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THE DEGENERATE PRINCIPAL SERIES REPRESENTATIONS OF EXCEPTIONAL GROUPS OF TYPE E_8 OVER p-ADIC FIELDS

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Abstract. In this paper, we study the reducibility of degenerate principal series of the split, simple, simply-connected exceptional group of type E_8. Furthermore, we calculate the maximal semi-simple subrepresentation and quotient of these representations for almost all cases.

1. Introduction

This paper is the final part in our project of studying the degenerate principal series of exceptional groups of type E_n. This paper is about E_8, which, as often noted by David Kazhdan, is the smallest, split, simple, simply-connected, adjoint and simply laced group. In fact, this essentially completes the study of degenerate principal series of simple p-adic groups up to isogeny.

More precisely, let F be a non-Archimedean local field and let G denote the split simple group of type E_8. For a maximal parabolic subgroup P of G with a Levi subgroup M and a 1-dimensional representation Ω of M, we consider the following two questions:

- Is the normalized parabolic induction $Ind_P^G(\Omega)$ irreducible?
- If $Ind_P^G(\Omega)$ is reducible, what is the length of its maximal semi-simple subrepresentation and quotient?

We completely answer the first one in Theorem 4.1, and almost completely answer the second. In fact, there are only two pairs (P, Ω) (out of hundreds of cases) in which we were only able to show that the maximal semi-simple subrepresentation is of length at most 2. In both of these cases, we show that the irreducible spherical subquotient is a subrepresentation and describe the

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other possible irreducible subrepresentation in terms of its Langlands data. Further, we describe a decisive test to determine the length of the maximal semi-simple subrepresentation for each of these cases, which would hopefully could be realized when stronger computing machines would be more commonly available. These two cases are detailed in Subsection 5.2.

In order to answer the above questions, we use the algorithm described in [HS20, Section 3] and [HS21, Section 3]. This provides an answer to both questions for almost all pairs (P, Ω) For the remaining cases, not determined by the algorithm, further study is performed in Section 5. This project uses a script implemented in the Sagemath environment [The22].

The study of local degenerate principles series is useful for various reasons. One of which is the study of the degenerate residual spectrum of the adelic group of the same type. Data regarding the degenerate residual spectrum, and the analytic behavior of degenerate Eisenstein series, can then be used to study other automorphic representations and functorial lifts.

Some examples are:

- The study of the degenerate residual spectrum of the simple group of type F_4 is the topic of the first author's PhD dissertation and relies on the study of the local degenerate principal series of F_4, performed in [CJ10].
- The study of degenerate residual representations of $Spin_8$, with P being the Heisenberg parabolic subgroup, was performed by the second author in [Seg18, Seg19]. The results of these paper was later used to study the exceptional θ-lift between G_2 and $\widetilde{SL_2}$ (also known as the Rallis-Schiffmann lift) in [GS]. This, in turn, followed the work Kudla, Rallis and Piatetski-Shapiro which, along similar lines, studied the local degenerate principal series and global degenerate Eisenstein series of groups of type Sp_{2n} and using that to study the Howe correspondence.
- The study of the degenerate residual spectrum of the adelic groups of type E_n is a work in progress as a joint project, [HS], of both authors. This work relies on the results of this paper as well as [HS20] and [HS21].

This paper is structured as follows:

- In Section 2, we recall basic notations and properties from representation theory of p-adic groups. We further recall basic data on the exceptional group of type E_8.
- In Section 3 we describe the tools and algorithm used by us in the proof of Theorem 4.1.
- In Section 4 we state our main theorem, Theorem 4.1, and list all cases which can be resolved using our algorithm. This algorithm constitutes

of reducibility and irreducibility tests as well as some tests to check if the representation admits a unique irreducible subrepresentation.

- In Section 5 we go over the exceptional cases which could not have been fully resolved by our algorithm. We resolve most of these cases completely and make some progress towards the resolution of the remaining two cases.

Finally, we wish to address a question which was broached to us following [HS20] and [HS21]. The reducibility of a non-unitary degenerate principal series representation can be determined by the local Shahidi coefficients which would seem to make our algorithm obsolete. However, the algorithm, presented in Section 3, is useful for various other reasons such as:

- It allows us to determine the reducibility for unitary cases too.
- While local Shahidi coefficients can inform us regarding the reducibility of non-unitary degenerate principal series, the output of the reducibility test is also useful when studying global phenomena such as the Siegel-Weil identity (and indeed this data is used in [HS]).
- Data from the reducibility and irreducibility tests is useful for studying the structure of reducible degenerate principal series and in particular for studying its socle and cosocle.

We also wish to point out that comparing the lists of non-unitary reducible degenerate principal series determined by our method with that given by Shahidi's method, was useful for debugging purposes. Indeed, as in the E_6 and E_7 case, our algorithm was decisive for all non-unitary cases and agreed with the results of Shahidi's method. The only cases where the algorithm was unable to determine the reducibility of the degenerate principal series were a few unitary ones.

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2. Preliminaries

This section has three parts. In the first part, we fix notations for this paper. This part is organized as an enumerated list in order to make the look up of notations easier. Also, at the end of this section we introduce the split group of type E_8.

2.1. *Groups, Characters and Representations.* In this subsection, we fix notations and recall basic facts about the groups, characters and representations involved in this paper. For a more detailed discussion, the reader is encouraged to consider [HS20, Section 2] and [HS21, Section 2].

2.1.1. *Groups.*

1. Let F be a non-Archimedean local field with norm $|\cdot|$. Let q denote the cardinality of its residue field and let ϖ denote a uniformizer of F.
2. Let G denote the F-points of a split simply-connected reductive group.
3. Let T be a maximal split torus of G.
4. Let B be a Borel subgroup of G such that $T \subset B$.
5. Let Φ_G denote the roots of G with respect to T and let $\Phi_G^+ \subset \Phi_G$ denote the positive roots of G with respect to B.
6. Let $\Delta_G = \{\alpha_1, ..., \alpha_n\}$ be the set of simple roots of Φ_G with respect to B.
7. Let $n = |\Delta_G| = \dim_F(T)$ denote the rank of G.
8. Let $\Phi_G^\vee = \{\alpha^\vee \mid \alpha \in \Phi_G\}$ denote the set of coroots of G with respect to T.
9. We use $\langle \cdot, \cdot \rangle$ to denote the usual pairing between characters and cocharacters of T.
10. Let $\bar{\omega}_{\alpha_1}, ..., \bar{\omega}_{\alpha_n}$ denote the fundamental weights of T which satisfy
$$\left\langle \bar{\omega}_{\alpha_i}, \alpha_j^\vee \right\rangle = \delta_{i,j}.$$
11. Let $W = \langle s_i \mid 1 \le i \le n \rangle$ denote the Weyl group of G with respect to T, generated by the simple reflections s_i associated with the simple roots α_i.
12. For $\Theta \subset \Delta_G$, let $P_\Theta = \langle B, s_i \mid \alpha_i \in \Theta \rangle = M_\Theta \cdot U_\Theta$ be the standard parabolic subgroup of G associated to Θ. We denote its Levi subgroup by M_Θ and its unipotent radical by U_Θ.
13. Let Φ_{M_Θ}, $\Phi_{M_\Theta}^+$ and Δ_{M_Θ} denote the roots, positive roots and simple roots of M_Θ with respect to T and $B \cap M_\Theta$ respectively.
14. Let $W_{M_\Theta} = \langle s_i \mid \alpha_i \in \Theta \rangle$ denote the Weyl group of M_Θ with respect to T.
15. Let $P_i = P_{\Delta_G \setminus \{\alpha_i\}}$ and $M_i = M_{\Delta_G \setminus \{\alpha_i\}}$ denote a maximal (proper) standard parabolic subgroup of G and its Levi subgroup.
16. For maximal standard Levi subgroups M_i and M_j of G, we write $M_{i,j} = M_i \cap M_j$.
17. We denote the rank 1 Levi subgroups by $L_i = M_{\{\alpha_i\}}$.

2.1.2. *Characters.*

1. Let $\mathbf{X}(G) = \{\Omega : G \to \mathbb{C}^\times\}$ denote the complex manifold of continuous characters of G, we use additive notations for this group, that is
$$(\Omega_1 + \Omega_2)(g) = \Omega_1(g) \cdot \Omega_2(g).$$
We usually use the letter Ω to denote elements in $\mathbf{X}(M)$, for a non-minimal Levi subgroup M of G, while using λ to denote an element of $\mathbf{X}(T)$. Also, note that W acts on $\mathbf{X}(T)$ via its action on T.
2. We denote the set of unramified elements $\Omega \in \mathbf{X}(T)$ by $\mathbf{X}^{un}(T)$.
3. Let $\mathbf{1}_G \in \mathbf{X}(G)$ denote the trivial character of G.

4. We say that $\chi \in \mathbf{X}(T)$ has finite order if there exists $k \in \mathbb{N}$ such that $\chi^k = \mathbf{1}_T$. The order of χ, denoted by $ord(\chi)$, is the minimal $k \in \mathbb{N}$ such that $\chi^k = \mathbf{1}_T$.

   In particular, every element $\Omega \in \mathbf{X}(F^\times)$ can be written as $\Omega = s + \chi$, where $s \in \mathbb{C}$ and χ is of finite order. Namely,
$$\Omega(x) = \chi(x)|x|^s \quad \forall x \in F^\times.$$
   It holds that $\mathbf{X}^{un}(F^\times)$ can be described by all characters of the forms $|x|^s$ for some $s \in \mathbb{C}$.
5. We write $\mathrm{Re}(\Omega)$ for the character
$$\mathrm{Re}(\Omega)(x) = |x|^{\mathrm{Re}(s)}$$
   and $\mathrm{Im}(\Omega)$ for the character satisfying
$$\Omega = \mathrm{Re}(\Omega) + \mathrm{Im}(\Omega).$$
6. We say that $\lambda \in \mathbf{X}(T)$ is **anti-dominant** if
$$\mathrm{Re}(\langle \lambda, \alpha_i^\vee \rangle) \leq 0 \quad \forall 1 \leq i \leq n.$$
   Note that every W_G-orbit in $\mathbf{X}(T)$ contains at least one anti-dominant element and all anti-dominant elements in the same W_G-orbit have an equal real part. As a convention, we denote an anti-dominant element by $\lambda_{a.d.}$.
7. Let $\Omega_{i,s,\chi}$ denote the character of a maximal Levi subgroup M_i of G associated with $(s+\chi) \circ \bar{\omega}_{\alpha_i}$, where $s \in \mathbb{C}$ and $\chi \in \mathbf{X}(F^\times)$ is of finite order. Note that if G is simple, then any element in $\mathbf{X}(M_i)$ can be written this way.

2.1.3. *Representations.*

1. Let $Rep(G)$ denote the category of admissible representations of G.
2. As above, $\mathbf{1}_G$ denotes the trivial representation of G.
3. Let $i_M^G : Rep(M) \to Rep(G)$ and $r_M^G : Rep(G) \to Rep(M)$ denote the functors of normalized parabolic induction and Jacquet functor, adjunct by the Frobenius reciprocity:

$$\mathrm{Hom}_G\left(\pi, i_M^G \sigma\right) \cong \mathrm{Hom}_M\left(r_M^G \pi, \sigma\right). \tag{2.1}$$

4. For $\pi, \sigma \in Rep(G)$, such that σ is irreducible, let $mult(\sigma, \pi)$ denote the multiplicity of σ in the Jordan-Hölder series of π.
5. Let $\mathfrak{R}(G)$ denote the Grothendieck ring of $Rep(G)$ and let $[\pi]$ denote the image of $\pi \in Rep(G)$ in $\mathfrak{R}(G)$. Recall that $\mathfrak{R}(G)$ admits a partial order such that $\pi_1 \leq \pi_2$ if $mult(\sigma, \pi_1) \leq mult(\sigma, \pi_2)$ for every irreducible $\sigma \in Rep(G)$.
6. We remind the reader that, for Levi subgroups L and M of G and $\sigma \in Rep(M)$, the composition $\left[r_L^G i_M^G \sigma\right]$ is given by the *geometric*

lemma ([BZ77, Lemma 2.12], [Cas74, Theorem 6.3.6]):

$$\left[r_L^G i_M^G \sigma\right] = \sum_{w\in W^{M,L}} \left[i_{L'}^L \circ w \circ r_{M'}^M \sigma\right], \tag{2.2}$$

where:
- $W^{M,L} = \left\{w \in W \mid w\left(\Phi_M^+\right) \subseteq \Phi_G^+,\ w^{-1}\left(\Phi_L^+\right) \subseteq \Phi_G^+\right\}$ is the set of shortest representatives in W of the double coset space $W_L\backslash W_G/W_M$.
- For $w \in W^{M,L}$ we write $M' = M\cap w^{-1}Lw$ and $L' = wMw^{-1}\cap L$.

7. For $\pi \in Rep\,(G)$, we write $\left[r_T^G\pi\right] = \sum\limits_{i=1}^{l} n_i \times [\lambda_i]$ for certain $\lambda_i \in \mathbf{X}\,(T)$ such that $mult\left(\lambda_i, r_T^G\pi\right) = n_i > 0$. Since $\dim_{\mathbb{C}}\left(r_T^G\pi\right)$ is finite, there are only finitely many such λ_i. We call such λ_i **the exponents of** π.
8. The representations $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$ are called **degenerate principal series**. The exponent $\lambda_0 = r_T^{M_i}\left(\Omega_{M_i,s,\chi}\right)$ is called the **initial exponent** of π.
9. We say that $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$ is **regular** if $\mathrm{Stab}_W\left(\lambda_0\right) = \{1\}$, where $\lambda_0 = r_T^{M_i}\left(\Omega_{M_i,s,\chi}\right)$.
10. Let $w_{0,i}$ denote the longest element in $W^{M_i,T}$. It holds that $w_{0,i}\cdot\left(i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)\right) = i_{M_j}^G\left(\Omega_{M_j,-s,\overline{\chi}}\right)$, where $\overline{\chi}$ is the complex conjugate of χ and $M_j = w_{0,i}M_iw_{0,i}^{-1}$. We recall that $M_j = M_i$, except when G is of type A_n, D_{2n+1} or E_6.

    We call $i_{M_j}^G\left(\Omega_{M_j,-s,\overline{\chi}}\right)$ the **invert representation** of $i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$ and note that the invert representation has the same irreducible constituents but in an "inverted order". That is, $i_{M_j}^G\left(\Omega_{M_j,-s,\overline{\chi}}\right)$ admits a Jordan-Hölder series whose irreducible quotients appear an inverted order than that of $i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$. When $\chi = \mathbf{1}$, the invert was defined in [Jan95, Remark 2.2.5] as a variation of the Iwahori-Matsumoto involution.
11. The following is a well known fact, commonly referenced as a "central character argument", see [HS20, Lemma. 3.12] for a proof. Let $\sigma \in Rep\,(G)$ be irreducible and let $\lambda \in \mathbf{X}\,(T)$, then

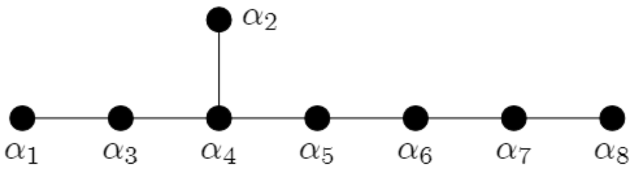
$$\lambda \leq r_T^G\sigma \implies \sigma \hookrightarrow i_T^G\lambda. \tag{2.3}$$

2.2. *The Exceptional Group of Type E_8.* Let G be the split, semi-simple, simply-connected group of type E_8. In this subsection we describe the structure of G. We fix a Borel subgroup B and a maximal split torus $T \subset B$ with notations as in Subsection 2.1. The set of roots, Φ_G, contains 240 roots. The group G is generated by symbols

$$\{x_\alpha(r)\ :\ \alpha \in \Phi_G, r \in F\}$$

subject to the Chevalley relations as in [Ste68, Section 6].

We label the simple roots Δ_G and the Dynkin diagram of G using the Bourbaki labelling:



Recall that for $\Theta \subset \Delta_G$ we denote by M_Θ the standard Levi subgroup of G such that $\Delta_M = \Theta$. We let M_i denote the Levi subgroup of the maximal parabolic subgroup $P_i = P_{\Delta_G \setminus \{\alpha_i\}}$.

Lemma 2.1. *Under these notations, it holds that:*

1. $M_1 \cong \left\{ g \in GSpin_{14}(F) \;\middle|\; \det(g) \in (F^\times)^2 \right\}$.
2. $M_2 \cong \left\{ g \in GL_8(F) \;\middle|\; \det(g) \in (F^\times)^2 \right\}$.
3. $M_3 \cong \{(g_1, g_2) \in GL_2(F) \times GL_7(F) \mid \det(g_1) = \det(g_2)\}$.
4. $M_4 \cong \{(g_1, g_2, g_3) \in GL_3(F) \times GL_2(F) \times GL_5(F) \mid$
   $\det(g_1) = \det(g_2) = \det(g_3)\}$.
5. $M_5 \cong \{(g_1, g_2) \in GL_5(F) \times GL_4(F) \mid \det(g_1) = \det(g_2)\}$.
6. $M_6 \cong \left\{ (g_1, g_2) \in GSpin_{10}(F) \times GL_3(F) \;\middle|\; \det(g_1) = \det(g_2) \in (F^\times)^2 \right\}$.
7. $M_7 \cong \{(g_1, g_2) \in GE_6(F) \times GL_2(F) \mid \det(g_1) = \det(g_2)\}$.
8. $M_8 \cong GE_7(F)$.

Where det *denotes the similitude factors on the relevant groups (in particular, the similitude factor on GL_n is the usual determinant).*

We record here, for $1 \leq i \leq 8$, the cardinality of $W^{M_i,T}$, the set of shortest representatives of W_G/W_{M_i}

i	1	2	3	4	5	6	7	8
$\lvert W^{M_i,T}\rvert$	2,160	17,280	69,120	483,840	241,920	60,480	6,720	240

We also mention that $|W_G| = 696,729,600$. Every $\lambda \in \mathbf{X}(T)$ is of the form

$$\lambda = \sum_{i=1}^{8} \Omega_i \circ \bar{\omega}_{\alpha_i}.$$

As a shorthand, we will write

$$\begin{pmatrix} & & \Omega_2 & & & & \\ \Omega_1 & \Omega_3 & \Omega_4 & \Omega_5 & \Omega_6 & \Omega_7 & \Omega_8 \end{pmatrix} = \sum_{i=1}^{8} \Omega_i \circ \bar{\omega}_{\alpha_i}.$$

3. The Algorithm

In this section, we recall the algorithm used by us to study the degenerate principal series $\pi = i_{M_i}^{G}\left(\Omega_{M_i,s,\chi}\right)$. This account follows similar lines to those in [HS20, Section 3] and [HS21, Section 3]. This algorithm was implemented by the authors using the Sagemath environment [The22] and was used to prove our main theorem Theorem 4.1. In fact, most cases were determined using this algorithm with a few exceptional cases listed and treated in Section 5.

We identify the data defining π by a triple of numbers $[i, s, ord\left(\chi\right)]$. In particular, the reducibility and lengths of the maximal semi-simple subrepresentation and quotient depend only on this triple (and are uniform among χs with the same order). As explained in [HS20, Remark 3.1], we may assume, without loss of generality, that $s \in \mathbb{R}$.

The algorithm has the following parts:

1. Determine all non-regular such π - as explained below, there is a finite number of such cases!
2. Determine all reducible regular π - there are only finitely many such cases!
3. Apply reducibility tests to non-regular π (these may be inconclusive).
4. Apply an irreducibility test to non-regular π (this too may be inconclusive). This test uses the so called **branching rule calculation** introduced in Subsection 3.4.
5. Determine if π admits a unique irreducible subrepresentation. As will be explained in Subsection 3.5, for $s < 0$, π admits a unique irreducible quotient and for $s > 0$, π admits a unique irreducible subrepresentation.

Here, regular is as in item (9) of Subsection 2.1.3.

We point out that, by contragredience, it is enough to consider only the cases where $s \leq 0$ since the invert representation of π admits a Jordan-Hölder series with same irreducible quotients appearing but in inverted order.

3.1. *Regularity and the Regular Case.* Recall that we say that $\pi = i_{M_i}^{G}\left(\Omega_{M_i,s,\chi}\right)$ is **regular** if $\mathrm{Stab}_W\left(\lambda_0\right) = \{1\}$, where $\lambda_0 = r_T^{M_i}\left(\Omega_{M_i,s,\chi}\right)$. We say that π is **non-regular** otherwise. Recall, from [HS20, Remark 3.1], that there are only finitely many non-regular degenerate principal series representations and only finitely many reducible regular degenerate principal series representations.

Our first order of business is to describe an algorithm for locating all non-regular degenerate principal series and determine all reducible regular degenerate principal series.

