbb{C}^m)$ defined on its maximal domain, the Sobolev space $W_2^l(\Omega)$, by

$$\nabla_l u = (D^\alpha u)_{|\alpha|=l}, \qquad u \in W_2^l(\Omega).$$

The Underlying Spaces

$W_2^l(\Omega)$ consists of those functions $u \in L_2(\Omega)$ whose distributional derivatives $D^\alpha u$ belong to $L_2(\Omega)$ for all $\alpha \in \mathbb{Z}_+^N, |\alpha| \leq l$ and with norm

$$\|u\|_{W_2^l(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^2_{L_2(\Omega)} \right)^{1/2}, \qquad (3.1)$$

$W_2^l(\Omega)$ becomes a Hilbert space that is continuously embedded in $L_2(\Omega)$.

$\overset{\circ}{W}{}_2^l(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the space $W_2^l(\Omega)$.

More Spaces

The space $\overset{\circ}{L}{}^{l}_{p}(\Omega)$, $(1 \le p < \infty)$ is defined as the completion of $C_0^\infty(\Omega)$ under the metric corresponding to

$$\|u\|_{p,l} := \|\nabla_l u\|_{L_p(\Omega)} = \left(\int_\Omega \left(\sum_{|\alpha|=l} |D^\alpha u(x)|^2 \right)^{p/2} \mathrm{d}\,x \right)^{1/p}, \quad u \in C_0^\infty(\Omega).$$

The elements of $\overset{\circ}{L}{}^{l}_{p}(\Omega)$ can be realized as locally integrable functions on Ω vanishing at the boundary $\partial\Omega$ and having distributional derivatives of order l in $L_p(\Omega)$.
Moreover, these functions, after modification on a set of zero measure, are absolutely continuous on every line which is parallel to the coordinate axes.

The Principal Symbol

On Ω there is defined an $m \times m$ matrix valued measurable function a, more precisely, $a(x) = [a_{\alpha\beta}(x)]$, $|\alpha|, |\beta| = l$, $x \in \Omega$, where the scalar valued functions $a_{\alpha,\beta}$ are measurable on Ω for all multi-indices $|\alpha|, |\beta| = l$.
(C1) *For almost all (with respect to the n-dimensional standard Lebesgue measure) $x \in \Omega$, the matrix $a(x)$ is nonnegative (positive semidefinite), that is,*

$$\sum_{|\alpha|,|\beta|=l} a_{\alpha\beta}(x)\overline{\eta}_\beta \eta_\alpha \geq 0, \text{ for all } \eta = (\eta_\alpha)_{|\alpha|=l} \in \mathbb{C}^m.$$

According to the condition (C1), there exists an $m \times m$ matrix valued measurable function b on Ω, such that

$$a(x) = b(x)^* b(x), \text{ for almost all } x \in \Omega,$$

where $b(x)^*$ denotes the Hermitian conjugate matrix of the matrix $b(x)$.

Conditions

(C2) *There is a nonnegative measurable function c on Ω such that, for almost all $x \in \Omega$ and all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N$,*

$$|b(x)\widetilde{\xi}| \geq c(x)|\tilde{\xi}|,$$

where $\tilde{\xi} = (\xi^\alpha)_{|\alpha|=l}$ is the vector in $\mathbb{C}^m$ with $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_N^{\alpha_N}$.

(C3) *All the entries $b_{\alpha\beta}$ of the $m \times m$ matrix valued function b are functions in $L_{1,\mathrm{loc}}(\Omega)$.*

(C4) *The function c in (C2) has the property that $1/c \in L_2(\Omega)$.*

The Operator T

Under the conditions (C1)–(C4), we consider the operator T acting from $L_2(\Omega)$ to $L_2(\Omega;\mathbb{C}^m)$ and defined by

$$(Tu)(x) = b(x)\nabla_l u(x), \quad \text{for almost all } x \in \Omega, \tag{3.2}$$

on its domain

$$\mathrm{Dom}(T) = \{u \in \overset{\circ}{W}{}_2^{\,l}(\Omega) \mid b\nabla_l u \in L_2(\Omega;\mathbb{C}^m)\}. \tag{3.3}$$

The Problem

Our aim is to describe, in view of the abstract model, the triplet of closely embedded Hilbert spaces $(\mathcal{D}(T); L_2(\Omega); \mathcal{R}(T^*))$ associated with the operator T defined at (3.2) and (3.3).

In terms of these results, we obtain information about weak solutions for the corresponding operator equation involving the Hamiltonian operator $H = T^*T$ of the triplet, which in fact is a Dirichlet boundary value problem in $L_2(\Omega)$ with homogeneous boundary values.