We recall, from [HS20, §3.1], the following algorithm to finding all non-regular degenerate principal series. Recall that

$$r_T^{M_i}\left(\Omega_{M_i,s,\mathbf{1}}\right) = s \circ \bar{\omega}_{\alpha_i} - \rho_B + \rho_{P_i}$$

and that
$$\mathrm{Stab}_W(\lambda) = \mathrm{Stab}_W(\mathrm{Re}(\lambda)) \cap \mathrm{Stab}_W(\mathrm{Im}(\lambda))$$
for $\lambda \in \mathbf{X}(T)$

1. First, for a fixed Levi subgroup M_i, we find the set X_i of all values $s \in \mathbb{R}$ such that $\mathrm{Stab}_W\left(r_T^{M_i}(\Omega_{M_i,s,\mathbf{1}})\right)$ is non-trivial, namely,
$$X_i = \left\{ \langle s \circ \bar{\omega}_{\alpha_i} - \rho_B + \rho_{P_i}, \beta^\vee \rangle = 0 \;\middle|\; \beta \in \Phi^+,\ s \in \mathbb{R} \right\}.$$
2. We then find the set Y_i of possible orders of characters χ such that $\mathrm{Stab}_W\left(\mathrm{Im}\left(r_T^{M_i}(\Omega_{M_i,s,\mathbf{1}})\right)\right)$ is non-trivial (and note that this is independent of s). We take
$$Y_i = \left\{ m \;\middle|\; \begin{array}{c} w \in W^{G,M_i} \\ \bar{\omega}_{\alpha_i} - w \cdot \bar{\omega}_{\alpha_i} = \sum_{\beta \in \Delta} n_\beta \bar{\omega}_\beta \\ m \mid \gcd(\{n_\beta \mid \beta \in \Delta\}) \end{array} \right\}$$
3. The set $Z_i = X_i \times Y_i$ is a set of candidates of pairs (s,m) such that $i_{M_i}^G(\Omega_{M_i,s,\chi})$ is non-regular if χ is of order m. We now determine which pairs $(s,m) \in Z_i$ indeed yield non-regular representations.

   In order to do this, fix a pair $(s,m) \in Z_i$, a character χ of order m and $\lambda_{a.d.}$ an anti-dominant element in the orbit of $r_T^{M_i}(\Omega_{M_i,s,\chi})$.

   We recall that $\mathrm{Stab}_W(\mathrm{Re}(\lambda_{a.d.}))$ is generated by the simple reflections w_j such that
$$\left\langle \lambda_{a.d.}, \alpha_j^\vee \right\rangle = 0.$$
   Thus, $\mathrm{Stab}_W(\lambda_{a.d.})$ is non-trivial if and only if there exists $w \in \mathrm{Stab}_W(\mathrm{Re}(\lambda_{a.d.}))$ such that $w \cdot \lambda_{a.d.} = \lambda_{a.d.}$.

We now turn to determine which regular representations are reducible. Assume that $\pi = i_{M_i}^G(\Omega_{M_i,s,\chi})$ is a regular degenerate principal series representation and let λ_0 denote its initial exponent. As explained in [HS20, §3.1], π is irreducible if and only if
$$\langle \lambda_0, \alpha^\vee \rangle \neq 1$$
for all $\alpha \in \Phi_G \setminus \Phi_{M_i}$. Evidently, if π is reducible, then $ord(\chi) \in Y_i$.

Thus, in order to find all regular reducible degenerate principal series representations, we go through all $m \in Y_i$, and for any χ of order m, we find the set
$$\{ s \in \mathbb{R} \mid \exists \beta \in \Phi_G \setminus \Phi_{M_i} :\ \langle \lambda_0, \beta^\vee \rangle = 1 \}.$$
Finally, we check for which such s, $i_{M_i}^G(\Omega_{M_i,s,\chi})$ is regular. It turns out that for each i and $m \in Y_i$, there is exactly one $s \leq 0$ such that $i_{M_i}^G(\Omega_{M_i,s,\chi})$ regular and reducible.

We point out that the process described above is fairly efficient. Difficulties arise in the latter parts of the algorithm which will be introduced below.

For the rest of this section, we assume that $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$ is non-regular.

3.2. *Reducibility Tests.* We now turn to describe a test for the reducibility of a non-regular $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$. As mentioned in the introduction, for non-unitary π there are simpler methods for determining reducibility. The tests described below are mainly useful to determine the reducibility of a unitary π. Also, the data provided by the calculation described below is useful for other purposes such as establishing Siegel-Weil like identities.

Recall, from [Tad98, Lemma 3.1], Tadić's criterion for reducibility:

LEMMA 3.1. *Let $\pi = i_M^G(\Omega)$. Assume there exist smooth representations Π and π' of G of finite length and a Levi subgroup L of G such that*

1. $\pi \leq \Pi$, $\pi' \leq \Pi$.
2. $r_L^G\pi + r_L^G\pi' \nleq r_L^G\Pi$.
3. $r_L^G\pi \nleq r_L^G\pi'$.

Then, π is reducible and admits a common irreducible subquotient with π'.

Thus, in order to prove that a representation $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$ is reducible, we provide another $\pi' \in Rep\left(G\right)$ such that π and π' share a common irreducible constituent, while $\pi \neq \pi'$. We always take $\Pi = i_T^G\lambda_{a.d.}$, where $\lambda_{a.d.}$ is an anti-dominant exponent of π.

In most cases, it is enough to consider $\pi' = i_{M_j}^G\left(\Omega_{M_j,t,\chi^l}\right)$, where l is a totative[1] of $ord\left(\chi\right)$. In other cases, one takes $\pi' = i_{M_{j_1,j_2}}^G\left(\Omega_{s_1,s_2,\chi,k_1,k_2}\right)$, where $\Omega_{s_1,s_2,\chi,k_1,k_2} \in \mathbf{X}\left(M_{j_{1,2}}\right)$ is associated with

$$\left(s_1+\chi^{k_1}\right)\circ\bar{\omega}_{\alpha_{j_1}} + \left(s_2+\chi^{k_2}\right)\circ\bar{\omega}_{\alpha_{j_2}}, \tag{3.1}$$

such that at least one of k_1 and k_2 is a totative of $ord\left(\chi\right)$.

Considering Lemma 3.1, the conditions $\pi \leq \Pi$ and $\pi' \leq \Pi$ hold by construction. As for the other conditions in the lemma:

- If π and π' share a common anti-dominant exponent then $r_L^G\pi + r_L^G\pi' \nleq r_L^G\Pi$ since for an anti-dominant exponent of π,
$$mult\left(\lambda_{a.d.}, r_T^G\pi\right) = mult\left(\lambda_{a.d.}, r_T^G\pi'\right) = mult\left(\lambda_{a.d.}, r_T^G\Pi\right).$$
- In order to verify that $r_T^G\pi \nleq r_T^G\pi'$, it is enough to find an exponent λ of π such that $mult\left(\lambda, r_T^G\pi\right) > mult\left(\lambda, r_T^G\pi'\right)$.

If π and π' satisfy these conditions, then π is reducible and contains an irreducible subquotient in common with π' (which contains an anti-dominant exponent).

It remains to explain how to find candidates for π' for a given $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$. We start by looking for π' of the form $i_{M_j}^G\left(\Omega_{M_j,t,\chi^l}\right)$. This is done as follows:

[1] That is, an integer $0 < l \leq ord\left(\chi\right)$ which is coprime with $ord\left(\chi\right)$

- We first consider the set Ξ of all non-regular degenerate principal series $i_{M_i}^G(\Omega_{M_i,s,\chi})$ of G. For each $\pi\in\Xi$, we calculate its set of anti-dominant exponents.
- We say that π and π' are in the same orbit if they have a common anti-dominant exponent. In fact, if they have an anti-dominant exponent in common then all of their anti-dominant exponents are in common. Thus, being in the same orbit is an equivalence relation. We partition $\Xi=\cup_{\lambda_{a.d.}}\Xi_{\lambda_{a.d.}}$ with respect to this equivalence relation.
- For each $\Xi_{\lambda_{a.d.}}$ and $\pi,\pi'\in\Xi_{\lambda_{a.d.}}$, we go over the exponents λ of π and check, using Equation (2.2), if the condition $mult\left(\lambda, r_T^G\pi\right) > mult\left(\lambda, r_T^G\pi'\right)$ holds. If this condition holds, then the conditions of Lemma 3.1 are satisfied.

The process described above would demonstrate the reducibility of most reducible degenerate principal series. In most cases where the process fails to produce a representation π' which demonstrates that π is reducible (in particular, if $\Xi_{\lambda_{a.d.}}=\{\pi\}$ is a singelton), the representation π is irreducible, but not always. We shall detail an irreducibility test in Subsection 3.4, but before that we wish to explain how to find π' in the few cases where π is reducible but the above process does not yield π'. Such a representation π' arise from an induction from a 1-dimensional representation of a non-maximal Levi subgroup. In particular, in every case in the proof of Theorem 4.1, we were able to find π' in the form of an induction from a 1-dimensional representation on Levi subgroup of co-rank 1 (maximal Levi) or 2.

In order to find such π', we go through all exponents of π. We write the exponent λ as

$$\lambda=\sum_{i=1}^{n} r_i\bar{\omega}_{\alpha_i},$$

where $r_i\in\mathbf{X}\left(F^\times\right)$, and write $\Theta_\lambda=\{\alpha_i\mid r_i=-1\}$. If $1\le|\Theta_\lambda|\le n$, then λ can be identified with the initial exponent of $\pi'=i_{M_{\Theta_\lambda}}^G\Omega_\lambda$, where Ω_λ is a character of M_{Θ_λ} which satisfies $r_T^{M_{\Theta_\lambda}}\Omega_\lambda=\lambda$.

As above, for such π', the conditions $\pi\le\Pi$, $\pi'\le\Pi$ and $r_L^G\pi+r_L^G\pi'\not\le r_L^G\Pi$ are satisfied by construction. If there exists an exponent λ' of π such that $mult\left(\lambda, r_T^G\pi\right) > mult\left(\lambda, r_T^G\pi'\right)$, then the condition $r_L^G\pi\not\le r_L^G\pi'$ is also satisfied. This condition may be tested using Equation (2.2) by going over all exponents λ' of π.

We note that searching for π' becomes more difficult (in terms of longer calculation run-time) with the following factors:

- The list of possible λ roughly increases with the size of the support of $r_T^G\pi$.
- The list of possible λ' roughly increases with the size of the support of $r_T^G\pi'$ which, in turn, roughly increases as Θ_λ decreases.

Luckily, it turns out that for every reducible π, one can find λ and π' with $|\Theta_\lambda|$ equals $n-1$ or $n-2$. The case of $n-1$ is the case $\pi' = i_{M_j}^G\left(\Omega_{M_j,t,\chi^l}\right)$ discussed above. And so this discussion is relevant for searching examples with $|\Theta_\lambda| = n-2$, namely $\pi' = i_{M_{j_1,j_2}}^G\left(\Omega_{s_1,s_2,\chi,k_1,k_2}\right)$.

3.3. *Branching Rule Calculations.* In this subsection we describe a process, named **branching rule calculation**, which is useful tool for both testing irreducibility and determining the length of the maximal semi-simple subrepresentation of π. In fact, this process is further used in [HS, Appendix A] for a more detailed study of the Jordan-Hölder series of certain degenerate principal series representations.

For an irreducible subquotient σ of a degenerate principal series representation π, we would wish to determine $\left[r_T^G\sigma\right]$. While this is in many cases very difficult, we are, however, able to give effective lower bounds to $\left[r_T^G\sigma\right]$.

More precisely, let

$$\mathcal{S} = \left\{f : \mathbf{X}\left(T\right) \to \mathbb{N} \mid f \text{ has a finite support}\right\}.$$

We note that $\mathcal{S}$ is endowed with a natural partial order. For any $\sigma \in Rep\left(G\right)$, let $f_\sigma \in \mathcal{S}$ be defined by

$$f_\sigma\left(\lambda\right) = mult\left(\lambda, r_T^G\sigma\right).$$

In particular, for $\sigma = \pi$, f_π can be calculated using Equation (2.2).

A sequence of functions $\left\{f_j\right\}_{j=1}^k$ is called a **σ-dominated sequence** if it satisfies the condition $f_1 \leq ... \leq f_k \leq f_\sigma$. A **branching rule calculation** is a process by which we construct such a sequence from a known initial function $f_1 \leq f_\sigma$.

In fact, in our implementation, we usually construct a **unital σ-dominated sequence** $\left\{f_j\right\}_{j=1}^k$. That is, a σ-dominated sequence which further satisfies $f_1 = \delta_\lambda$ for some $\lambda \leq r_T^G\sigma$, where δ_λ stands for the Kronecker delta function given by

$$\delta_\lambda\left(\lambda'\right) = \begin{cases} 1, & \lambda' = \lambda \\ 0, & \lambda' \neq \lambda \end{cases}.$$

We now explain the algorithm by which we construct such a sequence and later we will detail how this algorithm can be implemented in a computer script.

We start with a known initial function $f_1 \leq f_\sigma$ and proceed with the following recursive process:

1. Assume that we have already constructed a σ-dominated sequence $f_1 < ... < f_l$.
2. We choose $\lambda' \in \operatorname{supp}\left(f_l\right)$ and a Levi subgroup L of G such that L admits a unique irreducible representation τ such that $\lambda' \leq r_T^L\tau$.

3. For a choice of (λ', L, τ) as above, for any $\mu \in \mathbf{X}(T)$, let

$$g_{\lambda',L,\tau}(\mu) = \max\left\{ f_l(\mu), \left\lceil \frac{f_l(\lambda')}{mult\left(\lambda', r_T^L\tau\right)} \right\rceil \cdot mult\left(\mu, r_T^L\tau\right) \right\}$$
$$= \max\left\{ f_l(\mu), \left\lceil \frac{f_l(\lambda')}{f_\tau(\lambda')} \right\rceil \cdot f_\tau(\mu) \right\}. \tag{3.2}$$

4. If $f_l < g_{\lambda',L,\tau}$ for some choice of (λ', L, τ) as in step (2), set $f_{l+1} = g_{\lambda',L,\tau}$ and go back to step (1). Otherwise, we take $k = l$ and the process terminates.

The terminal function f_k in this sequence provides a lower bound to the multiplicities of exponents appearing in $r_T^G\sigma$.

Remark. The uniqueness condition on τ in item (2) can be slightly relaxed. In fact, one actually only needs that $\left[r_T^L\tau\right]$ be unique.

A few natural questions arise from the description of this algorithm which we will now answer:

- *How does one choose such an initial function f_1?* We usually start with $f_1 = \delta_\lambda$ for a known exponent $\lambda \leq r_T^G\sigma$. In fact, for most applications later in this paper, we consider an irreducible representation σ which admits an anti-dominant exponent $\lambda_{a.d.}$ and use $f_1 = \delta_{\lambda_{a.d.}}$.
- *How could one go through all possible triples (λ', L, τ)?* In our implementation we used a database of Levi subgroups and irreducible representation of these Levi subgroups which contain a unique exponent. This database, listed in Appendix A, is not exhaustive but contains all the examples which were required for the calculations performed for this paper.
- *Does the order of choices of triples (λ', L, τ) matter?* If one starts with the same initial function f_0 and uses the same database of triples (λ', L, τ), then the final element in the sequence will be the same regardless of the order of choices of triples (λ', L, τ).
- *Why is the sequence generated this way indeed σ-dominated?* We assume that $f_1 < ... < f_l$ is σ-dominated and consider the triple (λ', L, τ) chosen as above and let f_{l+1} be constructed as in step (3). We need to show that $f_l \leq f_{l+1} \leq f_\sigma$.
  - If μ is not an exponent of τ, then $f_{l+1}(\mu) = f_l(\mu) \leq f_\sigma(\mu)$.
  - If μ is an exponent of σ, then, by construction, $f_l(\mu) \leq f_{l+1}(\mu)$. In order to show that $f_{l+1}(\mu) \leq f_\sigma(\mu)$ we write

$$\left[r_L^G\sigma\right] = \sum_k r_k\left[\tau_k\right],$$

where τ_k are the irreducible representations of L appearing in the Jordan-Hölder series of $r_L^G\sigma$. Since the Jacquet functor is a functor, it follows that

$$\left[r_T^G\sigma\right] = \sum_k r_k \left[r_T^L\tau_k\right].$$

By assumption, λ' appears only in one of the τ_k, say $\tau_1 = \tau$. Thus, $r_1 f_\tau(\mu) \leq f_\sigma(\mu)$. On the other hand,

$$f_l(\lambda') \leq r_1 f_\tau(\lambda')$$

Hence, $f_{l+1}(\mu) \leq f_\sigma(\mu)$.

Other accounts of this process can be found in [HS20, Subsection 3.3] and [HS21, Subsection 3.3]. Furthermore, in Appendix B we give an example of a branching rule calculation performed in the group E_8. A pair of examples of such calculations, in the case of $SL_4(F)$, can be found in [HS21, Appendix C] and an example performed in the group E_6 can be found in [HS20, Appendix B].

We now wish to shortly explain how the branching rule calculation can be implemented in the computer. In fact, whenever we say "using a branching rule calculation" later in the text, we mean that such a calculation was performed using a computer script in [The22].

We start by considering the following equivalence relation on $\mathbf{X}(T)$. We say that $\lambda, \lambda' \in \mathbf{X}(T)$ are **A_1-equivalent** if there exist sequences $\lambda_1, ..., \lambda_k \in \mathbf{X}(T)$ and $j_1, ..., j_{k-1} \in \{1, ..., n\}$ such that

1. $\lambda_1 = \lambda$ and $\lambda_k = \lambda'$.
2. $\lambda_{t+1} = w_{i_t} \cdot \lambda_t$.
3. $\left\langle \lambda_t, \alpha_{i_t}^\vee \right\rangle \neq \pm 1$.

By Equation (A.2), $f_\sigma(\lambda) = f_\sigma(\lambda')$. Thus, it is more convenient (and efficient) to apply branching rules on equivalence classes of this relation.

The branching rule calculation starts with an irreducible representation σ and an exponent $\lambda_1 \leq r_T^G\sigma$ on which we perform the calculation.

During a branching rule calculation, we keep track of the following lists:

- A list of A_1-equivalence classes $\kappa_1, ..., \kappa_t$ such that the exponents in each κ_j are exponents of σ.
- A list of known lower bounds $n_1, ..., n_t$ to the multiplicities of the exponents in each of these equivalence classes.
- A list of "flagged" exponents consisting of either of the following cases:
  - Exponents that were added to the support but no branching rules were applied to them yet.
  - Exponents to which we have applied branching rules before but the lower bound we have for their multiplicities was increased due to the application of branching rules to other exponents, for such exponents we would wish to apply the relevant branching rules again, in case this will lead to an update in the lower bounds of multiplicities of other exponents.

The first two lists encode the functions $f_1, ..., f_l, ...$ in the σ-dominated sequence we construct. It should be noted that a list in Sagemath (as in Python) is a dynamic object and elements can be added or removed from a list.

At each step of the calculation, we keep only the data in the most recent function f_l as there is no use for f_{l-1} in subsequent calculations. The support of f_l is given by the union $\kappa_1 \cup ... \cup \kappa_t$ of A_1-equivalence classes and its values are given by

$$f_l(\lambda) = \begin{cases} n_j, & \lambda \in \kappa_j \\ 0, & \lambda \notin \kappa_1 \cup ... \cup \kappa_t \end{cases} \tag{3.3}$$

The idea is that at each step of the process, when building f_{l+1} from f_l, we update the list of equivalency classes κ_j or the list of multiplicities and update the list of flags accordingly.

We initiate the calculation by determining the A_1-equivalency class κ_1 of λ_1 and setting $f_1 \in \mathcal{S}$ by

$$f_1(\lambda) = \begin{cases} 1, & \lambda \in \kappa_1 \\ 0, & \lambda \notin \kappa_1 \end{cases}.$$

The calculation then proceeds by the following recursive process which constructs f_{l+1} from f_l:

1. Assume that we have a list of A_1-equivalency classes $\kappa_1, ..., \kappa_t$ and a list of multiplicities $n_1, ..., n_t$ such that f_l is given by Equation (3.3). We also assume that the list of flags is non-empty.
2. We go to the first exponent λ in the flag list and go over the list of branching rules listed in Appendix A (except for the rule given by Equation (A.2) which was already used when constructing the κ_js). For each branching rule that can be applied to λ (that is, λ fits the relevant pattern in Appendix A) we proceed as follows:
   - Let L be the Levi subgroup of G and let τ be the irreducible representation of L associated to λ by the branching rule so that $(\lambda, L.\tau)$ will be a triple as in the algorithm above.
   - Calculate $g_{\lambda',L,\tau}$. If $f_l < g_{\lambda',L,\tau}$ and then for any $\lambda' \in supp\,(g_{\lambda',L,\tau})$:
     - If $\lambda' \in supp\,(g_{\lambda',L,\tau})$ while $\lambda' \notin supp\,(f_l)$, then calculate the A_1-equivalency class of λ' and add it to the list of equivalency classes. Also add the exponents in this class to the flag list. This step needs to be done only for the first such representative of this class (it is natural for more than one representative to appear together when applying a branching rule to a certain exponent).
     - If $\lambda \in supp\,(f_l)$ but $f_l(\lambda') < g_{\lambda',L,\tau}(\lambda')$, then update the multiplicity of the class of λ' and add its exponents to the flag list (if not there).

3. Finally erase λ from the flag list and if it is not empty, return to step (1). Otherwise, the calculation is terminated.

From the resulting list of equivalency classes and multiplicities, one constructs f_k according to Equation (3.3).

3.4. *Irreducibility Test.* In this subsection we explain how to test for the irreducibility of $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$ using branching rule calculations.

Let $\lambda_{a.d.}$ denote an anti-dominant exponent of π and let π_0 denote an irreducible subquotient of π such that $\lambda_{a.d.} \le r_T^G \pi_0$. Then, construct a sequence $f_0 \le f_1 \le ... \le f_k \le f_{\pi_0}$ following the process described in Subsection 3.3.

- If $f_k = f_\pi$, then $f_{\pi_0} = f_\pi$ and it follows that $\pi = \pi_0$ is irreducible since all subquotients of π have their cuspidal support along B. In particular, π is irreducible.
- In fact, in order to prove that π is irreducible, it is enough to show that
$$\begin{aligned} mult\left(\lambda_0, r_T^G\pi_0\right) &= mult\left(\lambda_0, r_T^G\pi\right) \\ mult\left(\lambda_1, r_T^G\pi_0\right) &= mult\left(\lambda_1, r_T^G\pi\right), \end{aligned}$$
where λ_0 is the initial exponent of π and λ_1 is its terminal exponent, namely the initial exponent of the invert of π (see (8) and (10) in Subsection 2.1.3). This would imply that π_0 is both the unique irreducible subrepresentation and quotient of π and hence $\pi = \pi_0$ is irreducible.

  In other words, in order to show that π is irreducible, it is enough to show that
$$f_k\left(\lambda_0\right) = f_\pi\left(\lambda_0\right), \quad f_k\left(\lambda_1\right) = f_\pi\left(\lambda_1\right).$$

Remark. Most branching rules which appear in Appendix A arose from examples where we were able to prove that $f_k\left(\lambda_0\right) = f_\pi\left(\lambda_0\right)$ and $f_k\left(\lambda_1\right) = f_\pi\left(\lambda_1\right)$ but were unable to prove that $f_k = f_\pi$ directly. The latter equality follows from irreducibility.

3.5. *Irreducible Subrepresentations.* Below, we describe a few methods to determine the length of the maximal semi-simple subrepresentation of the degenerate principal series representation $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$.

1. If $mult\left(\lambda_0, r_T^G\pi\right) = 1$, where λ_0 is the initial exponent of π, then π admits a unique irreducible subrepresentation.
2. Let $\lambda_{a.d.}$ denote an anti-dominant exponent of π and let π_0 denote an irreducible subquotient of π such that $\lambda_{a.d.} \le r_T^G\pi_0$. If
$$mult\left(\lambda_0, r_T^G\pi_0\right) = mult\left(\lambda_0, r_T^G\pi\right), \tag{3.4}$$
then π_0 is the unique irreducible subrepresentation of π.

   This can, theoretically, be done with exponents other than $\lambda_{a.d.}$ but in practice, this is enough.

The condition in Equation (3.4) can, in many cases, be verified using branching rule calculations. Also, this argument is relevant only when $s \leq 0$, since otherwise π would be irreducible (and thus would not be the subject of investigation for this part of the algorithm).

3. Given a unitary degenerate principal series $\pi = i_{M_i}^G (\Omega_{M_i,0,\chi})$, then π is semi-simple of length at most 2 by [Ban02, Lemma 5.2]. In particular, if it is reducible, it is of length 2.

We note that, in a number of cases, the length of the semi-simple subrepresentation cannot be determined by these methods. This can happen for cases where the length of the semi-simple subrepresentation can be either 1 or more. These cases are listed in Theorem 4.1 and are dealt with in Section 5 using various other methods.

Remark. We point out that for $s < 0$, π admits a unique irreducible subrepresentation and for $s < 0$, π admits a unique irreducible quotient. This is shown in [BJ08, Theorem 6.3] but also follows simply from the fact that for $s > 0$, the initial exponent always appears in $r_T^G \pi$ with multiplicity 1.

3.6. *Remark on Calculating Multiplicities of Exponents Using the Geometric Lemma.* We now wish to describe a relatively efficient algorithm to calculate $m_{\pi,\lambda} = mult\left(\lambda, r_T^G \pi\right)$, where λ is an exponent of $\pi = i_{M_i}^G (\Omega_{M_i,s,\chi})$ using Equation (2.2).

From Equation (2.2) we have

$$\left[r_T^G i_M^G \sigma\right] = \sum_{w \in W^{M,L}} \left[w \circ r_T^M \sigma\right]$$

and hence

$$m_{\pi,\lambda} = \left|\left\{w \in W^{M_i,T} \;\middle|\; w \cdot \lambda_0 = \lambda\right\}\right|,$$

where $\lambda_0 = r_T^{M_i} (\Omega_{M_i,s,\chi})$. However, instead of going through all $w \in W^{M_i,T}$ and count for which of them the equality $w \cdot \lambda_0 = \lambda$ holds, one could proceed as follows:

- We fix an anti-dominant element $\lambda_{a.d.}$ in the orbit of λ_0 and the shortest element $w_0 \in W^{M_i,T}$ such that $w_0 \cdot \lambda_0 = \lambda_{a.d.}$. This can be done efficiently using the method described in [LCL92, §5.3.2] and [HS20, Lemma 3.4].
- Similarly, we fix the shortest word $w_1 \in W$ such that $w_1 \cdot \lambda = \lambda_{a.d.}$.
- We also point out that it is simple to determine $Stab_W (\lambda_{a.d.})$.
- We note that every $w \in W^{M_i,T}$ such that $w \cdot \lambda_0 = \lambda$ can be written in the form $w = w_1^{-1} u w_0$ where $u \in \mathrm{Stab}_W (\lambda_{a.d.})$.
- Thus,

$$m_{\pi,\lambda} = \left|\left\{u \in \mathrm{Stab}_W (\lambda_{a.d.}) \;\middle|\; w_1^{-1} u w_0 \in W^{M_i,T}\right\}\right|.$$

This is significantly more efficient than calculating $\left|\left\{w \in W^{M_i,T} \;\middle|\; w \cdot \lambda_0 = \lambda\right\}\right|$ especially since, given a reduced expression for $w \in W$, it is easy to determine whether $w \in W^{M_i,T}$ since

$$w \in W^{M_i,T} \quad \Longleftrightarrow \quad l\left(ws_i\right) < l\left(w\right).$$

3.7. *Reducibility and Irreducibility via Shahidi's Local Coefficients.* We finish this section by recalling a method to determine the reducibility or irreducibility of degenerate principal series using Shahidi's local coefficients. Following [MS98, Theorem 2.1, Proposition 3.3, Remark 3.4, §4] and [Sha90, §7], we have:

- $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$ is reducible if and only if $D_G\left(\pi\right) = i_{M_i}^G\left(\Omega_{M_i,s,\chi} \otimes \mathrm{St}_{M_i}\right)$, where St_{M_i} is the **Steinberg** representation of M_i, is reducible.
- For $s > 0$, the generalized principal series $i_{M_i}^G\left(\Omega_{M_i,s,\chi} \otimes \mathrm{St}_{M_i}\right)$ is reducible if and only if

  $$\mathcal{L}\left(1-s, St_{M_i,\chi}, r\right)^{-1} = 0,$$

  where $St_{M_i,\chi}$ is the Steinberg representation of M_i twisted by χ and r is the action of the dual group ${}^L M_i$ on the Lie algebra of the dual unipotent radical ${}^L N_i$.
- The expression $\mathcal{L}\left(1-s, St_{M_i,\chi}, r\right)^{-1}$ is, in fact, a polynomial in q^{-s} that is, up to a unit of $\mathbb{C}\left[q^{-s}, q^s\right]$, equal to the numerator of the simplified form of

$$\prod_{\substack{\gamma \in \Phi_G^+ \\ w\cdot\gamma \notin \Phi_G^+}} \frac{\mathcal{L}\left(\left\langle s\bar{\omega}_{\alpha_i} - \rho_B + \rho_{P_i}, \gamma^\vee\right\rangle - 1, \chi^{\left\langle \bar{\omega}_{\alpha_i}, \gamma^\vee\right\rangle}\right)}{\mathcal{L}\left(\left\langle s\bar{\omega}_{\alpha_i} - \rho_B + \rho_{P_i}, \gamma^\vee\right\rangle, \chi^{\left\langle \bar{\omega}_{\alpha_i}, \gamma^\vee\right\rangle}\right)}, \tag{3.5}$$

  where

  $$\mathcal{L}\left(s,\chi\right) = \frac{1}{1-q^{-s}\chi\left(\varpi\right)}.$$

- Alternatively, one can use Gindikin-Karpelevich factors to determine the reducibility points of $i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$. Namely, let $J_{w_{0,i}}\left(s,\chi\right)$ be the Gindikin-Karpelevich factor associated with the interwining-operator

  $$M_{w_{0,i}}\left(s\right) : i_T^G \lambda_{i,s,\chi} \to i_T^G w_{0,i} \cdot \lambda_{i,s,\chi},$$

  where $\lambda_{i,s,\chi} = r_T^{M_i}\left(\Omega_{i,s,\chi}\right)$. This factor can be calculated by

$$J_w\left(s,\chi\right) = \prod_{\substack{\gamma \in \Phi_G^+ \\ w\cdot\gamma \notin \Phi_G^+}} \frac{\mathcal{L}\left(\left\langle s\bar{\omega}_{\alpha_i} - \rho_B + \rho_{P_i}, \gamma^\vee\right\rangle, \chi^{\left\langle \bar{\omega}_{\alpha_i}, \gamma^\vee\right\rangle}\right)}{\mathcal{L}\left(\left\langle s\bar{\omega}_{\alpha_i} - \rho_B + \rho_{P_i}, \gamma^\vee\right\rangle + 1, \chi^{\left\langle \bar{\omega}_{\alpha_i}, \gamma^\vee\right\rangle}\right)}. \tag{3.6}$$

- Thus, in order to find all reducible values of $s < 0$ such that $\pi = i_{M_i}^G\left(\Omega_{M_i,s,\chi}\right)$ is reducible, one should calculate the zeros of the denominator of the simplified expressions of $J_w\left(s,\chi\right)$ as a rational function in q^{-s}.

The above is proven in [MS98] for unramified χ but is well known for general characters too though we are unable to locate a reference for this statement. However, our results are independent from this result and it was only used by us for double-checking our results.

As an example, we consider the case of $\pi = i_{M_i}^G(\Omega_{M_8,s,\chi})$. The set $\left\{\alpha \in \Phi_G^+ \mid w \cdot \alpha \notin \Phi_G^+\right\}$ is of cardinality 57. Hence, to begin with, $J_w(s,\chi)$ is given by a quotient of 57 $\mathcal{L}$-functions in the numerator and 57 $\mathcal{L}$-functions in the denominator. However, once the simplified form of $J_w(s,\chi)$ is computed, one finds that

$$J_w(s,\chi) = \frac{\mathcal{L}\left(s-\frac{9}{2},\chi\right)\mathcal{L}\left(s-\frac{17}{2},\chi\right)\mathcal{L}\left(s-\frac{27}{2},\chi\right)\mathcal{L}\left(2s,\chi^2\right)}{\mathcal{L}\left(s+\frac{29}{2},\chi\right)\mathcal{L}\left(s+\frac{19}{2},\chi\right)\mathcal{L}\left(s+\frac{11}{2},\chi\right)\mathcal{L}\left(2s+1,\chi^2\right)}$$

and indeed, the reducibility points $s<0$ of $i_{M_i}^G(\Omega_{M_8,s,\chi})$ are shown in Theorem 4.1 to be $s \in \left\{-\frac{29}{2},-\frac{19}{2},-\frac{11}{2},-\frac{1}{2}\right\}$ for $\chi = \mathbf{1}$ and $s = -\frac{1}{2}$ for χ of order 2.

As explained in the introduction, while this method solves the question of reducibility for $s \neq 0$, it does not solve this question for unitary degenerate principal series and it does not address the question of the maximal semi-simple subrepresentation of a reducible $i_{M_i}^G(\Omega_{M_i,s,\chi})$. This is why all the tools described above in this section are truly required and supply us with innovative data. Furthermore, the algorithm above supply useful data for other applications (such as studying the Jordan-Hölder series of π and various Siegel-Weil like identities).

4. The Main Theorem

In this section, we state our main theorem which lists all non-regular and all reducible degenerate principal series of G. We further determine the length of the maximal semi-simple subrepresentation and quotient of almost all cases.

We point out that the theorem is stated for $\pi = i_{M_i}^G(\Omega_{M_i,s,\chi})$ with $s \leq 0$ only. This is enough since:

- By contragredience, $i_{M_i}^G(\Omega_{M_i,s,\chi})$ is regular (or reducible) if and only if $i_{M_i}^G(\Omega_{M_i,-s,\overline{\chi}})$ is.
- The maximal semi-simple subrepresentation of π is the invert of the maximal semi-simple quotient of the invert of π (see (10) in Subsection 2.1.3).
- By Subsection 3.5, the $\pi = i_{M_i}^G(\Omega_{M_i,s,\chi})$ admits a unique irreducible quotient when $s<0$.

For most cases of non-regular degenerate principal series $\pi = i_{M_i}^G(\Omega_{M_i,s,\chi})$, the questions of reducibility and length of the maximal semi-simple subrepresentation can be resolved using the algorithm from Section 3. The rest of this section will be devoted to the output of our implementation of the algorithm in Sagemath. The remaining cases are dealt with in Section 5.

Theorem 4.1. *Let* $\pi = i_{M_i}^G(\Omega_{M_i,s,\chi})$ *with* $s \le 0$ *and let* $k = ord(\chi)$.

1. *The following tables Tables 1-8 below lists all triples* $[i,s,k]$ *such that* π *is either non-regular or reducible. In particular, for each triple* $[i,s,k]$ *the entry in the ith table for this value of* s *and* k *will be*
   - ***irr.*** *for non-regular and irreducible* π.
   - ***red.*** *for non-regular and reducible* π.
   - ***red.**** *for regular and reducible* π.

   For any triple $[i,s,k]$*, not appearing in the tables, the degenerate principal series* $i_{M_i}^G(\Omega_{M_i,s,\chi})$*, with* $ord(\chi) = k$*, is regular and irreducible.*
2. *All* $\pi = i_{M_i}^G(\Omega_{M_i,s,\chi})$ *admit a unique irreducible subrepresentation, with the exception of:*
   (a) $[i,s,k]$ *is one of* $[1,-5/2,1]$, $[3,-1/2,2]$ $[6,0,1]$, $[6,0,2]$ *and* $[7,-3/2,1]$. *In these cases, the representation* π *admits a maximal semi-simple subrepresentation of length* 2.
   (b) $[i,s,k]$ *is one of* $[2,-1/2,1]$ *and* $[5,-1/2,1]$*, in which case the length of the maximal semi-simple subrepresentation of* π *is at most* 2.

1. For $P = P_1$

s \ $ord(\chi)$	1	2
$-\frac{23}{2}$	*red.**	
$-\frac{21}{2}$	*irr.*	
$-\frac{19}{2}$	*irr.*	
$-\frac{17}{2}$	*red.*	
$-\frac{15}{2}$	*irr.*	
$-\frac{13}{2}$	*red.*	
$-\frac{11}{2}$	*red.*	
$-\frac{9}{2}$	*irr.*	
$-\frac{7}{2}$	*red.*	*red.**
-3	*irr.*	*irr.*
$-\frac{5}{2}$	*red.*	*irr.*
-2	*irr.*	*irr.*
$-\frac{3}{2}$	*irr.*	*irr.*
-1	*irr.*	*irr.*
$-\frac{1}{2}$	*red.*	*red.*
0	*irr.*	*irr.*

s \ $ord(\chi)$	1	2

TABLE 1. P_1-Reducibility Points

2. For $P = P_2$

s \ $ord(\chi)$	1	2	3
$-\frac{17}{2}$	$red.^*$		
$-\frac{15}{2}$	$irr.$		
$-\frac{13}{2}$	$red.$		
$-\frac{11}{2}$	$red.$		
$-\frac{9}{2}$	$red.$		
$-\frac{7}{2}$	$red.$	$red.^*$	
-3	$irr.$	$irr.$	
$-\frac{5}{2}$	$red.$	$red.$	
-2	$irr.$	$irr.$	
$-\frac{3}{2}$	$red.$	$red.$	$red.^*$
$-\frac{7}{6}$	$irr.$		$irr.$
-1	$irr.$	$irr.$	
$-\frac{5}{6}$	$irr.$		$irr.$
$-\frac{1}{2}$	$red.$	$red.$	$irr.$
$-\frac{1}{6}$	$irr.$		$irr.$
0	$irr.$	$irr.$	

TABLE 2. P_2-Reducibility Points

3. For $P = P_3$

s \ $ord(\chi)$	1	2	3	4
$-\frac{13}{2}$	$red.^*$			
$-\frac{11}{2}$	$red.$			
$-\frac{9}{2}$	$red.$			
$-\frac{7}{2}$	$red.$	$red.^*$		
-3	$irr.$	$irr.$		

s \ $ord\,(\chi)$	1	2	3	4
$-\frac{5}{2}$	*red.*	*red.*		
-2	*red.*	*red.*		
$-\frac{3}{2}$	*red.*	*red.*	*red.**	
$-\frac{7}{6}$	*red.*		*red.*	
-1	*red.*	*red.*		*red.**
$-\frac{5}{6}$	*irr.*		*irr.*	
$-\frac{3}{4}$	*irr.*	*irr.*		*irr.*
$-\frac{1}{2}$	*red.*	*red.*	*irr.*	*irr.*
$-\frac{1}{4}$	*irr.*	*irr.*		*irr.*
$-\frac{1}{6}$	*irr.*		*irr.*	
0	*irr.*	*irr.*		*irr.*

TABLE 3. P_3-Reducibility Points

4. For $P = P_4$

s \ $ord\,(\chi)$	1	2	3	4	5	6
$-\frac{9}{2}$	*red.**					
$-\frac{7}{2}$	*red.*					
$-\frac{5}{2}$	*red.*	*red.**				
-2	*red.*	*red.*				
$-\frac{3}{2}$	*red.*	*red.*	*red.**			
$-\frac{7}{6}$	*red.*		*red.*			
-1	*red.*	*red.*		*red.**		
$-\frac{5}{6}$	*red.*		*red.*			
$-\frac{3}{4}$	*red.*	*red.*		*red.*		
$-\frac{1}{2}$	*red.*	*red.*	*red.*	*red.*	*red.**	*red.**
$-\frac{1}{3}$	*irr.*	*irr.*	*irr.*			*irr.*
$-\frac{3}{10}$	*red.*				*red.*	
$-\frac{1}{4}$	*irr.*	*irr.*		*irr.*		
$-\frac{1}{6}$	*irr.*	*irr.*	*irr.*			*irr.*
$-\frac{1}{10}$	*irr.*				*irr.*	
0	*irr.*	*irr.*	*irr.*	*irr.*		*irr.*

TABLE 4. P_4-Reducibility Points

5. For $P = P_5$

s \ $ord(\chi)$	1	2	3	4	5
$-\frac{11}{2}$	$red.^*$				
$-\frac{9}{2}$	$red.$				
$-\frac{7}{2}$	$red.$				
$-\frac{5}{2}$	$red.$	$red.^*$			
-2	$red.$	$red.$			
$-\frac{3}{2}$	$red.$	$red.$	$red.^*$		
$-\frac{7}{6}$	$red.$		$red.$		
-1	$red.$	$red.$		$red.^*$	
$-\frac{5}{6}$	$red.$		$red.$		
$-\frac{3}{4}$	$irr.$	$irr.$		$irr.$	
$-\frac{1}{2}$	$red.$	$red.$	$red.$	$red.$	$red.^*$
$-\frac{3}{10}$	$irr.$				$irr.$
$-\frac{1}{4}$	$irr.$	$irr.$		$irr.$	
$-\frac{1}{6}$	$irr.$		$irr.$		
$-\frac{1}{10}$	$irr.$				$irr.$
0	$irr.$	$irr.$		$irr.$	

TABLE 5. P_5-Reducibility Points

6. For $P = P_6$

s \ $ord(\chi)$	1	2	3	4
-7	$red.^*$			
-6	$red.$			
-5	$red.$			
-4	$red.$			
-3	$red.$	$red.^*$		
$-\frac{5}{2}$	$red.$	$red.$		
-2	$red.$	$red.$	$red.^*$	
$-\frac{5}{3}$	$irr.$		$irr.$	
$-\frac{3}{2}$	$irr.$	$irr.$		
$-\frac{4}{3}$	$irr.$		$irr.$	

s \ $ord(\chi)$	1	2	3	4
-1	$red.$	$red.$	$red.$	
$-\frac{2}{3}$	$irr.$		$irr.$	
$-\frac{1}{2}$	$red.$	$red.$		$red.^*$
$-\frac{1}{3}$	$irr.$		$irr.$	
$-\frac{1}{4}$	$irr.$	$irr.$		$irr.$
0	$red.$	$red.$	$irr.$	$irr.$

TABLE 6. P_6-Reducibility Points

7. For $P = P_7$

s \ $ord(\chi)$	1	2	3
$-\frac{19}{2}$	$red.^*$		
$-\frac{17}{2}$	$red.$		
$-\frac{15}{2}$	$irr.$		
$-\frac{13}{2}$	$irr.$		
$-\frac{11}{2}$	$red.$		
$-\frac{9}{2}$	$red.$	$red.^*$	
-4	$irr.$	$irr.$	
$-\frac{7}{2}$	$irr.$	$irr.$	
-3	$irr.$	$irr.$	
$-\frac{5}{2}$	$red.$	$red.$	
-2	$irr.$	$irr.$	
$-\frac{3}{2}$	$red.$	$irr.$	
-1	$irr.$	$irr.$	
$-\frac{1}{2}$	$red.$	$red.$	$red.^*$
$-\frac{1}{6}$	$irr.$		$irr.$
0	$irr.$	$irr.$	

TABLE 7. P_7-Reducibility Points

8. For $P = P_8$

s \ $ord(\chi)$	1	2
$-\frac{29}{2}$	*red.**	
$-\frac{27}{2}$	*irr.*	
$-\frac{25}{2}$	*irr.*	
$-\frac{23}{2}$	*irr.*	
$-\frac{21}{2}$	*irr.*	
$-\frac{19}{2}$	*red.*	
$-\frac{17}{2}$	*irr.*	
$-\frac{15}{2}$	*irr.*	
$-\frac{13}{2}$	*irr.*	
$-\frac{11}{2}$	*red.*	
$-\frac{9}{2}$	*irr.*	
$-\frac{7}{2}$	*irr.*	
$-\frac{5}{2}$	*irr.*	
$-\frac{3}{2}$	*irr.*	
$-\frac{1}{2}$	*red.*	*red.**
0	*irr.*	*irr.*

TABLE 8. P_8-Reducibility Points

REMARK. According to [GS05], the minimal representation of E_8 is the unique irreducible subrepresentation of $[8,-19/2,1]$ (Note that P_8 is the Heisenberg parabolic subgroup of E_8). From the data provided by Tadić's reducibility criterion, it follows that the minimal representation is also a subquotient of the following cases: $[1,-17/2,1]$, $[2,-13/2,1]$ and $[4,-7/2,1]$. Indeed, it is isomorphic to their unique irreducible subrepresentations.