The Problem

This problem is associated to the differential sesqui-linear form

$$a[u,v] = \int_\Omega \langle a(x)\nabla_l(x), \nabla_l(x)\rangle \,\mathrm{d}\, x$$
$$= \sum_{|\alpha|=|\beta|=l} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x) \overline{D^\alpha v(x)} dx, \quad u, v \in C_0^\infty(\Omega), \tag{3.4}$$

which, as will be seen, can be extended up to elements of $\mathcal{D}(T)$.
The problem can be reformulated as follows : *given $f \in \mathcal{D}(T)^*$ (which is canonically identified withe $\mathcal{R}(T^*)$), find $v \in \mathcal{D}(T)$ such that*

$$a[u,v] = \langle u, f\rangle \text{ for all } u \in \mathcal{D}(T), \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathcal{D}(T)$ and $\mathcal{D}(T)^*$.

The problem in (3.5) can be considered only for $u \in \overset{\circ}{W}{}_2^l(\Omega)$, or, even more restrictively, only for $u \in C_0^\infty(\Omega)$.

The Main Result

Theorem

For Ω a domain in $\mathbb{R}^N$ and $l \in \mathbb{N}$, let $a(x) = [a_{\alpha\beta}(x)] = b(x)^* b(x)$, $|\alpha|, |\beta| = l$, $x \in \Omega$, satisfy the conditions (C1)–(C4), and consider the differential sesqui-linear form

$$\begin{aligned} a[u, v] &= \int_\Omega \langle a(x) \nabla_l(x), \nabla_l(x) \rangle \,\mathrm{d}\, x \\ &= \sum_{|\alpha|=|\beta|=l} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x) \overline{D^\alpha v(x)} \,\mathrm{d}\, x, \quad u, v \in C_0^\infty(\Omega), \end{aligned}$$

The Main Result

Theorem (Continuation)

Then:

(1) *The operator* T *acting from* $L_2(\Omega)$ *to* $L_2(\Omega;\mathbb{C}^m)$ *and defined by* $(Tu)(x) = b(x)\nabla_l u(x)$ *for* $x \in \Omega$ *and* $u \in \mathrm{Dom}(T) = \{u \in \overset{\circ}{W}{}_2^{l}(\Omega) \mid b\nabla_l u \in L_2(\Omega;\mathbb{C}^m)\}$ *is closed, densely defined, and injective.*

(2) *The pre-Hilbert space* $\mathrm{Dom}(T)$ *with norm* $|u|_T = (\int_\Omega |b(x)\nabla_l u(x)|^2 \,\mathrm{d}\, x)^{\frac{1}{2}}$*, has a unique Hilbert space completion, denoted by* $\mathcal{H}_a^l(\Omega)$*, that is continuously embedded into* $\overset{\circ}{L}{}_1^{l}(\Omega)$*.*

The Main Result

Theorem (Continuation)

(3) *The conjugate space of* $\overset{\circ}{\mathcal{H}}{}_a^{l}(\Omega)$, *denoted by* $\overset{\circ}{\mathcal{H}}{}_a^{-l}(\Omega)$, *can be realized in such a way that, for any* $f \in \overset{\circ}{\mathcal{H}}{}_a^{-l}(\Omega)$ *there exist elements* $g \in L_2(\Omega; \mathbb{C}^m)$ *such that*

$$f(u) = \int_\Omega \langle g(x), b(x)\nabla_l u(x)\rangle \, \mathrm{d}\, x, \quad u \in \overset{\circ}{W}{}_2^{l}(\Omega), \tag{3.6}$$

and

$$\|f\|_{\overset{\circ}{\mathcal{H}}{}_a^{-l}(\Omega)} = \inf\{\|g\|_{L_2(\Omega;\mathbb{C}^m)} \mid g \in L_2(\Omega; \mathbb{C}^m) \text{ such that (3.6) holds }\}.$$

The Main Result

Theorem (Continuation)

(4) $(\overset{\circ}{\mathcal{H}}{}^{l}_{a}(\Omega); L_2(\Omega); \overset{\circ}{\mathcal{H}}{}^{-l}_{a}(\Omega))$ *is a triplet of closely embedded Hilbert spaces.*

(5) *For every* $f \in \overset{\circ}{\mathcal{H}}{}^{-l}_{a}(\Omega)$ *there exists a unique* $v \in \mathcal{H}^{l}_{a}(\Omega)$ *that solves the Dirichlet problem associated to the sesquilinear form a, in the sense that*

$$a[u, v] = \langle u, f \rangle \text{ for all } u \in \mathcal{H}^{l}_{a}(\Omega).$$

More precisely, $v = \widetilde{H}^{-1} f$, *where* $\widetilde{H}$ *is the unitary operator acting between* $\overset{\circ}{\mathcal{H}}{}^{l}_{a}(\Omega)$ *and* $\overset{\circ}{\mathcal{H}}{}^{-l}_{a}(\Omega)$ *that uniquely extends the positive selfadjoint operator* $H = T^* T$ *in* $L_2(\Omega)$.

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