PROOF. We separate the proof into four parts: proof of reducibility (for all reducible π), proof of irreducibility (for most irreducible cases), proof of unique irreducible subrepresentation (for most cases) and exceptional cases. The last part is dealt with in Section 5 while the remainder of this section is devoted to the first three parts (which use the algorithm from Section 3).

4.1. *Reducibility.* For most reducible cases, it is enough to find $\pi' \neq \pi$ which shares an irreducible subquotient with π. As explained in Subsection 3.2, for most cases, one can find $\pi' = i_{M_j}^G\left(\Omega_{M_j,t,\chi^l}\right)$ which satisfy the required conditions. In the following tables we list, for each reducible non-regular $[i,s,ord(\chi)]$ a triple $\left[j,t,ord\left(\chi^l\right)\right] = [j,t,ord(\chi)]$ which provides a representation $\pi' \neq \pi$ sharing a common irreducible subquotient with π. Furthermore, we list an exponent λ of π such $m_{\pi,\lambda} > m_{\pi',\lambda}$, where

$$m_{\pi,\lambda} = mult\left(\lambda, r_T^G \pi\right), \quad m_{\pi',\lambda} = mult\left(\lambda, r_T^G \pi'\right).$$

In the table, the exponent λ is seperated into real and imaginary part, and should be read as follows:

- Fix a character χ of the prescribed order.
- Let $\mathrm{Re}(\lambda)$ be given by $[k_1, ..., k_8]$ and let $\mathrm{Im}(\lambda)$ be given by $[m_1, ..., m_8]$.
- Then

$$\lambda = \sum_{i=1}^{8} \left(k_i \bar{\omega}_{\alpha_i} + \chi^{m_i} \circ \bar{\omega}_{\alpha_i}\right)$$

i	s	$o(\chi)$	j	t	$\mathrm{Re}(\lambda)$	$\mathrm{Im}(\lambda)$	$m_{\pi,\lambda}$	$m_{\pi',\lambda}$
1	$-\frac{17}{2}$	1	2	$-\frac{13}{2}$	$[-1,-19,11,7,-8,7,-8,-1]$	$[0,0,0,0,0,0,0,0]$	1	0
1	$-\frac{13}{2}$	1	7	$-\frac{11}{2}$	$[-1,13,-1,-1,-5,-8,7,-1]$	$[0,0,0,0,0,0,0,0]$	1	0
1	$-\frac{11}{2}$	1	8	$-\frac{5}{2}$	$[-4,-5,-1,-1,-1,6,-1,6]$	$[0,0,0,0,0,0,0,0]$	1	0
1	$-\frac{5}{2}$	1	3	$-\frac{5}{2}$	$[2,1,-1,-2,-1,-1,3,-1]$	$[0,0,0,0,0,0,0,0]$	2	1
1	$-\frac{1}{2}$	1	7	$-\frac{3}{2}$	$[-5,-1,-1,5,-1,-5,4,-5]$	$[0,0,0,0,0,0,0,0]$	1	0
2	$-\frac{13}{2}$	1	1	$-\frac{17}{2}$	$[6,-1,-15,14,-1,-5,-13,12]$	$[0,0,0,0,0,0,0,0]$	1	0
2	$-\frac{11}{2}$	1	8	$-\frac{13}{2}$	$[-1,11,11,-7,-5,-4,-1,-1]$	$[0,0,0,0,0,0,0,0]$	1	0
2	$-\frac{9}{2}$	1	8	$-\frac{3}{2}$	$[-1,4,12,-5,-1,-1,-10,4]$	$[0,0,0,0,0,0,0,0]$	1	0
2	$-\frac{7}{2}$	1	1	$-\frac{7}{2}$	$[-8,-1,7,-1,-2,-1,-4,12]$	$[0,0,0,0,0,0,0,0]$	1	0
2	$-\frac{7}{2}$	2	1	$-\frac{7}{2}$	$[-1,4,-4,3,-4,-4,3,4]$	$[0,1,1,1,1,0,0,1]$	1	0
2	$-\frac{5}{2}$	1	1	$-\frac{1}{2}$	$[2,-5,-1,-2,5,-1,-3,5]$	$[0,0,0,0,0,0,0,0]$	1	0
2	$-\frac{1}{2}$	1	6	-1	$[4,-3,-1,2,-3,-1,3,-1]$	$[0,0,0,0,0,0,0,0]$	2	1
3	$-\frac{11}{2}$	1	1	$-\frac{19}{2}$	$[-6,6,12,-1,-23,11,6,5]$	$[0,0,0,0,0,0,0,0]$	1	0
3	$-\frac{9}{2}$	1	8	$-\frac{15}{2}$	$[-9,-1,15,-7,-1,8,-16,7]$	$[0,0,0,0,0,0,0,0]$	2	0
3	$-\frac{7}{2}$	1	8	$-\frac{1}{2}$	$[-1,-11,9,1,6,-8,-1,-4]$	$[0,0,0,0,0,0,0,0]$	1	0
3	$-\frac{5}{2}$	1	1	$-\frac{5}{2}$	$[-4,1,-5,3,4,-3,-5,3]$	$[0,0,0,0,0,0,0,0]$	1	0
3	-2	1	7	-1	$\left[-3,4,-1,-\frac{3}{2},-\frac{7}{2},8,-4,-\frac{3}{2}\right]$	$[0,0,0,0,0,0,0,0]$	1	0
3	-2	2	7	-1	$\left[5,-1,-6,5,-9,\frac{15}{2},\frac{1}{2},-\frac{3}{2}\right]$	$[0,0,0,0,0,1,1,1]$	1	0
3	$-\frac{3}{2}$	1	2	$-\frac{3}{2}$	$[-1,-1,-4,-1,4,2,-3,2]$	$[0,0,0,0,0,0,0,0]$	3	0
3	$-\frac{7}{6}$	1	2	$-\frac{5}{6}$	$\left[-1,-1,\frac{5}{3},-2,\frac{2}{3},-\frac{4}{3},\frac{5}{3},\frac{2}{3}\right]$	$[0,0,0,0,0,0,0,0]$	1	0
3	$-\frac{1}{2}$	1	6	0	$[2,3,1,-2,-2,3,-4,0]$	$[0,0,0,0,0,0,0,0]$	2	0

i	s	$o(\chi)$	j	t	$\mathrm{Re}(\lambda)$	$\mathrm{Im}(\lambda)$	$m_{\pi,\lambda}$	$m_{\pi',\lambda}$
4	$-\frac{7}{2}$	1	1	$-\frac{17}{2}$	$[-10,-3,-11,7,13,-14,7,-3]$	$[0,0,0,0,0,0,0,0]$	1	0
4	$-\frac{5}{2}$	1	8	$-\frac{1}{2}$	$[2,-8,-8,7,-1,7,-8,-1]$	$[0,0,0,0,0,0,0,0]$	1	0
4	-2	1	7	-3	$\left[2,\frac{5}{2},7,-\frac{7}{2},-1,\frac{7}{2},-9,-1\right]$	$[0,0,0,0,0,0,0,0]$	1	0
4	-2	2	7	-3	$\left[\frac{13}{2},4,-4,3,-4,-4,-2,\frac{7}{2}\right]$	$[1,0,0,0,0,0,0,1]$	1	0
4	$-\frac{3}{2}$	1	7	$-\frac{1}{2}$	$[-1,-2,2,-5,6,1,-5,2]$	$[0,0,0,0,0,0,0,0]$	1	0
4	$-\frac{3}{2}$	2	7	$-\frac{1}{2}$	$[-6,-7,5,4,-5,2,5,-3]$	$[1,0,1,0,0,1,0,1]$	1	0
4	$-\frac{7}{6}$	1	6	$-\frac{4}{3}$	$\left[-1,\frac{17}{3},\frac{17}{3},-\frac{19}{3},-1,-\frac{5}{3},\frac{7}{3},\frac{10}{3}\right]$	$[0,0,0,0,0,0,0,0]$	1	0
4	-1	1	3	-1	$\left[-3,-1,-\frac{3}{2},4,-2,-\frac{3}{2},2,-3\right]$	$[0,0,0,0,0,0,0,0]$	1	0
4	-1	2	3	-1	$\left[\frac{1}{2},\frac{7}{2},-2,\frac{1}{2},-\frac{9}{2},1,4,-2\right]$	$[1,1,0,1,1,0,0,0]$	1	0
4	$-\frac{5}{6}$	1	6	$-\frac{1}{3}$	$\left[-\frac{2}{3},\frac{2}{3},-4,\frac{10}{3},-\frac{11}{3},\frac{8}{3},\frac{2}{3},-\frac{5}{3}\right]$	$[0,0,0,0,0,0,0,0]$	2	0
4	$-\frac{5}{6}$	3	6	$-\frac{1}{3}$	$\left[\frac{1}{3},5,1,-\frac{11}{3},-1,\frac{8}{3},-1,-3\right]$	$[2,0,0,2,0,1,0,0]$	1	0
4	$-\frac{3}{4}$	1	3	$-\frac{1}{4}$	$\left[\frac{11}{4},-\frac{5}{4},\frac{7}{4},-\frac{11}{4},\frac{5}{2},-1,-\frac{1}{4},-\frac{11}{4}\right]$	$[0,0,0,0,0,0,0,0]$	1	0
4	$-\frac{3}{4}$	2	3	$-\frac{1}{4}$	$\left[\frac{7}{2},-\frac{11}{4},-\frac{11}{4},\frac{7}{4},\frac{5}{2},-1,-\frac{17}{4},\frac{7}{4}\right]$	$[0,1,1,1,0,0,1,1]$	1	0
4	$-\frac{1}{2}$	1	5	$-\frac{1}{2}$	$[3,-1,-1,0,1,-2,-1,2]$	$[0,0,0,0,0,0,0,0]$	62	34
4	$-\frac{3}{10}$	1	5	$-\frac{1}{10}$	$\left[-\frac{4}{5},\frac{6}{5},\frac{7}{5},-\frac{8}{5},-\frac{2}{5},-\frac{1}{5},-\frac{3}{5},2\right]$	$[0,0,0,0,0,0,0,0]$	1	0
5	$-\frac{9}{2}$	1	8	$-\frac{21}{2}$	$[14,-14,-18,17,-8,13,-14,5]$	$[0,0,0,0,0,0,0,0]$	1	0
5	$-\frac{7}{2}$	1	1	$-\frac{13}{2}$	$[-1,8,3,-1,-12,9,-10,9]$	$[0,0,0,0,0,0,0,0]$	1	0
5	$-\frac{5}{2}$	1	1	$-\frac{7}{2}$	$[-2,2,4,-5,-1,-1,14,-10]$	$[0,0,0,0,0,0,0,0]$	1	0
5	-2	1	1	-1	$\left[-1,4,-\frac{1}{2},-1,\frac{13}{2},-\frac{21}{2},3,-4\right]$	$[0,0,0,0,0,0,0,0]$	1	0
5	-2	2	1	-1	$\left[-1,-\frac{9}{2},3,\frac{1}{2},-1,-1,-\frac{7}{2},\frac{17}{2}\right]$	$[0,1,0,1,0,0,1,1]$	1	0
5	$-\frac{3}{2}$	1	2	$-\frac{3}{2}$	$[-1,2,4,-1,-1,-5,1,1]$	$[0,0,0,0,0,0,0,0]$	1	0
5	-1	1	6	$-\frac{1}{2}$	$\left[\frac{11}{2},\frac{1}{2},-2,-\frac{3}{2},-\frac{1}{2},\frac{5}{2},-\frac{3}{2},-2\right]$	$[0,0,0,0,0,0,0,0]$	1	0
5	-1	2	6	$-\frac{1}{2}$	$\left[-1,\frac{5}{2},-\frac{3}{2},\frac{1}{2},-1,3,-\frac{9}{2},-1\right]$	$[0,1,1,1,0,0,1,0]$	1	0
5	$-\frac{5}{6}$	1	3	$-\frac{1}{6}$	$\left[4,-\frac{8}{3},-\frac{2}{3},-\frac{1}{3},-1,-\frac{1}{3},4,-1\right]$	$[0,0,0,0,0,0,0,0]$	1	0
5	$-\frac{7}{6}$	1	2	$-\frac{1}{6}$	$\left[2,\frac{4}{3},-\frac{14}{3},\frac{1}{3},\frac{4}{3},-1,\frac{5}{3},-3\right]$	$[0,0,0,0,0,0,0,0]$	1	0
5	$-\frac{7}{6}$	3	2	$-\frac{1}{6}$	$\left[\frac{11}{3},\frac{8}{3},-\frac{4}{3},\frac{1}{3},-\frac{11}{3},-1,\frac{10}{3},-\frac{8}{3}\right]$	$[2,2,2,1,1,0,1,1]$	1	0
6	-6	1	8	$-\frac{23}{2}$	$[-7,-3,-1,17,-18,11,-17,5]$	$[0,0,0,0,0,0,0,0]$	1	0
6	-5	1	7	$-\frac{13}{2}$	$[-20,5,5,5,-1,-5,-1,-4]$	$[0,0,0,0,0,0,0,0]$	1	0
6	-4	1	8	$-\frac{7}{2}$	$[7,-6,-6,-2,13,-14,8,5]$	$[0,0,0,0,0,0,0,0]$	1	0
6	-3	1	1	$-\frac{7}{2}$	$[-2,2,4,-5,-1,-1,14,-10]$	$[0,0,0,0,0,0,0,0]$	1	0
6	$-\frac{5}{2}$	1	1	-2	$\left[-1,4,4,-\frac{7}{2},-2,-\frac{7}{2},\frac{5}{2},\frac{3}{2}\right]$	$[0,0,0,0,0,0,0,0]$	1	0
6	$-\frac{5}{2}$	2	1	-2	$\left[\frac{17}{2},4,\frac{5}{2},-\frac{15}{2},\frac{5}{2},-\frac{7}{2},\frac{5}{2},-\frac{7}{2}\right]$	$[1,0,1,1,1,1,1,1]$	1	0
6	-2	1	7	$-\frac{1}{2}$	$[-1,-1,-1,-1,5,-7,1,8]$	$[0,0,0,0,0,0,0,0]$	1	0
6	-1	1	2	$-\frac{1}{2}$	$[-1,7,3,-8,3,1,-2,1]$	$[0,0,0,0,0,0,0,0]$	1	0
6	0	1	3	$-\frac{1}{2}$	$[-1,-1,-3,2,-1,4,-1,-4]$	$[0,0,0,0,0,0,0,0]$	2	1
7	$-\frac{17}{2}$	1	8	$-\frac{25}{2}$	$[15,-1,-1,6,-26,19,-1,-1]$	$[0,0,0,0,0,0,0,0]$	1	0
7	$-\frac{11}{2}$	1	1	$-\frac{13}{2}$	$[-1,-17,-1,13,-9,8,-9,8]$	$[0,0,0,0,0,0,0,0]$	1	0
7	$-\frac{9}{2}$	1	8	$-\frac{1}{2}$	$[2,-8,-8,7,-1,7,-8,-1]$	$[0,0,0,0,0,0,0,0]$	1	0
7	$-\frac{5}{2}$	1	1	$-\frac{5}{2}$	$[-1,-3,-1,2,-3,-1,13,-10]$	$[0,0,0,0,0,0,0,0]$	1	0
7	$-\frac{3}{2}$	1	1	$-\frac{1}{2}$	$[-1,-1,-1,6,-7,3,-4,-1]$	$[0,0,0,0,0,0,0,0]$	1	0
8	$-\frac{19}{2}$	1	1	$-\frac{17}{2}$	$[8,-9,-9,8,-1,-1,-7,6]$	$[0,0,0,0,0,0,0,0]$	1	0
8	$-\frac{11}{2}$	1	1	$-\frac{13}{2}$	$[-1,-1,-12,11,-1,-1,-10,9]$	$[0,0,0,0,0,0,0,0]$	1	0

TABLE 9. Data for the proof of the reducibility of $i_{M_i}^{G}\left(\Omega_{M_i,s,\chi}\right)$ using maximal Levi subgroups.

In the remaining non-regular reducible cases, one needs to find π' of the form $\pi' = i^G_{M_{j_1,j_2}} \Omega_{s_1,s_2,\chi,k_1,k_2}$, where $\Omega_{s_1,s_2,\chi,k_1,k_2}$ is given by Equation (3.1). In the following table we list the data $\left[\overline{j}, \overline{s}, \overline{k}\right] = \left[\left[j_1, j_2\right], \left[s_1, s_2\right], \left[k_1, k_2\right]\right]$ for these cases as well as an exponent λ of π such that $m_{\pi,\lambda} > m_{\pi',\lambda}$.

i	s	$o(\chi)$	$\overline{j}$	$\overline{s}$	$\overline{k}$	$\mathrm{Re}(\lambda)$	$\mathrm{Im}(\lambda)$	$m_{\pi,\lambda}$	$m_{\pi',\lambda}$
1	$-\frac{7}{2}$	1	$[1,8]$	$\left[-\frac{7}{2},-2\right]$	$[0,0]$	$[4,-1,-1,-1,-1,-1,-1,3]$	$[0,0,0,0,0,0,0,0]$	1	0
2	$-\frac{3}{2}$	1	$[1,8]$	$\left[\frac{1}{2},-2\right]$	$[0,0]$	$[3,3,4,-4,-1,-1,-2,1]$	$[0,0,0,0,0,0,0,0]$	2	0
2	$-\frac{3}{2}$	2	$[2,8]$	$[-2,0]$	$[1,1]$	$[-1,4,-4,-1,-1,1,4,-1]$	$[0,1,0,0,0,1,1,0]$	1	0
3	$-\frac{3}{2}$	2	$[6,7]$	$\left[1,-\frac{7}{2}\right]$	$[0,1]$	$[-1,-5,-1,3,-3,4,-1,-1]$	$[0,0,0,1,1,0,0,1]$	2	0
3	$-\frac{7}{6}$	3	$[4,5]$	$\left[\frac{3}{2},-\frac{19}{6}\right]$	$[0,1]$	$\left[-1,\frac{5}{3},-1,-\frac{1}{3},\frac{1}{3},-\frac{4}{3},\frac{1}{3},-\frac{4}{3}\right]$	$[0,2,0,2,1,2,1,2]$	2	0
3	-1	1	$[6,8]$	$[-1,0]$	$[0,0]$	$\left[\frac{1}{2},-3,-1,\frac{1}{2},\frac{3}{2},\frac{1}{2},-\frac{3}{2},-2\right]$	$[0,0,0,0,0,0,0,0]$	2	0
3	-1	2	$[6,7]$	$[-3,4]$	$[0,1]$	$\left[\frac{1}{2},-3,-1,\frac{1}{2},\frac{3}{2},\frac{1}{2},-\frac{3}{2},-2\right]$	$[1,0,0,1,1,1,1,0]$	2	0
4	$-\frac{7}{6}$	3	$[1,6]$	$\left[6,-\frac{10}{3}\right]$	$[0,2]$	$\left[-1,\frac{4}{3},-\frac{4}{3},\frac{1}{3},0,-1,-\frac{5}{3},\frac{4}{3}\right]$	$[0,1,2,1,0,0,1,1]$	2	1
4	$-\frac{3}{4}$	4	$[5,6]$	$\left[-\frac{11}{4},\frac{17}{4}\right]$	$[3,3]$	$\left[-\frac{9}{4},-\frac{13}{4},\frac{1}{2},\frac{7}{4},\frac{1}{2},-\frac{3}{2},\frac{1}{2},-1\right]$	$[1,1,2,1,2,2,2,0]$	2	1
4	$-\frac{1}{2}$	2	$[3,8]$	$\left[-\frac{1}{2},\frac{3}{2}\right]$	$[1,0]$	$[-1,-1,4,-1,1,-3,1,-2]$	$[0,1,1,0,1,1,0,0]$	2	0
4	$-\frac{1}{2}$	3	$[3,8]$	$\left[-\frac{1}{2},\frac{3}{2}\right]$	$[1,1]$	$[0,-1,-1,0,0,1,-2,5]$	$[1,2,2,1,2,2,1,0]$	2	0
4	$-\frac{1}{2}$	4	$[2,6]$	$[-1,0]$	$[3,3]$	$[-1,-2,2,1,-2,2,-4,3]$	$[0,3,1,1,3,1,2,2]$	2	0
4	$-\frac{3}{10}$	5	$[2,5]$	$\left[-\frac{11}{10},-\frac{1}{5}\right]$	$[2,1]$	$\left[-\frac{2}{5},\frac{6}{5},-1,\frac{2}{5},-\frac{6}{5},-\frac{1}{5},\frac{3}{5},0\right]$	$[3,1,0,2,4,4,3,0]$	2	0
5	$-\frac{5}{6}$	3	$[2,3]$	$\left[-\frac{7}{6},0\right]$	$[2,0]$	$[0,-1,-2,1,2,-2,-1,1]$	$[2,0,2,1,1,2,0,1]$	2	0
5	$-\frac{1}{2}$	1	$[3,8]$	$\left[-\frac{1}{2},\frac{3}{2}\right]$	$[0,0]$	$[-2,3,-1,2,-3,0,2,-3]$	$[0,0,0,0,0,0,0,0]$	2	0
5	$-\frac{1}{2}$	2	$[3,7]$	$\left[-\frac{3}{2},\frac{3}{2}\right]$	$[1,1]$	$[-1,1,1,1,-4,1,1,-1]$	$[1,1,1,1,1,1,1,0]$	2	0
5	$-\frac{1}{2}$	3	$[3,8]$	$\left[-\frac{1}{2},\frac{3}{2}\right]$	$[1,0]$	$\left[-\frac{5}{3},-1,-\frac{4}{3},\frac{8}{3},-\frac{7}{3},\frac{4}{3},-1,-\frac{1}{3}\right]$	$[2,0,1,1,1,2,0,1]$	2	0
5	$-\frac{1}{2}$	4	$[2,5]$	$\left[\frac{1}{2},-1\right]$	$[3,3]$	$[-1,1,2,-2,0,1,-2,0]$	$[0,2,1,2,1,1,2,1]$	2	0
6	-2	2	$[2,3]$	$\left[-\frac{3}{2},-1\right]$	$[1,0]$	$[-1,-6,2,-3,6,-4,6,-3]$	$[0,0,1,0,1,1,0,1]$	1	0
6	-1	3	$[1,7]$	$\left[-2,\frac{5}{2}\right]$	$[1,1]$	$[-2,2,-2,1,0,-2,1,-1]$	$[2,1,2,1,0,2,1,0]$	2	0
6	$-\frac{1}{2}$	1	$[2,8]$	$\left[-\frac{1}{2},-1\right]$	$[0,0]$	$\left[-\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-1,-1,\frac{3}{2},5\right]$	$[0,0,0,0,0,0,0,0]$	2	0
6	$-\frac{1}{2}$	2	$[1,2]$	$\left[1,-\frac{3}{2}\right]$	$[0,1]$	$\left[\frac{7}{2},3,-\frac{9}{2},\frac{3}{2},-1,-\frac{3}{2},-1,\frac{3}{2}\right]$	$[1,0,1,1,0,1,0,1]$	2	0
6	0	2	$[3,7]$	$\left[-\frac{3}{2},\frac{1}{2}\right]$	$[1,1]$	$[-1,-1,-3,2,3,-4,1,-1]$	$[0,0,0,1,0,0,1,0]$	2	1
7	$-\frac{1}{2}$	1	$[6,8]$	$\left[-\frac{7}{2},\frac{15}{2}\right]$	$[0,0]$	$[-4,-9,3,5,-1,-1,-1,-1]$	$[0,0,0,0,0,0,0,0]$	1	0
7	$-\frac{1}{2}$	2	$[3,8]$	$\left[-\frac{5}{2},\frac{3}{2}\right]$	$[1,0]$	$[-1,3,-2,1,-3,-2,4,-1]$	$[0,0,1,1,0,1,1,0]$	1	0
8	$-\frac{1}{2}$	1	$[1,8]$	$\left[-\frac{11}{2},-1\right]$	$[0,0]$	$[2,-1,-1,-1,-1,-1,-1,4]$	$[0,0,0,0,0,0,0,0]$	1	0

TABLE 10. Data for the proof of the reducibility of $i^G_{M_i}\left(\Omega_{M_i,s,\chi}\right)$ using non-maximal Levi subgroups.

4.2. *Irreducibility.* For most irreducible non-regular $\pi = i^G_{M_i}\left(\Omega_{M_i,s,\chi}\right)$, it is possible to prove irreducibility using the branching rules calculation, as explained in Subsection 3.4. The only exceptions were those of the cases $[4,0,1]$ and $[4,0,2]$, which will be dealt with in Section 5.

4.3. *Unique Irreducible Subrepresentation.* Here too, for most reducible non-regular $\pi = i^G_{M_i}\left(\Omega_{M_i,s,\chi}\right)$, one can use the algorithm in Subsection 3.5 to determine the length of the maximal semi-simple subrepresentation. There are, however, two kinds of exceptions:

- π with unique irreducible subrepresentation which could not have been determined using the algorithm: $[2,-5/2,1]$, $[3,-3/2,1]$, $[3,-3/2,2]$, $[4,-1/2,1]$, $[4,-1/2,2]$, $[4,-3/2,1]$, $[5,-3/2,1]$, $[6,-2,1]$ and $[6,-2,2]$.
- Non-unitary π with a maximal semi-simple subrepresentation of length 2: $[1,-5/2,1]$, $[3,-1/2,2]$ and $[7,-3/2,1]$.

Also, there are the cases $[2,-1/2,1]$ and $[5,-1/2,1]$ where we are able to show that the maximal semi-simple subrepresentation is of length at most 2. In these cases we further show that the irreducible spherical subquotient is a subrepresentation and describe the other candidate subrepresentation in terms of its Langlands data. □

4.4. *Some Remarks Regarding the Algorithm Runtime.* We finish this section by shortly commenting on the runtime of our algorithm to arrive at the data in this section and in the following one. As explained in Section 3, the algorithm is made of various parts, we list them:

1. Generating the object E_8, the list of reducible regular points and non-regular points was a relatively simple matter. We were able to perform this within a few days on a laptop with 16GB RAM Memory and 8 processors.
2. Organizing the non-regular degenerate principal series into the equivalence classes $\Xi = \cup_{\lambda_{a.d.}} \Xi_{\lambda_{a.d.}}$ (see Subsection 3.2) and determining reducibility by comparison of different degenerate principal series within each class, is also a simple matter. This step was calculated in a few hours on the same computer.

   It should be reminded here that there were still cases which are reducible but there was no data showing their reducibility arising from another degenerate principal series. For each of these cases, we were able to find such data arising from an induction from a Levi subgroup of co-rank 2. Calculating each of these cases could have taken hours, days or weeks as the search for such data requires two indented loops going through the exponents of π. For these calculations we used a computer with 16 processors and 64 GB of RAM memory.
3. Applying branching rule calculation in order to determine irreducibility or the existence of unique irreducible subrepresentations was the most tedious process as each case could have taken days or weeks to calculate on the 16 processors and 64 GB of RAM memory computer.

   After improving the algorithm by using the partition of $\left[r_T^G\pi\right]$ into A_1-equivalency class (see Subsection 3.4) instead of using the A_1 branching rule (as we have done in previous cases), runtime of these cases was decreased significantly. Also, when we obtained a computer with 256 GB of RAM memory and 32 processors, we were able to further parallelize the cases.

4. Some of the arguments in Subsection 5.1 rely on data from various branching rule calculations, these were the calculations already performed in the previous step.

5. Exceptional Cases

In this section, we finish the proof of Theorem 4.1 by resolving the cases not solved by the algorithm of Section 3.

5.1. *Fully Resolved Cases.* In this subsection, we deal with all cases listed in Subsection 4.2 and Subsection 4.3 except for $[2,-1/2,1]$ and $[5,-1/2,1]$. We organize the discussion according to similarities between the arguments used for the various cases, such as:

- Using a chain of isomorphisms to embed π in a parabolic induction that admits a unique irreducible subrepresentation in common with π.
- Using a chain of isomorphisms to embed π in a parabolic induction with two irreducible subrepresentations and then calculating its intersection with π.
- Using R-groups of unitary principal series representations of Levi subgroups of G.

When $\chi = \mathbf{1}_{F^\times}$, the representation $\pi = i_{M_i}^G \Omega_{i,s,\chi}$ admits a unique anti-dominant exponent and the *Orthogonality Rule* in [HS21, Appendix A.1] implies that there exists a unique irreducible subquotient π_0 of π such that $\lambda_{a.d.} \leq r_T^G \pi_0$. However, when $ord(\chi) > 1$, this is not necessarily true as can be seen from [HS21, Propositions 4.7 and 4.8] or Proposition 5.6 below, for example. In the following lemma, we verify the existence of a unique such π_0 for certain $\pi = i_{M_i}^G \Omega_{i,s,\chi}$ with $ord(\chi) = 2$.

Lemma 5.1. *Let χ be a character of $F^\times$ of order 2. In the following cases, $[i,s,ord(\chi)]$, there exists a unique irreducible subquotient π_0 of $\pi = i_{M_i}^G \Omega_{i,s,\chi}$ such that $r_T^G \pi_0$ contains an anti-dominant exponent $\lambda_{a.d.}$ of π:*

$$[3,-1/2,2], [3,-3/2,2], [4,-1/2,2], [5,-1/2,2], [6,-2,2].$$

Proof. We argue by a similar argument to that of [HS21, Proposition 4.7(3)]. We do this by first studying the restricition $\left(i_T^L \lambda_{a.d.}\right)\Big|_{L^{der}}$ of $i_T^L \lambda_{a.d.}$ to L^{der}, where L^{der} is the derived group of the Levi subgroup L associated with

$$\Theta(\lambda_{a.d.}) = \{\alpha \in \Delta \mid \mathrm{Re}(\langle \lambda_{a.d.}, \alpha^\vee \rangle) = 0\}.$$

Since G is simply-connected and simply-laced and L is of type $A_{n_1} \times ... \times A_{n_l}$, it follows that in these cases

$$L^{der} = \prod_{k=1}^{l} SL_{n_k}(F), \quad n_k \in \mathbb{N}.$$

$[i, s, ord(\chi)]$	$\lambda_{a.d.}$	$\Theta(\lambda_{a.d.})$	L^{der}	$len\left(\left(i_T^L\lambda_{a.d.}\right)\Big\|_{L^{der}}\right)$
$[3,-1/2,2]$	$\begin{pmatrix} & & \chi & & & & \\ \chi & \chi & \chi & -1 & \chi & \chi & -1 \end{pmatrix}$	$\{1,2,3,4,6,7\}$	$SL_5 \times SL_3$	1
$[3,-3/2,2]$	$\begin{pmatrix} & & \chi & & & & \\ \chi & \chi & -1 & \chi & \chi & -1 & 0 \end{pmatrix}$	$\{1,2,3,5,6,8\}$	$SL_3 \times SL_3 \times SL_2 \times SL_2$	2
$[4,-1/2,2]$	$\begin{pmatrix} & & \chi & & & & \\ \chi & \chi & \chi & -1 & \chi & \chi & 0 \end{pmatrix}$	$\{1,2,3,4,6,7,8\}$	$SL_5 \times SL_4$	1
$[5,-1/2,2]$	$\begin{pmatrix} & & \chi & & & & \\ \chi & \chi & \chi & -1 & \chi & \chi & \chi \end{pmatrix}$	$\{1,2,3,4,6,7,8\}$	$SL_5 \times SL_4$	1
$[6,-2,2]$	$\begin{pmatrix} & & \chi & & & & \\ \chi & \chi & -1 & 0 & \chi & -1 & -1 \end{pmatrix}$	$\{1,2,3,5,6\}$	$SL_3 \times SL_3 \times SL_2$	2

TABLE 11. Data for the proof of Lemma 5.1.

Hence, the length of $\left(i_T^L\lambda_{a.d.}\right)\Big|_{L^{der}}$ can be determined from the representation theory of $SL_n(F)$ (see [GK81, Tad92] for details). In the following table, we list one anti-dominant exponent $\lambda_{a.d.}$, the set $\Theta(\lambda_{a.d.})$ and $len\left(\left(i_T^L\lambda_{a.d.}\right)\Big|_{L^{der}}\right)$ for each of these cases:

In the cases $[3,-1/2,2]$, $[4,-1/2,2]$ and $[5,-1/2,2]$ the representation $\left(i_T^L\lambda_{a.d.}\right)\Big|_{L^{der}}$ is irreducible and hence, so is $i_T^L\lambda_{a.d.}$.

In the other two cases, $[3,-3/2,2]$ and $[6,-2,2]$, the representation $\left(i_T^L\lambda_{a.d.}\right)\Big|_{L^{der}}$ has length 2. Since L is generated by L^{der} and T, the reducibility of $i_T^L\lambda_{a.d.}$ is determined by the action of T on the irreducible pieces of $\left(i_T^L\lambda_{a.d.}\right)\Big|_{L^{der}}$.

More precisely, we have

$$L = \begin{cases} \left\langle L^{der}, \alpha_4^\vee(x_1), \alpha_7^\vee(x_2) \mid x_1, x_2 \in F\right\rangle, & [i,s,ord(\chi)] = [3,-3/2,2] \\ \left\langle L^{der}, \alpha_4^\vee(x_1), \alpha_7^\vee(x_2)\, \alpha_8^\vee(x_3) \mid x_1, x_2, x_3 \in F\right\rangle, & [i,s,ord(\chi)] = [6,-2,2] \end{cases}$$

The representation $\left(i_T^L\lambda_{a.d.}\right)\Big|_{L^{der}}$ of L^{der} can be written as

$$\begin{cases} \sigma_\chi^{(2)} \otimes \sigma_{\chi,\chi}^{(3)} \otimes \sigma_{\chi,\chi}^{(3)} \otimes \sigma_0^{(2)}, & [i,s,ord(\chi)] = [3,-3/2,2] \\ \sigma_\chi^{(2)} \otimes \sigma_{\chi,\chi}^{(3)} \otimes \sigma_{0,\chi}^{(3)}, & [i,s,ord(\chi)] = [6,-2,2], \end{cases}$$

where $\sigma_{\chi,\chi}^{(3)}$ and $\sigma_{0,\chi}^{(3)}$ are irreducible unitary principal series representations of $SL_3(F)$, $\sigma_0^{(2)}$ is an irreducible unitary principal series representation of $SL_2(F)$ and $\sigma_\chi^{(2)} = \sigma_1 \oplus \sigma_{-1}$ is a reducible unitary representation of $SL_2(F)$ of length 2. However, from the representation theory of $SL_2(F)$ and $GL_2(F)$, it follows that $\alpha_4^\vee(\varpi)\cdot\sigma_\epsilon = \sigma_{-\epsilon}$, where $\epsilon = \pm 1$. Thus $i_T^L\lambda_{a.d.}$ is an irreducible representation of L in both cases. See [HS21, Proposition 4.7(3)] for more details.

We note that

$$\mathrm{Re}\left(\langle \lambda, \alpha^\vee\rangle\right) < 0 \quad \forall \alpha \notin \Delta_\lambda.$$

Hence, by Langlands' unique irreducible subrepresentation theorem (see [Jan98, Section 1]),

$$i_T^G \lambda_{a.d.} \cong i_L^G \left(i_T^L \lambda_{a.d.} \right)$$

admits a unique irreducible subrepresentation π_0 which appears in $i_T^G \lambda_{a.d.}$ with multiplicity 1. We now argue that

$$mult \left(\lambda_{a.d.}, r_T^G \pi_0 \right) = mult \left(\lambda_{a.d.}, r_T^G \left(i_T^G \lambda_{a.d.} \right) \right).$$

Assume to the contrary that there exists a subquotient $\tau \neq \pi_0$ of $i_T^G \lambda_{a.d.}$ such that $\lambda_{a.d.} \leq r_T^G (\tau)$. By a central character argument, see Equation (2.3), it holds that $\tau \hookrightarrow i_T^G \lambda_{a.d.}$. Since π_0 is the unique irreducible subrepresentation of $i_T^G \lambda_{a.d.}$, it follows that $\tau \cong \pi_0$, contradicting the multiplicity 1 property of π_0. □

Remark. If one chooses a different anti-dominant element $\lambda'_{a.d.}$ in the orbit of $\lambda_{a.d.}$ in Table 11, they would have the same real part. Thus $\Theta \subset \Delta$ and L^{der} are invariant under the choice of representative of the orbit. Further more, $\left(i_T^L \lambda'_{a.d.} \right) \Big|_{L^{der}}$ is isomorphic to $\left(i_T^L \lambda_{a.d.} \right) \Big|_{L^{der}}$

Proposition 5.2. *In the following cases, $[i, s, ord(\chi)]$, the representation $\pi = i_{M_i}^G \Omega_{i,s,\chi}$ admits a unique irreducible subrepresentation:*

$$[3, -3/2, 2], [6, -2, 2]$$

Proof. Let $\lambda_{a.d.}$ be an anti-dominant exponent of π and π_0 be the unique irreducible subquotient of π such that $\lambda_{a.d.} \leq r_T^G \pi_0$ as in Table 11 and Lemma 5.1. One checks, using a branching rule calculation (performed on the computer as explained in Subsection 3.3), that in these cases

$$mult \left(\lambda_{a.d.}, r_T^G \pi_0 \right) = \left| \mathrm{Stab}_W \left(\lambda_{a.d.} \right) \right| \quad \Rightarrow \quad mult \left(\lambda_0, r_T^G \pi_0 \right) = mult \left(\lambda_0, r_T^G \pi \right) = 2.$$

From which the claim follows. Here,

$$\left| \mathrm{Stab}_W \left(\lambda_{a.d.} \right) \right| = \begin{cases} 8, & [i, s, ord(\chi)] = [3, -3/2, 2] \\ 4, & [i, s, ord(\chi)] = [6, -2, 2]. \end{cases}$$

□

In the following, we make a repeated use of a certain inclusion argument. It is thus convenient to state it in general at this point. For that matter, let $\Omega_0 = \Omega_{i,s,\chi}$ and $\pi = i_{M_i}^G \Omega_0$. For $j \neq i$ let $M_{i,j} = M_i \cap M_j$ and $\Omega_1 = r_{M_{i,j}}^{M_i} \Omega_0$. By Frobenius reciprocity,

$$\Omega_0 \hookrightarrow i_{M_{i,j}}^{M_i} \Omega_1.$$

and hence, by induction in stages,

$$\pi = i_{M_i}^G \Omega_0 \hookrightarrow i_{M_i}^G \left(i_{M_{i,j}}^{M_i} \Omega_1 \right) \cong i_{M_{i,j}}^G \Omega_1 \cong i_{M_j}^G \left(i_{M_{i,j}}^{M_j} \Omega_1 \right). \tag{5.1}$$

PROPOSITION 5.3. *In the following cases, $[i,s,ord\,(\chi)]$, the representation $\pi = i_{M_i}^{G}\Omega_{i,s,\chi}$ admits a unique irreducible subrepresentation:*

$[2,-5/2,1]\,,[3,-3/2,1]\,,[4,-1/2,1]\,,[4,-3/2,1]\,,[5,-1/2,2]\,[5,-3/2,1]\,,[6,-2,1]\,.$

PROOF. Since the proof is similar for all cases, we start by outlining it and conclude by listing the relevant data for each of the cases in Table 12.

For $[i,s,ord\,(\chi)]$ as above, let $\lambda_0 = r_T^{M_i}\Omega_{i,s,\chi}$. For $j \neq i$ as in Table 12, let $M_{i,j} = M_i \cap M_j$ and $\Omega_1 = r_{M_{i,j}}^{M_i}\Omega_0$. It holds that

$$\lambda_0 = r_T^{M_i}\Omega = r_T^{M_{i,j}}\Omega_1.$$

Assume that $i_{M_{i,j}}^{M_j}\Omega_1$ is an irreducible representation of M_j, this can be verified by a branching rule calculation in M_j or by referring to previous knowledge on the representation theory of M_j as explained below in Subsection 5.1. Let λ_1 be an M_j-anti-dominant exponent of $i_{M_{i,j}}^{M_j}\Omega_1$, that is $\lambda_1 \leq r_T^{M_j}\left(i_{M_{i,j}}^{M_j}\Omega_1\right)$ and

$$\mathrm{Re}\left(\langle\lambda_1,\alpha_l^\vee\rangle\right) \leq 0 \quad \forall l \neq j. \tag{5.2}$$

By a central character argument, see Equation (2.3), it holds that

$$i_{M_{i,j}}^{M_j}\Omega_1 \hookrightarrow i_T^{M_j}\lambda_1.$$

Hence, by Equation (5.1) and induction in stages,

$$\pi \hookrightarrow i_{M_j}^{G}\left(i_{M_{i,j}}^{M_j}\Omega_1\right) \hookrightarrow i_T^{G}\lambda_1.$$

By Frobenius reciprocity, any irreducible subrepresentation τ of $i_T^G\lambda_1$ must satisfy

$$\lambda_1 \leq r_T^G\tau.$$

On the other hand, let $\lambda_{a.d.}$ be an anti-dominant exponent of π and let π_0 denote the unique irreducible subquotient of π such that $\lambda_{a.d.} \leq r_T^G\pi_0$. If $ord\,(\chi) = 1$ the existence of π_0 is automatic, and if $ord\,(\chi) = 2$, it follows from Lemma 5.1.

In each of these cases, a branching rule calculation (performed on the computer as explained in Subsection 3.3) implies that

$$mult\left(\lambda_1, r_T^G\pi\right) = mult\left(\lambda_1, r_T^G\pi_0\right).$$

It follows that π_0 is the unique irreducible subquotient of π which admits λ_1 as an exponent. In other words, π_0 is the unique irreducible subquotient of π such that

$$\pi_0 \hookrightarrow i_T^G\lambda_1.$$

On the other hand, since

$$mult\left(\lambda_{a.d.}, r_T^G\pi\right) = mult\left(\lambda_{a.d.}, r_T^G i_T^G\lambda_1\right) = mult\left(\lambda_{a.d.}, r_T^G\pi_0\right),$$

i	s	$ord\,(\chi)$	j	λ_0	λ_1	$\lambda_{a.d.}$
2	$-\frac{5}{2}$	1	1	$\begin{pmatrix} & & 5 & & & & \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & -1 & & & & \\ 2 & -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}$
3	$-\frac{3}{2}$	1	7	$\begin{pmatrix} & & -1 & & & & \\ -1 & 4 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}$
4	$-\frac{1}{2}$	1	8	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & 3 & -1 & -1 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$
4	$-\frac{3}{2}$	1	6	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & 2 & -1 & -1 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 \end{pmatrix}$
5	$-\frac{1}{2}$	2	8	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & 4+\chi & -1 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & \chi & & & & \\ \chi & \chi & -1+\chi & \chi & \chi & \chi & 1+\chi \end{pmatrix}$	$\begin{pmatrix} & & \chi & & & & \\ \chi & \chi & \chi & -1 & \chi & \chi & \chi \end{pmatrix}$
5	$-\frac{3}{2}$	1	7	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & 3 & -1 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & -1 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}$
6	-2	1	8	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & -1 & 4 & -1 & -1 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & -1 & -1 & 2 \end{pmatrix}$	$\begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 \end{pmatrix}$

TABLE 12. Data for the proof of Proposition 5.3.

it follows that π_0 appears in $i_T^G\lambda_1$, and in π, with multiplicity 1. We conclude that π_0 is the unique irreducible subrepresentation of π.

For the case $[5,-1/2,2]$, χ denotes in Table 12 a character of $F^\times$ of order 2.

$\square$

REMARK. The above argument, as well as other arguments below, used the fact that $\sigma = i_{M_{i,j}}^{M_j}\Omega_1$ is an irreducible representation of M_j. One way to prove the irreducibility of σ is using branching rule calculations but it can also be inferred from previous works. We wish to allow for a common reference for following arguments and thus we treat all values of j and not only 1, 6, 7 and 8.

More precisely, σ is a degenerate principal series of M_j and if its restriction $\sigma\Big|_{M_j^{der}}$ to M_j^{der} is irreducible, then so is σ. For different values of j, one can find the list of reducible and irreducible degenerate principal series of M_j^{der} in the following sources:

- [BJ03] for $j=1$.
- [Tad92] for $j=2,3,4,5$
- [BJ03] and [Tad92] for $j=6$.
- [HS20, HS21] for $j=7,8$.

It should be noted here that [BJ03] deals only with representations of orthogonal groups, while $M_1^{der} = Spin_{14}$ and $M_6^{der} = Spin_{10}\times SL_2$. That is, in order to prove irreducibility, one should also take the isogeny map into account, using [Tad92].

A slight modification of the argument made in Proposition 5.3 can be used to prove:

PROPOSITION 5.4. *The representation $\pi = i_{M_i}^G \Omega_{4,-\frac{1}{2},\chi}$, where χ has order 2, admits a unique irreducible subrepresentation.*

PROOF. Let $\Omega_0 = \Omega_{4,-\frac{1}{2},\chi}$ and let

$$\lambda_0 = r_T^{M_4}\Omega_0 = \begin{pmatrix} & & -1 & & & & \\ -1 & -1 & 3+\chi & -1 & -1 & -1 & -1 \end{pmatrix}.$$

We fix an anti-dominant exponent

$$\lambda_{a.d.} = \begin{pmatrix} & & \chi & & & & \\ \chi & \chi & \chi & -1 & \chi & \chi & 0 \end{pmatrix}$$

of π. Let $\Omega_1 = r_{M_{3,4}}^{M_4}\Omega_0$. By Equation (5.1),

$$\pi \hookrightarrow i_{M_3}^G\left(i_{M_{3,4}}^{M_3}\Omega_1\right).$$

The representation $i_{M_{3,4}}^{M_3}\Omega_1$ of M_3 is an irreducible degenerate principal series of M_3. This can be verified by a branching rule calculation in M_7 or by a similar argument to that of Subsection 5.1. Furthermore, by invertedness, it holds that $i_{M_{3,4}}^{M_3}\Omega_1 \cong i_{M_{3,7}}^{M_3}\Omega_2$, where

$$r_T^{M_{3,7}}\Omega_2 = \begin{pmatrix} & & -1 & & & & \\ -1 & 3 & -1 & -1 & -1 & 2+\chi & -1 \end{pmatrix}$$

and $i_{M_{3,7}}^{M_3}\Omega_2$ is the invert of the irreducible $i_{M_{3,4}}^{M_3}\Omega_1$. It follows, by induction in stages, that

$$\pi \hookrightarrow i_{M_3}^G\left(i_{M_{3,7}}^{M_3}\Omega_2\right) \cong i_{M_7}^G\left(i_{M_{3,7}}^{M_7}\Omega_2\right). \tag{5.3}$$

The M_7-anti-dominant exponent of $i_{M_{3,7}}^{M_7}\Omega_2$ (see Equation (5.2)) is given by

$$\lambda_1 = \begin{pmatrix} & & 0 & & & & \\ -1 & 0 & -1 & 0 & 0 & 2+\chi & -1 \end{pmatrix}.$$

By a central character argument, see Equation (2.3),

$$i_{M_{3,7}}^{M_7}\Omega_1 \hookrightarrow i_T^{M_7}\lambda_1.$$

and hence

$$\pi \hookrightarrow i_T^G\lambda_1.$$

On the other hand, for the unique irreducible subrepresentation π_0 of π such that $\lambda_{a.d.} \le r_T^G\pi_0$ as in Lemma 5.1, a branching rule calculation shows that

$$mult\left(\lambda_1, r_T^G\pi\right) = mult\left(\lambda_1, r_T^G\pi_0\right) = 48.$$

Arguing as in Proposition 5.3, it follows that π admits a unique irreducible subrepresentation. ∎

PROPOSITION 5.5. *The representation $\pi = i_{M_4}^G\Omega_{4,0,\chi}$, where $\chi^2 = 1$, is irreducible.*

k	i_k	j_k	$r_T^{M_{i_k.j_k}}\Omega_k$	i_{k+1}	$r_T^{M_{j_{k+1},i_{k+1}}}\Omega_{k+1}$
1	2	4	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & \frac{7}{2}+\chi & -1 & -1 & -1 & -1 \end{pmatrix}$	6	$\begin{pmatrix} & & \frac{7}{2}+\chi & & & & \\ -1 & -1 & -1 & -1 & \frac{5}{2}+\chi & -1 & -1 \end{pmatrix}$
2	6	2	$\begin{pmatrix} & & \frac{7}{2}+\chi & & & & \\ -1 & -1 & -1 & -1 & \frac{5}{2}+\chi & -1 & -1 \end{pmatrix}$	5	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & \frac{5}{2}+\chi & \frac{3}{2}+\chi & -1 & -1 \end{pmatrix}$
3	5	6	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & \frac{5}{2}+\chi & \frac{3}{2}+\chi & -1 & -1 \end{pmatrix}$	8	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & 4 & -1 & -1 & \frac{1}{2}+\chi \end{pmatrix}$

TABLE 13. Data for the proof of Proposition 5.5.

PROOF. Let $\Omega_0 = \Omega_{4,0,\chi}$ and let $\Omega_1 = r_{M_{2,4}}^{M_4}\Omega_0$. By Equation (5.1),

$$\pi \hookrightarrow i_{M_2}^G\left(i_{M_{2,4}}^{M_2}\Omega_1\right).$$

In what follows, we consider a sequence of isomorphisms, similar to the one in Equation (5.3). That is, we wish to write a sequence of isomorphisms:

$$i_{M_{i_k}}^G\left(i_{M_{i_k,j_k}}^{M_{i_k}}\Omega_k\right) \cong i_{M_{j_{k+1}}}^G\left(i_{M_{i_{k+1},j_{k+1}}}^{M_{j_{k+1}}}\Omega_{k+1}\right) \cong i_{M_{i_{k+1}}}^G\left(i_{M_{i_{k+1},j_{k+1}}}^{M_{i_{k+1}}}\Omega_{k+1}\right), \tag{5.4}$$

where

- $j_{k+1} = i_k$.
- Ω_k is a 1-dimensional representation of M_{i_k,j_k}.
- Ω_{k+1} is a 1-dimensional representation of $M_{i_k,j_{k+1}}$.
- $i_{M_{i_k,j_k}}^{M_{i_k}}\Omega_k$ is an irreducible representation of M_{i_k}.
- $i_{M_{j_{k+1},i_{k+1}}}^{M_{j_{k+1}}}\Omega_{k+1}$ is the invert representation of $i_{M_{i_k,j_k}}^{M_{i_k}}\Omega_k$.

The sequence in Equation (5.4) relies on an isomorphism

$$i_{M_{i_k,j_k}}^{M_{i_k}}\Omega_k \cong i_{M_{j_{k+1},i_{k+1}}}^{M_{j_{k+1}}}\Omega_{k+1} \tag{5.5}$$

of irreducible representations for each k. The irreducibility of each of the $i_{M_{i_k,j_k}}^{M_{i_k}}\Omega_k$ follows from a branching rule calculation in M_{i_k} and it can also be inferred from the references in Subsection 5.1.

We summarize this data for the sequence of isomorphisms in the following table

Namely, the following sequence of isomorphisms hold

$$\begin{aligned} & i_{M_2}^G\left(i_{M_{2,4}}^{M_2}\Omega_1\right) \cong i_{M_2}^G\left(i_{M_{2,6}}^{M_2}\Omega_2\right) \cong i_{M_6}^G\left(i_{M_{2,6}}^{M_6}\Omega_2\right) \\ & \cong i_{M_6}^G\left(i_{M_{5,6}}^{M_6}\Omega_3\right) \cong i_{M_5}^G\left(i_{M_{5,6}}^{M_5}\Omega_3\right) \cong i_{M_8}^G\left(i_{M_{5,8}}^{M_8}\Omega_4\right). \end{aligned} \tag{5.6}$$

In each step we use either induction in stages or invert a representation. That is:

- $i_{M_2}^G\left(i_{M_{2,4}}^{M_2}\Omega_1\right) \cong i_{M_2}^G\left(i_{M_{2,6}}^{M_2}\Omega_2\right)$ since $i_{M_{2,4}}^{M_2}\Omega_1$ and $i_{M_{2,6}}^{M_2}\Omega_1$ are two irreducible degenerate principal series representations of M_2 which are invert to each other.
- $i_{M_2}^G\left(i_{M_{2,6}}^{M_2}\Omega_1\right) \cong i_{M_6}^G\left(i_{M_{2,6}}^{M_6}\Omega_1\right)$ by induction in stages.
- etc.

It follows that

$$\pi \hookrightarrow i_{M_8}^G\left(i_{M_{5,8}}^{M_8}\Omega_4\right).$$

Let

$$\lambda_1 = \begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & 0 & \frac{5}{2}+\chi \end{pmatrix}$$

be the M_8-anti-dominant exponent of $i_{M_8}^G\left(i_{M_{5,8}}^{M_8}\Omega_4\right)$. By a central character argument, see Equation (2.3), it follows that

$$\pi \hookrightarrow i_{M_8}^G\left(i_{M_{5,8}}^{M_8}\Omega_4\right) \hookrightarrow i_T^G\lambda_1.$$

Let

$$\lambda_{a.d.} = \begin{pmatrix} & & 0 & & & & \\ 0 & -\frac{1}{2}+\chi & 0 & 0 & -\frac{1}{2}+\chi & 0 & 0 \end{pmatrix}$$

denote the anti-dominant exponent of π and let π_0 denote the unique (due to [HS21, Lemma A.1]) subquotient of π such that $\lambda_{a.d.} \leq r_T^G\pi_0$. A branching rule calculation yields that

$$mult\left(\lambda_1, r_T^G\pi_0\right) = 288 = mult\left(\lambda_1, r_T^G\pi\right)$$

Hence, π_0 is the unique subquotient of π which is a subrepresentation of $i_T^G\lambda_1$. Thus, π_0 is the unique irreducible subrepresentation of π. Since π is semi-simple, it follows that it is irreducible. ∎

Proposition 5.6. *In the following cases, $[i, s, ord\,(\chi)]$, the maximal semi-simple subrepresentation of $\pi = i_{M_i}^G\Omega_{i,s,\chi}$ has length 2:*

$$[1, -5/2, 1]\,, [3, -1/2, 2]\,.$$

Proof. We prove the two cases using a similar argument and so we deal with them simultaneously and point out where the arguments diverge. Let $\Omega_0 = \Omega_{i,s,\chi}$ and $\pi = i_{M_i}^G\Omega_0$. We point out that

$$mult\left(\lambda_0, r_T^G\pi\right) = 2,$$

where $\lambda_0 = r_T^{M_i}\Omega_0$. Hence, π admits a maximal semi-simple subrepresentation of length at most 2.

For each case, we write a similar sequence of isomorphisms as in Equation (5.4) (see also Equation (5.6) and Subsection 5.1), summarized in the following tables (analog to Table 13):

- Fix a character χ of order 2. In the case $[3, -1/2, 2]$:

k	i_k	j_k	$r_T^{M_{i_k,j_k}}\Omega_k$	i_{k+1}	$r_T^{M_{j_{k+1},i_{k+1}}}\Omega_{k+1}$
1	1	3	$\begin{pmatrix} & & & -1 & & & \\ -1 & 5+\chi & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$	2	$\begin{pmatrix} & & & 5+\chi & & & \\ 2+\chi & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$
2	2	1	$\begin{pmatrix} & & & 5+\chi & & & \\ 2+\chi & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$	8	$\begin{pmatrix} & & & 5 & & & \\ -1 & -1 & -1 & -1 & -1 & -1 & 4+\chi \end{pmatrix}$

Table 14. Data for the proof of Proposition 5.6 in the case $[3, -1/2, 2]$.

k	i_k	j_k	$r_T^{M_{i_k,j_k}}\Omega_k$	i_{k+1}	$r_T^{M_{j_{k+1},i_{k+1}}}\Omega_{k+1}$
1	2	1	$\begin{pmatrix} & & & -1 & & & \\ 8 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}$	8	$\begin{pmatrix} & & & 5 & & & \\ -1 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix}$

Table 15. Data for the proof of Proposition 5.6 in the case $[1, -5/2, 1]$.

- In the case $[1, -5/2, 1]$:

  In order to match notations with the previous case, we write $\Omega_3 = \Omega_2$ for this case.

The conclusion of the above discussion is that in both cases it holds that

$$\pi \hookrightarrow i_{M_{2,8}}^G \Omega_3 \cong i_{M_8}^G \left(i_{M_{2,8}}^{M_8} \Omega_3 \right).$$

The representation $i_{M_{2,8}}^{M_8}\Omega_3$ is reducible and admits a maximal semi-simple subrepresentation of length 2 (this is the case denoted by $[2, -1, 1]$ in [HS21]). It follows that $i_{M_{2,8}}^G\Omega_3$ admits a maximal semi-simple subrepresentation of length at least 2.

On the other hand,

$$mult \left(\lambda_1, r_T^G \left(i_{M_{2,8}}^G \Omega_3 \right) \right) = mult \left(\lambda_1, r_T^G \pi \right) = 2,$$

where $\lambda_1 = r_T^{M_{2,8}}\Omega_3$. Hence, $i_{M_{2,8}}^G\Omega_3$ admits a maximal semi-simple subrepresentation of length precisely 2 and both of these subrepresentation intersect π, from which the claim follows. ∎

The following is a slight variation of the previous argument.

Proposition 5.7. *The representation $\pi = i_{M_7}^G\Omega_{7,-3/2}$ admits a maximal semi-simple subrepresentation of length* 2.

Proof. Let $\Omega_0 = \Omega_{7,-3/2}$ and $\lambda_0 = r_T^G\Omega_0$. We note that

$$mult \left(\lambda_0, r_T^G \pi \right) = 2.$$

Hence, π admits a maximal semi-simple subrepresentation of length at most 2.

k	i_k	j_k	$r_T^{M_{i_k \cdot j_k}} \Omega_k$	i_{k+1}	$r_T^{M_{j_{k+1}, i_{k+1}}} \Omega_{k+1}$
1	3	7	$\begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & -1 & -1 & 7 & -1 \end{pmatrix}$	4	$\begin{pmatrix} & & -1 & & & & \\ -1 & 5 & -2 & -1 & -1 & -1 & -1 \end{pmatrix}$
2	4	3	$\begin{pmatrix} & & -1 & & & & \\ -1 & 5 & -2 & -1 & -1 & -1 & -1 \end{pmatrix}$	1	$\begin{pmatrix} & & -1 & & & & \\ -4 & -1 & 3 & -1 & -1 & -1 & -1 \end{pmatrix}$

TABLE 16. Data for the proof of Proposition 5.7.

We write a similar sequence of isomorphisms to that of Equation (5.4) (see also Equation (5.6) and Subsection 5.1), summarized in the following table (analog to Table 13):

It follows that

$$\pi \hookrightarrow i_{M_1}^{G} \left(i_{M_{1,4}}^{M_1} \Omega_3 \right).$$

Since $i_{M_{1,4}}^{M_1} \Omega_3$ is a reducible unitary degenerate principal series of M_1, it follows that $i_{M_{1,4}}^{M_1} \Omega_3 = \sigma_0 \oplus \sigma_1$, with $\lambda_{a.d.} \leq r_T^{M_1} \sigma_0$ and $\lambda_{a.d.} \nleq r_T^{M_1} \sigma_1$, where

$$\lambda_{a.d.} = \begin{pmatrix} & & 0 & & & & \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

On the other hand, $\lambda_1 \leq r_T^{M_1} \sigma_0, r_T^{M_1} \sigma_1$, where $\lambda_1 = r_T^{M_{1,4}} \Omega_3$. Using Equation (2.2), one checks that

$$mult \left(\lambda_1, r_T^G i_{M_{1,4}}^G \Omega_3 \right) = mult \left(\lambda_1, r_T^G \pi \right) = 2.$$

On the other hand,

$$mult \left(\lambda_1, r_T^G i_{M_1}^G \sigma_0 \right) = mult \left(\lambda_1, r_T^G i_{M_1}^G \sigma_1 \right) = 1.$$

It follows that $i_{M_1}^G \sigma_0$ and $i_{M_1}^G \sigma_1$ each contains a unique irreducible subrepresentation, π_0 and π_1, both of which intersect π. Thus,

$$\pi_0 \oplus \pi_1 \hookrightarrow \pi.$$

□

5.2. *Unresolved Cases.* We finish with two cases we were unable to fully resolve, these are the cases of $\pi = i_{M_2}^G \Omega_{2,-\frac{1}{2}}$ and $\pi = i_{M_5}^G \Omega_{5,-\frac{1}{2}}$. In both cases we show that π admits a maximal semi-simple subrepresentation of length 1 or 2 and that the spherical subquotient of π is a subrepresentation. We were, however, unable to determine the precise length of the socle. We do, however, outline computational methods to determine this in the future, when stronger computers are readily available.

Let $\pi = i_{M_i}^{G}\left(\Omega_{M_i,s,\chi}\right)$ with $s<0$. Also, let $\lambda_{a.d.}$ be an anti-dominant exponent of π and let π_0 be an irreducible subquotient of π such that $\lambda_{a.d.} \leq r_T^G\pi_0$. It seems to be the case that π_0 is a subrepresentation of π automatically. However, we were unable to find a general proof for that. While this usually follows from a simple branching rule calculation, this line of argument was insufficient in the following case and the claim requires a more circuitous proof.

Lemma 5.8. *Let $\pi = i_{M_2}^{G}\Omega_{2,-\frac{1}{2}}$ and let π_0 denote the unique irreducible subquotient of π such that $\lambda_{a.d.} \leq r_T^G\pi_0$, where*

$$\lambda_{a.d.} = \begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then, $\pi_0 \hookrightarrow \pi$.

Proof. Since $\lambda_{a.d.}$ is the anti-dominant exponent of π, the uniqueness of π_0 follows from [HS21, Lemma A.1]. In fact, it follows that π_0 appears in $i_T^G\lambda_{a.d.}$ with multiplicity 1.

We fix the following exponents of π

$$\begin{aligned}
\lambda_0 &= \begin{pmatrix} & & 7 & & & & \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}\\
\lambda_1 &= \begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & -1 & 5 & -1 & -1 \end{pmatrix}\\
\lambda_2 &= \begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -2 & 1 & 4 & -1 & -1 \end{pmatrix},
\end{aligned}$$

where λ_0 is its initial exponent.

By a branching rule calculation, we have

$$mult\left(\lambda_1, r_T^G\pi\right) = mult\left(\lambda_1, r_T^G\pi_0\right) = 2.$$

On the other hand, by a central character argument (see Equation (2.3) for details) it holds that $\pi_0 \hookrightarrow i_T^G\lambda_1$. We consider the normalized intertwining operator (see [HS21, Subsection 2D] for a short account on these operators and their properties)

$$i_T^G\lambda_1 \xrightarrow{N_w} i_T^G\lambda_0,$$

where

$$w = s_5s_4s_3s_2s_4s_5s_1s_3s_4s_2.$$

Note that it decomposes as $N_w\left(\lambda_1\right) = N_{w'}\left(\lambda_2\right) \circ N_{s_5}\left(\lambda_1\right)$, where

$$w' = s_4s_3s_2s_4s_5s_1s_3s_4s_2$$

and that $N_{w'}\left(\lambda_2\right)$ is an isomorphism between $i_T^G\lambda_2$ and $i_T^G\lambda_0$.

On the other hand, the operator

$$i_T^G\lambda_1 \xrightarrow{N_{s_5}} i_T^G\lambda_2,$$

is not an isomorphism. However, Equation (2.2) implies that

$$mult\left(\lambda_{a.d.}, r_T^G \ker\left(N_{s_5}\left(\lambda_1\right)\right)\right) = 0.$$

It follows that π_0 is not contained in the kernel of $N_{s_5}(\lambda_1)$ or $N_w(\lambda_1)$ and hence π_0 is a subrepresentation of $i_T^G\lambda_0$. In particular, since π_0 appears in $i_T^G\lambda_0$ with multiplicity 1, it is also a subrepresentation of π.

□

REMARK. By Equation (2.2), $mult\left(\lambda_0, r_T^G\pi\right) = 2$. Hence, the length of the socle of π is either 1 or 2. We point out that, by Table 9, π_0 is also an irreducible subquotient of $i_{M_6}^G\Omega_{6,-1}$. It follows from Equation (2.2) that $mult\left(\lambda_0, r_T^G i_{M_6}^G\Omega_{6,-1}\right) = 1$ and hence also $mult\left(\lambda_0, r_T^G\pi_0\right) = 1$, while on the other hand $mult\left(\lambda_0, r_T^G\pi\right) = 2$. Hence, π admits an irreducible subquotient $\pi_1 \neq \pi_0$ of π such that $\lambda_0 \leq r_T^G\pi_1$. It remains to determine whether π_1 is a subrepresentation of π or not. Namely, whether the length of the maximal semi-simple subrepresentation of π is 1 or 2.

It seems that the methods used above are futile for this case. One could technically use the method of [HS21, Proposition 4.9] to determine whether π_1 is a subrepresentation of π, or not. The method described there has two steps:

- Find a standard intertwining operator $N_w(\lambda_0)$, such that the kernel of the operator is composed of the irreducible subrepresentations of π other than π_0.
- Calculate the dimension d of the space of Iwahori-fixed vectors in the kernel of $N_w(\lambda_0)$. If $d = 0$, then π_0 is the unique irreducible subrepresentation and if $d > 0$ then π admits a socle of length at least 2.

This is, however, beyond the capabilities of the computers available to us as the minimal Weyl word that could be used for this is

$$w = s_5s_4s_3s_2s_4s_5s_1s_3s_4s_2.$$

which is of length 10. The issue is that the required computing time grows exponentially with the length of the Weyl word and the cardinalities of relevant Weyl groups and their coset spaces.

below, we suggest a slightly simpler method to determine this in hope that it could be used in the near future. In the discussion, we give further indications on the properties of π_1.

Let π, π_0, λ_0, λ_1 and $\lambda_{a.d.}$ be as in Lemma 5.8 and let w be as above.

Let $\Theta_0 = \{1,2,3,4,5\}$. Since $w \in W_{\Theta_0}$, the action of the intertwining operator $N_w(\lambda_0)$ factors through any of M_6, M_7 or M_8. We will describe a unified argument for a calculation that can be performed for each choice of

M_j, with $j \in \{6,7,8\}$, to factor through. We will then explain why factorizing through M_8 seems to be the most efficient.

Write $\Omega_0 = \Omega_{2,-\frac{1}{2},1}$ and $\Omega_1 = r^{M_2}_{M_{2,j}}(\Omega_0)$.

We now recall, from [HS20, HS21, BJ03], that

$$length\left(i^{M_j}_{M_{2,j}}\Omega_1\right) = \begin{cases} 2, & j = 6,7 \\ 3, & j = 8. \end{cases}$$

More precisely:

- If $j = 6$ or $j = 7$, $\tau = i^{M_j}_{M_{2,j}}\Omega_1$ is of length 2 and one may write a non-splitting exact sequence
$$\tau_0 \hookrightarrow \tau \twoheadrightarrow \tau_1,$$
where τ_0 and τ_1 are irreducible and τ_1 is spherical. In the case $j = 6$, the quotient τ_1 is in fact 1-dimensional.
- If $j = 8$, $\tau = i^{M_8}_{M_{2,8}}\Omega_1$ is of length 3 and one may write a non-splitting exact sequence
$$\tau_0 \hookrightarrow \tau \twoheadrightarrow \tau_1 \oplus \tau_{-1},$$
where τ_0, τ_{-1} and τ_1 are irreducible and τ_1 is spherical.

Since

$$mult\left(\lambda_1, r^{M_j}_T\tau\right) = mult\left(\lambda_1, r^{M_j}_T\tau_1\right) = 1,$$

we conclude that the kernel and image of the intertwining operator $N^{M_j}_w(\lambda_0)$ is given by:

$$I = \mathrm{Im}\left(N^{M_j}_w(\lambda_0)\right) = \tau_1, \quad K = \ker\left(N^{M_j}_w(\lambda_0)\right) = \begin{cases} \tau_0, & j = 6,7 \\ \tau_0 + \tau_{-1}, & j = 8. \end{cases}$$

Hence, it holds that

$$i^G_{M_j}(K) \hookrightarrow i^G_{M_{2,j}}(\Omega_1) \twoheadrightarrow i^G_{M_j}(I).$$

On the other hand, since

$$mult\left(\lambda_1, r^G_T\pi_0\right) = mult\left(\lambda_1, r^G_T i^G_{M_{2,j}}(\Omega_1)\right) = mult\left(\lambda_1, r^G_T\pi\right) = 2,$$

it follows that π_0 is the unique irreducible subrepresentation of $i^G_{M_j}(I)$.

It follows from the above discussion that π admits a unique irreducible subrepresentation if and only if $\pi \cap i^G_{M_j}(K) = 0$.

In order to determine if the intersection is indeed 0, we consider

$$\dim_{\mathbb{C}}\left[\pi^{\mathcal{J}} \cap i^G_{M_j}(K)^{\mathcal{J}}\right],$$

where $\mathcal{J}$ denotes the Iwahori subgroup of G. That is, we calculate the intersection of the 17,280-dimensional space $\pi^{\mathcal{J}}$ with $i^G_{M_j}(K)^{\mathcal{J}}$ whose dimension depends on j and is given by

$$\dim_{\mathbb{C}}\left(i^G_{M_j}(K)^{\mathcal{J}}\right) = \left[\dim_{\mathbb{C}}\left(r^{M_j}_T(K)\right)\right] \times \left|W^{M_j,T}\right|.$$

We collect relevant data in the following table:

j	6	7	8
$\dim_{\mathbb{C}} r_T^{M_j}(K)$	1919	15	120=105+15
$\left\|W^{M_j,T}\right\|$	60,480	6,720	240
$\dim_{\mathbb{C}}\left(i_{M_j}^G(K)^{\mathcal{J}}\right)$	116,061,120	100,800	28,800

Thus, while the choice of $j=6$ seems more intuitive, it is seems like using $j=8$ would be more efficient.

We have tried to produce the basis of $i_{M_j}^G(K)^{\mathcal{J}}$ on a computer with 256 GB of RAM memory and 32 processors and were not able to create it in reasonable time but this might be overcome by improving parallelization of the calculation. Having said that, even if a basis for the module

$$i_{M_j}^G(K)^{\mathcal{J}}$$

is generated, calculating the intersection $\pi^{\mathcal{J}} \cap i_{M_j}^G(K)^{\mathcal{J}}$ will probably also prove to be difficult, as we we wish to intersect a $17,280$ and $28,800$ dimensional subspaces of a $696,729,600$ dimensional space.

We now turn to deal with the other unresolved case, $\pi = i_{M_5}^G \Omega_{5,-\frac{1}{2}}$. In this case, a branching rule calculation shows that the irreducible spherical subquotient π_0 of π is a subrepresentation. However, from Equation (2.2) we have $mult\left(\lambda_0, r_T^G \pi\right) = 30$, where

$$\lambda_0 = \begin{pmatrix} & & & -1 & & & \\ -1 & -1 & -1 & 4 & -1 & -1 & -1 \end{pmatrix}$$

is the initial exponent of π. Thus, it is not immediately clear that the length of the socle of π is at most 2.

LEMMA 5.9. *The representation $\pi = i_{M_5}^G \Omega_{5,-\frac{1}{2}}$ admits a maximal semi-simple subrepresentation of length at most* 2.

PROOF. We consider the following exponents of π:

$$\begin{aligned} \lambda_0 &= \begin{pmatrix} & & & -1 & & & \\ -1 & -1 & -1 & 4 & -1 & -1 & -1 \end{pmatrix}, \\ \lambda_1 &= \begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \lambda_{a.d.} &= \begin{pmatrix} & & & 0 & & & \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Here, λ_0 is the initial exponent of π, $\lambda_{a.d.}$ is its anti-dominant exponent. Let π_0 denote the unique irreducible subquotient of π such that $\lambda_{a.d.} \leq r_T^G \pi_0$.

Since $mult\left(\lambda_0, r_T^G \pi\right) = 30$ and since a branching rule calculation yielded only $mult\left(\lambda_0, r_T^G \pi_0\right) \geq 4$, the claim is not immediate. However, this does imply that $\pi_0 \hookrightarrow \pi$.

On the other hand, [HS21, Lemma A.1] implies that for every $\sigma \in Rep\left(G\right)$, it holds that

$$288 \Big| mult\left(\lambda_1, r_T^G \sigma\right).$$

Let $\Omega_1 = r_{M_{5,8}}^{M_5} \Omega_{5,-1\frac{1}{2}}$ and note that $i_{M_{5,8}}^{M_8} \Omega_1$ is irreducible. Hence, reasoning as in Proposition 5.3, it follows that

$$\pi \hookrightarrow i_{M_8}^G \left(i_{M_{5,8}}^{M_8} \Omega_1\right) \hookrightarrow i_T^G \lambda_1$$

as $\lambda_1 = r_T^{M_{5,8}} \Omega_1$. By Equation (2.2),

$$mult\left(\lambda_1, r_T^G \pi\right) = 864$$

and on the other hand, a branching rule calculation yields,

$$mult\left(\lambda_1, r_T^G \pi_0\right) \geq 576.$$

Hence, π could have at most one more subquotient π_1 such that $\pi_1 \hookrightarrow i_T^G \lambda_1$. Thus, the length of the maximal semi-simple subrepresentation of π is at most 2. ∎

Remark. Attempting to follow the methods of [HS21, Proposition 4.9] in order to determine the length of the socle of π is, too, beyond the capabilities of current available computers and even more so. The exponent λ "closest" to λ_0 such that a branching rule calculation guarantees that $mult\left(\lambda, r_T^G \pi_0\right) = mult\left(\lambda, r_T^G \pi\right)$ is $\lambda = \lambda_{a.d.}$ and the shortest Weyl element w such that $w \cdot \lambda_0 = \lambda$ is

$$w = s_5 s_6 s_7 s_8 s_4 s_3 s_2 s_4 s_5 s_6 s_7 s_4 s_1 s_3 s_2 s_4 s_5 s_6 s_4 s_1 s_3 s_2 s_4 s_5$$

which is of length 24 and thus, the associated intertwining operator cannot be realistically generated in currently available computers. Also, since this word contains all 8 generators of W, the associated operator does not factor via a Levi subgroup and the method suggested in Subsection 5.2 is not applicable here.

Here, we are able to show that the same calculation can be performed with a Weyl element of length 21. While this is an improvement, this calculation still seems to be infeasible.

As noted above,

$$\pi \hookrightarrow i_T^G \lambda_1 = i_{M_4}^G \left(i_T^{M_4} \lambda_1\right).$$

We also note that, by [HS21, Lemma A.4 and Equation (OR)], $i_T^{M_4} \lambda_1$ is non-semi-simple of length 2, we write

$$\sigma_1 \hookrightarrow i_T^{M_4} \lambda_1 \twoheadrightarrow \sigma_0,$$

where σ_0 is spherical.

Let

$$\lambda_2 = \begin{pmatrix} & & 0 & & & & \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

We show that $i_T^{M_4}\lambda_2 = \sigma_0 \oplus \sigma_1$ and that π_i is the unique irreducible subrepresentation of $i_{M_4}^G\sigma_i$.

Indeed, let Ω_2 denote the 1-dimensional representation of L_8 such that $r_T^{L_8}\Omega_2 = \lambda_2$. By [NSS20, Corollary 4.4], both $i_{L_8}^{M_4}\Omega_2$ and $i_{L_8}^{M_4}\left(\Omega_2 \otimes St_{L_8}\right)$ admit a unique irreducible subrepresentation, where L_8 is the rank 1 Levi subgroup introduced in Subsection 2.1.1 item (17). Similarly, so do $i_{L_8}^G\Omega_2$ and $i_{L_8}^G\left(\Omega_2 \otimes St_{L_8}\right)$.

On the other hand, since

$$i_T^{L_8}\lambda_2 = \Omega_2 \oplus \left(\Omega_2 \otimes St_{L_8}\right),$$

it follows that $\sigma_0 = i_T^{L_8}\Omega_2$ and $\sigma_1 = i_T^{L_8}\left(\Omega_2 \otimes St_{L_8}\right)$. We thus conclude that π_i is the unique irreducible subrepresentation of $i_{M_4}^G\sigma_i$.

We now note that $\sigma_1 = \ker\left(N_{s_8(\lambda_1)}^{M_4}\right)$. On the other hand, $\pi \hookrightarrow i_T^G\lambda_1$. If π_1 is a subrepresentation of π, then the kernel of $N_{w'}\left(\lambda_0\right)\Big|_\pi$ is non-trivial, where

$$w' = s_8s_4s_3s_2s_4s_5s_6s_7s_4s_1s_3s_2s_4s_5s_6s_4s_1s_3s_2s_4s_5.$$

Here, too, all 8 generators of W appear and thus, the method suggested in Subsection 5.2 cannot be applied to this case.

In order to calculate the dimension of the kernel $N_{w'}\left(\lambda_0\right)\Big|_\pi$ using the method of [HS21, Proposition 4.9], one first need to generate the element $n_{w'} \in \mathcal{J}$ associated with $N_{w'}$ and then apply the basis elements of $\pi^{\mathcal{J}}$. However, using a computer with 256 GB of RAM memory and 36 processors did not allow us to generate $n_{w'} \in \mathcal{J}$ due to the length of w'. In particular, the calculation runs out of memory (even with an addition of an extra 1 TB of SWAP memory) when using Weyl elements of length lower than 10. Thus, it seems likely a much more powerful computer is needed for this particular task, if such already exists.

Appendix A. Branching Rules Database

In this section, we retain the notations of Subsection 2.1 and list data on irreducible representations of Levi subgroups which are used for the branching rule calculations performed for this paper.

That is, we wish to list triples of (λ, M, τ) where λ is a character of T, M is a Levi subgroup of G and τ is the unique irreducible representation of M such that $\lambda \leq r_T^M\tau$.

We start, by a general rule. For a character λ of T, set

$$\Theta_\lambda = \left\{\alpha \in \Delta_L \mid \langle\lambda, \alpha^\vee\rangle = 0\right\}.$$

It holds that $M = M_{\Theta_\lambda}$ admits a unique irreducible representation τ such that $\lambda \leq r_T^M \tau$, as shown in [NSS20, Lemma 2.3], which satisfies

$$\left[r_T^{M_{\Theta_\lambda}} \tau\right] = \left|W_{M_{\Theta_\lambda}}\right| \times [\lambda] \,.$$

This can be decoded by the *Orthogonality Rule* which states that, for any irreducible representation σ of G it holds that

$$\lambda \leq \sigma \Rightarrow \left|W_{M_{\Theta_\lambda}}\right| \times [\lambda] \leq r_T^G \sigma. \tag{A.1}$$

For other branching rules however, instead of listing the data by Levi subgroups M of G, it is more convenient to list them by simple factors L of the derived subgroups M^{der}. Moreover, we recall that the uniqueness of τ is not required, only the uniqueness of $\left[r_T^L \tau\right]$ is. Thus we list the semi-simplified Jacquet functors $\left[r_T^L \tau\right]$ of irreducible representations. Finally, it would be more convenient to write these in the "intrinsic coordinates" of a maximal split torus of L instead of those of the torus of G.

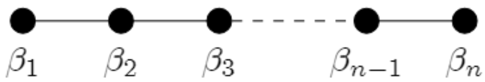
That is, we list triples of data $\left(L, \lambda, \left[r_T^G \tau\right]\right)$, where:

- L is a simple, split and simply-connected p-adic group (whose Dynkin diagram is a sub-Dynkin diagram of E_8), by abuse of notations we fix a maximal split torus T of L.
- λ is a character of T.
- $\left[r_T^L \tau\right]$ is the unique semi-simplified Jacquet functor of an irreducible representation τ of L such that $\lambda \leq r_T^G \tau$.

We separate the list of branching rules by the type of L. Most of these rules are listed in [HS21, Appendix A] while the rest can be deduced from the results of [HS20, HS21, BJ03, Jan96, Jan93, BZ76].

For convenience of applying the branching rules, we list the elements appearing in $\left[r_T^L \tau\right]$ using the action of the Weyl group $W = W_L$ of L on the characters of T. Also, we point out the each rules can be written with several variations, either due to automorphisms of the Dynkin diagram (if L is of type A_n, D_n or E_6) but also due to the Aubert involution (see [Ban02]), **we list only one variation of each rule**.

A.1. *L of type A_n.* Let L be a simple group of type A_n. We think of it as a simple factor in the derived group M^{der} of a Levi subgroup M of G. We fix a labeling for the Dynkin diagram of type A_n and use this labeling to formulate the branching rules arising from L of type A_n instead of the labeling inherited from that of the Dynkin diagram of type E_8 given in Subsection 2.2. This labeling is given by



below, we list the branching rules arising from L of type A_n which were implemented by us in the context of this paper, this list is by no means exhaustive and there are many branching rules arising from other irreducible representations of groups of type A_n:

- If L of type A_1, it admits a unique simple root β_1. For λ such that $\langle\lambda,\beta_1^\vee\rangle \neq \pm 1$, there exist a unique irreducible representation τ such that $\lambda \leq r_T^L \tau$. In fact, it holds that

$$\left[r_T^L(\tau)\right] = [\lambda] + [s_{\beta_1}\cdot\lambda].$$

This could be encoded as follows

$$\lambda \leq r_T^L(\tau),\ \langle\lambda,\beta_1^\vee\rangle \neq \pm 1 \Longrightarrow [\lambda] + [s_{\beta_1}\cdot\lambda] \leq \left[r_T^L(\tau)\right]. \tag{A.2}$$

In what follows, branching rules will be encoded in this fashion.

We point out that this rule is correct since $L = GL_2(F)\times(F^\times)^m$. If L was of the form $SL_2(F)\times(F^\times)^{m+1}$, then $i_T^L(\lambda)$ would also have been reducible if $\langle\lambda,\beta_1^\vee\rangle$ is a non trivial quadratic character. However, there are no such Levi subgroups under our assumptions on G.

- For group L of type A_n with $n\geq 2$, we have the following rule:

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle\lambda,\beta_1^\vee\rangle = \pm 1, \\ \langle\lambda,\beta_k^\vee\rangle = 0 \quad \forall\ 2\leq k\leq n \end{cases} \Longrightarrow \left[r_T^L(\tau)\right] = \sum_{w\in W^{L,T}} (n-l(w))\cdot(n-1)!\,[w\cdot\lambda],$$

where $M = M_{\{\beta_2\ldots\beta_n\}}$.

For example, in type A_2, this rule can be written as

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle\lambda,\beta_1^\vee\rangle = \pm 1, \\ \langle\lambda,\beta_2^\vee\rangle = 0 \end{cases} \Longrightarrow 2\times[\lambda] + [s_{\beta_1}\cdot\lambda] \leq \left[r_T^L(\tau)\right]. \tag{A.3}$$

In type A_3, this rule can be written as

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle\lambda,\beta_1^\vee\rangle = 1, \\ \langle\lambda,\beta_2^\vee\rangle = \langle\lambda,\beta_3^\vee\rangle = 0 \end{cases} \Longrightarrow 6\times[\lambda] + 4\times[s_{\beta_1}\cdot\lambda] + 2\times[s_{\beta_2}s_{\beta_1}\cdot\lambda] \leq \left[r_T^L(\tau)\right]$$

- For L of type A_3 we have an additional rule:

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle\lambda,\beta_1^\vee\rangle = 1, \\ \langle\lambda,\beta_2^\vee\rangle = 0, \\ \langle\lambda,\beta_3^\vee\rangle = -1 \end{cases} \Longrightarrow 2\times[\lambda] + [s_{\beta_1}\cdot\lambda] + [s_{\beta_3}\cdot\lambda] + 2\times[s_{\beta_1}s_{\beta_3}\cdot\lambda] \leq \left[r_T^L(\tau)\right].$$

- In type A_4, we have the following two additional rules:

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle \lambda, \beta_2^\vee \rangle = 1, \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 1,3,4 \end{cases}$$
$$\implies 12 \times [\lambda] + 8 \times [s_{\beta_2} \cdot \lambda] + 4 \times [s_{\beta_3} s_{\beta_2} \cdot \lambda] + 4 \times [s_{\beta_1} s_{\beta_2} \cdot \lambda] + 2 \times [s_{\beta_1} s_{\beta_3} s_{\beta_2} \cdot \lambda] \leq \left[r_T^L(\tau)\right].$$

and

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle \lambda, \beta_1^\vee \rangle = 1, \\ \langle \lambda, \beta_4^\vee \rangle = -1, \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 2,3 \end{cases}$$
$$\implies 6 \times [\lambda] + 4 \times [s_{\beta_1} \cdot \lambda] + 4 \times [s_{\beta_4} \cdot \lambda] + 4 \times [s_{\beta_1} s_{\beta_4} \cdot \lambda] + 2 \times [s_{\beta_3} s_{\beta_4} \cdot \lambda]$$
$$+2 \times [s_{\beta_2} s_{\beta_1} \cdot \lambda] + 4 \times [s_{\beta_4} s_{\beta_2} s_{\beta_1} \cdot \lambda] + 4 \times [s_{\beta_1} s_{\beta_3} s_{\beta_4} \cdot \lambda] \leq \left[r_T^L(\tau)\right].$$

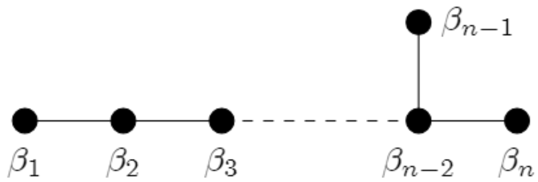
- In type A_5 we have the following two rules:

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle \lambda, \beta_3^\vee \rangle = 1, \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 1,2,4,5 \end{cases}$$
$$\implies 36 \times [\lambda] + 24 \times [s_{\beta_3} \cdot \lambda] + 12 \times [s_{\beta_2} s_{\beta_3} \cdot \lambda] + 12 \times [s_{\beta_4} s_{\beta_3} \cdot \lambda] + 6 \times [s_{\beta_2} s_{\beta_4} s_{\beta_3} \cdot \lambda] \leq \left[r_T^L(\tau)\right].$$

and

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle \lambda, \beta_1^\vee \rangle = 1, \\ \langle \lambda, \beta_4^\vee \rangle = -1, \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 2,3,5 \end{cases}$$
$$\implies 12 \times [\lambda] + 12 \times [s_{\beta_5} s_{\beta_4} s_{\beta_2} s_{\beta_1} \cdot \lambda] + 8 \times [s_{\beta_1} \cdot \lambda] + 8 \times [s_{\beta_4} \cdot \lambda] + 8 \times [s_{\beta_1} s_{\beta_4} \cdot \lambda]$$
$$+8 \times [s_{\beta_4} s_{\beta_2} s_{\beta_1} \cdot \lambda] + 8 \times [s_{\beta_1} s_{\beta_3} s_{\beta_4} \cdot \lambda] + 8 \times [s_{\beta_5} s_{\beta_4} s_{\beta_1} \cdot \lambda] + 4 \times [s_{\beta_3} s_{\beta_4} \cdot \lambda]$$
$$+4 \times [s_{\beta_5} s_{\beta_4} \cdot \lambda] + 4 \times [s_{\beta_5} s_{\beta_1} \cdot \lambda] + 4 \times [s_{\beta_5} s_{\beta_1} s_{\beta_3} s_{\beta_4} \cdot \lambda] + 2 \times [s_{\beta_5} s_{\beta_3} s_{\beta_4} \cdot \lambda] \leq \left[r_T^L(\tau)\right].$$

A.2. *L of type D_n.* We fix the following labeling of the Dynkin diagram of type D_n.



If L is of type D_n we encode the branching rules in a similar fashion to that of type A_n.

- For L of type D_4, we have the following two rules:

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle \lambda, \beta_2^\vee \rangle = 1, \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 1,3,4 \end{cases}$$
$$\Longrightarrow 8 \times [\lambda] + 5 \times [s_{\beta_2} \cdot \lambda] + 2 \times [s_{\beta_1} s_{\beta_2} \cdot \lambda] + 2 \times [s_{\beta_3} s_{\beta_2} \cdot \lambda] + 2 \times [s_{\beta_4} s_{\beta_2} \cdot \lambda] + [s_{\beta_1} s_{\beta_3} s_{\beta_2} \cdot \lambda] + [s_{\beta_1} s_{\beta_4} s_{\beta_2} \cdot \lambda] + [s_{\beta_3} s_{\beta_4} s_{\beta_2} \cdot \lambda] \leq \left[r_T^L(\tau)\right].$$

and

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle \lambda, \beta_1^\vee \rangle = 1, \\ \langle \lambda, \beta_4^\vee \rangle = -1, \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 2,3 \end{cases}$$
$$\Longrightarrow 12 \times [\lambda] + 8 \times [s_{\beta_1} \cdot \lambda] + 8 \times [s_{\beta_4} \cdot \lambda] + 12 \times [s_{\beta_1} s_{\beta_4} \cdot \lambda] + 4 \times [s_{\beta_2} s_{\beta_4} \cdot \lambda] + 4 \times [s_{\beta_2} s_{\beta_1} \cdot \lambda] \leq \left[r_T^L(\tau)\right].$$

- For L of type D_5, we have the following two rules:

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle \lambda, \beta_5^\vee \rangle = 1, \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 1,2,3,4 \end{cases}$$
$$\Longrightarrow 120 \times [\lambda] + 96 \times [s_{\beta_5} \cdot \lambda] + 72 \times [s_{\beta_3} s_{\beta_5} \cdot \lambda] + 48 \times [s_{\beta_2} s_{\beta_3} s_{\beta_5} \cdot \lambda] + 48 \times [s_{\beta_4} s_{\beta_3} s_{\beta_5} \cdot \lambda] + 32 \times [s_{\beta_4} s_{\beta_2} s_{\beta_3} s_{\beta_5} \cdot \lambda] + 24 \times [s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\beta_5} \cdot \lambda] + 16 \times [s_{\beta_3} s_{\beta_2} s_{\beta_4} s_{\beta_3} s_{\beta_5} \cdot \lambda] + 16 \times [s_{\beta_1} s_{\beta_2} s_{\beta_4} s_{\beta_3} s_{\beta_5} \cdot \lambda] + 8 \times [s_{\beta_3} s_{\beta_1} s_{\beta_2} s_{\beta_4} s_{\beta_3} s_{\beta_5} \cdot \lambda] \leq \left[r_T^L(\tau)\right].$$

and

$$\begin{cases} \lambda \leq r_T^L(\tau), \\ \langle \lambda, \beta_3^\vee \rangle = 1, \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 1,2,4,5 \end{cases}$$
$$\Longrightarrow 24 \times [\lambda] + 16 \times [s_{\beta_3} \cdot \lambda] + 8 \times [s_{\beta_2} s_{\beta_3} \cdot \lambda] + 8 \times [s_{\beta_4} s_{\beta_3} \cdot \lambda] + 8 \times [s_{\beta_5} s_{\beta_3} \cdot \lambda] + 4 \times [s_{\beta_4} s_{\beta_5} s_{\beta_3} \cdot \lambda] + 4 \times [s_{\beta_2} s_{\beta_5} s_{\beta_3} \cdot \lambda] + 4 \times [s_{\beta_2} s_{\beta_4} s_{\beta_3} \cdot \lambda] + 2 \times [s_{\beta_4} s_{\beta_2} s_{\beta_5} s_{\beta_3} \cdot \lambda] + 2 \times [s_{\beta_3} s_{\beta_4} s_{\beta_2} s_{\beta_5} s_{\beta_3} \cdot \lambda] \leq \left[r_T^L(\tau)\right].$$

- For L of type D_6, we use one branching rule. For this rule, however, the list of exponents contains 30 different exponents of various multiplicities. Thus, we give a rough explanation on how to determine the Jacquet functor of τ in the relevant case, see [HS21, Lemma A.6] for more details. First, we consider the degenerate principal series representation σ of L corresponding to the notation $[D_6, 3, 0, 1]$. This is a direct sum of two irreducible representations, one spherical and the other is not. We consider the spherical irreducible constituent τ, this is the unique irreducible representation of L such that

$$\begin{cases} \lambda \leq r_T^L \tau \\ \langle \lambda, \beta_1^\vee \rangle = \langle \lambda, \beta_4^\vee \rangle = -1 \\ \langle \lambda, \beta_k^\vee \rangle = 0, \quad k = 2,3,5,6. \end{cases}$$

Let τ' denote the non-spherical constituent of σ. Using the method described in [HS21, Appendix B], one can show that

$$\begin{array}{l} \dim_{\mathbb{C}} \left(r_T^L \tau \right) = 155 \\ \dim_{\mathbb{C}} \left(r_T^L \tau' \right) = 5 \\ \dim_{\mathbb{C}} \left(r_T^L \sigma \right) = 160. \end{array}$$

The initial exponent of σ is given by

$$\lambda_0 = \begin{bmatrix} & & & -1 & \\ -1 & -1 & 3 & -1 & -1 \end{bmatrix}.$$

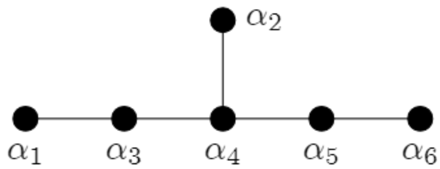
Applying Equation (A.2), together with the above, yields

$$\left[r_T^L \tau' \right] = [\lambda_0] + [s_{\beta_3} \cdot \lambda_0] + [s_{\beta_2} s_{\beta_3} \cdot \lambda_0] + [s_{\beta_4} s_{\beta_3} \cdot \lambda_0] + [s_{\beta_2} s_{\beta_4} s_{\beta_3} \cdot \lambda_0].$$

One then computes $\left[r_T^L \sigma \right]$ using Equation (2.2) and the structure of $\left[r_T^L \tau \right]$ follows by reducing the multiplicity of these 5 exponents by 1.

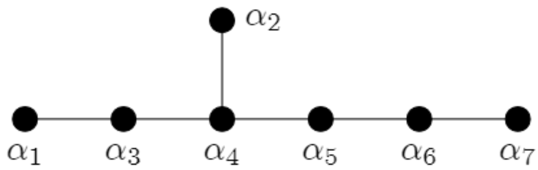
A.3. *L of type E_n.* In the branching rules for L of type E_n, the list of exponents is long. Thus, we only list the irreducible degenerate principal series of L from which we derive the branching rules used by us. One then needs to compute $\left[r_T^L \tau \right]$ using Equation (2.2) in order to determine the branching rule explicitly.

We fix the following labeling of the Dynkin diagram of group of type E_6.



The irreducible degenerate principal series of L of type E_6 whose Jacquet functor $\left[r_T^L \tau \right]$ contribute additional information to the one given previously are: $\left[E_6, 5, -\frac{1}{2}, 1 \right]$ and $\left[E_6, 3, -\frac{1}{2}, 1 \right]$.

We fix the following labeling of the Dynkin diagram of group of type E_7.



The irreducible degenerate principal series of L whose Jacquet functor $\left[r_T^L\tau\right]$ contribute additional information to the one given previously are: $[E_7, 4, 0, 1]$, $[E_7, 5, 0, 1]$ and $\left[E_7, 5, -\frac{3}{2}, 1\right]$.

Appendix B. Example of a Branching Rule Calculation

In this appendix, we would like to demonstrate how a branching rule calculation is performed by our algorithm. We consider the representation $\pi = i_{M_8}^G\left(\Omega_{M_8,-\frac{9}{2},\mathbf{1}}\right)$, namely the case denoted by $\left[8, -\frac{9}{2}, 1\right]$ in Section 4 and show how this calculation shows that this representation is irreducible.

As $r_T^G\pi$ has dimension 240 and contains 196 different isomorphism classes of exponents, we do not list the complete calculation and only follow the exponents required to show irreducibility. We start with an irreducible subquotient σ which contains the anti-dominant exponent $\lambda_{a.d.}$ of π and show that it contains the intial exponent λ_0 and terminal exponent λ_t of π with full multiplicity, that is

$$mult\left(\lambda_0, r_T^G\sigma\right) = mult\left(\lambda_0, r_T^G\pi\right) = 2, \quad mult\left(\lambda_t, r_T^G\sigma\right) = mult\left(\lambda_t, r_T^G\pi\right) = 1.$$

This implies that σ is both the unique irreducible subrepresentation and the unique irreducible quotient of π and hence $\pi = \sigma$ is irreducible.

In order to do this, we construct a σ-dominated sequence $f_1 \leq ..., \leq f_{10}$ so that $f_{10}(\lambda_0) = 2$ and $f_{10}(\lambda_t) = 1$. We start with $f_1 = \delta_{\lambda_{a.d.}}$. At each step of the calculation, the construction of f_{k+1} from f_k is based on one of the branching rules in Appendix A. In the table below we list the following data:

- An exponent λ' of σ such that $\lambda' \in supp(f_k)$.
- An exponent λ of σ such that $\lambda' \in supp(f_{k+1})$. This is the exponent on which we obtain new information in this step of the calculation.
- A Levi subgroup L of G and a branching rule which we apply on λ' in order to obtain information on the multiplicity of λ in σ (which will be encoded by $f_{k+1}(\lambda)$).
- A Weyl group element $w \in W^{M,T}$ such that $w \cdot \lambda' = \lambda$.
- We also list the value $m_{\pi,\lambda} = mult\left(\lambda, r_T^G\pi\right)$ calculated using Equation (2.2), see Subsection 3.6 for more information.

below the table we explain some of the steps and how to read the information in the table.

k	λ	$m_{\pi,\lambda}$	$f_k(\lambda)$	Rule (Levi)	λ'	w
1	$\lambda_{a.d.} = \begin{pmatrix} & & 0 & & & & \\ -1 & -1 & -1 & 0 & -1 & 0 & -1 \end{pmatrix}$	8	1	–	–	–
2	$\lambda_{a.d.} = \begin{pmatrix} & & 0 & & & & \\ -1 & -1 & -1 & 0 & -1 & 0 & -1 \end{pmatrix}$	8	8	OR $(M_{\{2,5,6\}})$	$\lambda_{a.d.}$	–
3	$\lambda_1 = \begin{pmatrix} & & -1 & & & & \\ -1 & -2 & 1 & -1 & -1 & 0 & -1 \end{pmatrix}$	4	4	A_2 $(M_{\{2,4\}})$	$\lambda_{a.d.}$	s_4
4	$\lambda_2 = \begin{pmatrix} & & -1 & & & & \\ -1 & -2 & 1 & -2 & 1 & -1 & -1 \end{pmatrix}$	2	2	A_2 $(M_{\{6,7\}})$	λ_1	s_6
5	$\lambda_0 = \begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & -1 & -1 & -1 & 9 \end{pmatrix}$	2	2	A_1	λ_2	u_1
6	$\lambda_3 = \begin{pmatrix} & & 0 & & & & \\ -1 & -1 & -1 & 0 & -1 & -1 & 1 \end{pmatrix}$	4	4	A_2 $(M_{\{7,8\}})$	$\lambda_{a.d.}$	s_8
7	$\lambda_4 = \begin{pmatrix} & & 0 & & & & \\ -1 & -1 & -1 & -1 & 1 & -2 & 1 \end{pmatrix}$	2	2	A_2 $(M_{\{5,6\}})$	λ_3	s_6
8	$\lambda_5 = \begin{pmatrix} & & -1 & & & & \\ -1 & -2 & 1 & -2 & 1 & -2 & 1 \end{pmatrix}$	1	1	A_2 $(M_{\{2,4\}})$	λ_4	s_4
9	$\lambda_t = \begin{pmatrix} & & -1 & & & & \\ -1 & -1 & -1 & -1 & -1 & -1 & 18 \end{pmatrix}$	1	1	A_1	λ_5	u_2

Here

$$\begin{aligned} u_1 =& s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_1 s_3 \\ u_2 =& s_8 s_7 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_6 s_7 s_8 s_1 s_3 s_4 s_5 s_6 s_2 s_4 s_5 s_3 s_4 s_1 s_3 \\ & \cdot s_2 s_4 s_5 s_6 s_7 s_6 s_5 s_4 s_3 s_2 s_4 s_5 s_1 s_3. \end{aligned}$$

- When the calculation begins, we start with $f_1 = \delta_{\lambda_{a.d.}}$. That is,
$$f_1(\mu) = \begin{cases} 1, & \mu = \lambda_{a.d.} \\ 0, & \mu \neq \lambda_{a.d.} \end{cases}$$
- In the first step, we consider the exponent $\lambda' = \lambda_{a.d.}$ and apply the orthogonality rule, Equation (A.1), with respect to the Levi subgroup $L = M_{\{\alpha_2,\alpha_5,\alpha_6\}}$. This rule implies that $8 \times \lambda_{a.d.} \leq r_T^G \sigma$. Hence, we have
$$f_2(\mu) = \begin{cases} 8, & \mu = \lambda_{a.d.} \\ 0, & \mu \neq \lambda_{a.d.} \end{cases}$$
- In the second step, we consider $\lambda' = \lambda_{a.d.}$ again and apply the branching rule associated with Levi groups of type A_2 (see Equation (A.3)), in this case $L = M_{\{\alpha_2,\alpha_4\}}$. Since $f_2(\lambda') = 8$, this implies that $mult\left(\tau, r_L^G \sigma\right) \geq 4$, where τ is the irreducible representation of L in this rule. The new data from this step in the calculation is, by Equation (3.2) and Equation (A.3), that
$$mult\left(\lambda_1, r_T^G \sigma\right) \geq \left\lceil \frac{f_2(\lambda')}{mult\left(\lambda', r_T^L \tau\right)} \right\rceil \cdot mult\left(\lambda_1, r_T^L \tau\right) = \left\lceil \frac{8}{2} \right\rceil \cdot 1 = 4,$$
since $\lambda_1 = w_4 \cdot \lambda_{a.d.}$.

In particular, we have

$$f_3(\mu) = \begin{cases} 8, & \mu = \lambda_{a.d.} \\ 4, & \mu = \lambda_1 \\ 0, & otherwise \end{cases}.$$

- The next step is similar with $\lambda' = \lambda_1$, $L = M_{\{\alpha_6,\alpha_7\}}$, $\lambda_2 = w_6 \cdot \lambda'$ and τ being the relevant representation of L for this case. In particular

$$mult\left(\lambda_2, r_T^G\sigma\right) \geq \left\lceil \frac{f_3(\lambda')}{mult\left(\lambda', r_T^L\tau\right)} \right\rceil \cdot mult\left(\lambda_2, r_T^L\tau\right) = \left\lceil \frac{4}{2} \right\rceil \cdot 1 = 2,$$

Hence

$$f_4(\mu) = \begin{cases} 8, & \mu = \lambda_{a.d.} \\ 4, & \mu = \lambda_1 \\ 2, & \mu = \lambda_2 \\ 0, & otherwise \end{cases}.$$

- The last step we detail is applying a sequence of branching rules of type A_1 starting at λ_2. As explained in Subsection 3.3, for any exponent which is A_1-equivalent to λ_2 has the same multiplicity in σ as λ_2 does. Namely,

$$f_5(\mu) = \begin{cases} 8, & \mu = \lambda_{a.d.} \\ 4, & \mu = \lambda_1 \\ 2, & \mu \sim \lambda_2 \\ 0, & otherwise \end{cases},$$

where $\sim$ denotes the A_1-equivalency relation on $\mathcal{S}$. In particular, the table also allows us, by reading the Weyl element u_1, to write the sequence of exponents in $\mathbf{X}(T)$ in the definition of $\sim$ in Subsection 3.3. This sequence is given by

$$\lambda_2, w_3 \cdot \lambda_2, w_1 w_3 \cdot \lambda_2, ..., u_1 \cdot \lambda_2.$$

- Applying all the steps listed in the table, we would arrive at the following function

$$f_{10}(\mu) = \begin{cases} 8, & \mu = \lambda_{a.d.} \\ 4, & \mu \sim \lambda_1 \\ 2, & \mu \sim \lambda_2 \\ 4, & \mu \sim \lambda_3 \\ 2, & \mu \sim \lambda_4 \\ 1, & \mu \sim \lambda_5. \end{cases}$$

We note that $f_{10} < f_\pi$ since the A_1-equivalency of λ_i has cardinality

i	1	2	3	4	5
Cardinality	3	17	1	2	168

and $1 \times 8 + 3 \times 4 + 17 \times 2 + 1 \times 4 + 2 \times 2 + 168 \times 1 = 230 < 240$. Still, the calculation implies that $\pi = \sigma$ as explained above.

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