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Milena Sošić

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THE INVERSE OF A QUANTUM BILINEAR FORM OF THE ORIENTED BRAID ARRANGEMENT

Milena Sošić
University of Rijeka, Croatia

Abstract. We follow here the results of Varchenko, who assigned to each weighted arrangement $\mathcal{A}$ of hyperplanes in n-dimensional real space a bilinear form, which he called the quantum bilinear form of the arrangement $\mathcal{A}$. We briefly explain the quantum bilinear form of the oriented braid arrangement in n-dimensional real space. The main concern of this paper is to compute the inverse of the matrix of the quantum bilinear form of the oriented braid arrangement in $\mathbb{R}^n$, $n \geq 2$. To solve this problem, in [5] the authors used some special matrices and their factorizations in terms of simpler matrices. So, to simplify some matrix calculations, we first introduce a twisted group algebra $\mathcal{A}(S_n)$ of the symmetric group S_n with coefficients in the polynomial ring in n^2 commutative variables and then use a natural representation of some elements of the algebra $\mathcal{A}(S_n)$ on the generic weight subspaces of the multiparametric quon algebra $\mathcal{B}$, which immediately gives the corresponding matrices of the quantum bilinear form.

1. Introduction

We first briefly explain the basic concepts of an arrangement and of the oriented braid arrangement in $\mathbb{R}^n$, $n \geq 2$. An arrangement is a finite set of hyperplanes in $\mathbb{R}^n$, $n \geq 1$. Connected components of the complement of the union of all hyperplanes of $\mathcal{A}$ are called regions (chambers or domains). An edge of $\mathcal{A}$ is any nonempty intersection of a subset of $\mathcal{A}$, including the empty intersection, where the space $\mathbb{R}^n$ can be regarded as the intersection of the empty set of hyperplanes. We denote by $L_{\mathcal{A}}$ the intersection poset consisting of all edges of $\mathcal{A}$, where $L_{\mathcal{A}}$ is partially ordered by reverse inclusion. We denote by $L'_{\mathcal{A}} = L_{\mathcal{A}} \backslash \mathbb{R}^n$ the intersection poset except $\mathbb{R}^n$. Let $R_{\mathcal{A}} = \mathbb{Z}[a_H \mid H \in \mathcal{A}]$

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be the commutative polynomial ring in variables a_H, $H \in \mathcal{A}$. First we assign a weight $a_H \in R_{\mathcal{A}}$ to each hyperplane H of $\mathcal{A}$, and then we define the weight of an edge $L \in L'_{\mathcal{A}}$ as the product of the weights of all hyperplanes containing L. Note that in particular the weight of the space $\mathbb{R}^n$ is equal to one, which is not considered here. Then a quantum bilinear form B associated to $\mathcal{A}$ is the bilinear form on the module $M_{\mathcal{A}}$ of all $R_{\mathcal{A}}$-linear combinations of regions of $\mathcal{A}$ defined by

$$B(P, Q) = \prod a_H \tag{1.1}$$

where the product runs over all hyperplanes $H \in \mathcal{A}$ separating regions P and Q. The matrix B with the entries (1.1) is a symmetric square matrix which Varchenko called the quantum bilinear form of the arrangement $\mathcal{A}$ and proved that the determinant of B is given by the formula

$$\det B = \prod_{L \in L'_{\mathcal{A}}} (1 - a_L^2)^{l(L)} \tag{1.2}$$

where a_L is the weight of the edge $L \in L'_{\mathcal{A}}$ and $l(L)$ is the multiplicity of the edge L, see [15] for more details.

We now consider the braid arrangement in a real affine space $\mathbb{R}^n$, $n \geq 2$, denoted by B_n, consisting of all diagonal hyperplanes

$$H_{ij} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}, \quad 1 \leq i < j \leq n.$$

Moreover, if we introduce the orientation of the braid arrangement, we obtain the oriented braid arrangement in a real affine space $\mathbb{R}^n$, $n \geq 2$, denoted by B_n^*, consisting of open half-spaces

$$H_{ij}^+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i > x_j\},$$
$$H_{ij}^- = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i < x_j\}$$

for all $1 \leq i < j \leq n$. Then to every open half-space H_{ij}^+ we associate a weight $q_{ij} = a(H_{ij}^+)$ and similarly to every open half-space H_{ij}^- we associate a weight $q_{ji} = a(H_{ij}^-)$ in the polynomial ring in variables q_{ij}, q_{ji}. Therefore, $q_{ji} \neq q_{ij}$ for all $1 \leq i < j \leq n$. In agreement with the fact that the braid arrangement B_n is the reflection arrangement of the symmetric group S_n, see [6, 2], the regions of B_n and also of B_n^* are directly connected to S_n, so that each region P_σ is in one-to-one correspondence with the corresponding permutation $\sigma \in S_n$, as follows

$$P_\sigma = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_{\sigma_1} < x_{\sigma_2} < \dots < x_{\sigma_n}\}.$$

Let us denote by B_n^* the quantum bilinear form associated to the oriented braid arrangement B_n^* in a real affine space $\mathbb{R}^n$, $n \geq 2$. Then the entries of B_n^* are the monomials of the form

$$B_n^*(P_\sigma, P_\tau) = \prod_{(a,b) \in I(\tau^{-1}\sigma)} q_{\sigma(a)\sigma(b)} \tag{1.3}$$

where $q_{\sigma(b)\sigma(a)} \neq q_{\sigma(a)\sigma(b)}$ and $I(\tau^{-1}\sigma) = \{(a,b) \mid a < b,\ \tau^{-1}\sigma(a) > \tau^{-1}\sigma(b)\}$ denotes the set of inversions of $\tau^{-1}\sigma$, c.f. [12, Proposition 3.2 and Proposition 3.5]. Then the matrix B_n^* with the entries (1.3) is non-symmetric. We call the matrix B_n^* the quantum bilinear form B_n^* of the oriented braid arrangement B_n^*. In the following we will explain the determination of the inverse of the matrix B_n^*. Before that we recall that the formula for the determinant of the quantum bilinear form B_n^* of the oriented braid arrangement B_n^* is given by

$$\det B_n^* = \prod_{\substack{T\in(Q;m)\\ 2\leq m\leq n}} (1-\sigma_T)^{(m-2)!\,(n-m+1)!} \tag{1.4}$$

(c.f. [12, Theorem 3.8]). Here $(Q;m) = \{T \subseteq Q \mid \text{Card}\, T = m\}$ denotes the set of all subsets T of the set $Q = \{l_1, l_2, \ldots, l_n\}$ of cardinality n such that the cardinality of T is equal to m, and

$$\sigma_T = \prod_{\{i,j\}\subseteq T} \sigma_{ij} = \prod_{i\neq j\in T} q_{ij}, \tag{1.5}$$

where $\sigma_{ij} := q_{ij}q_{ji}$ for $i < j$ and $\sigma_{ii} = 1$, which is consistent with $q_{ii} = 1$. Compare (1.4) with [5, Theorem 1.9.2], where the matrix B_n^* is denoted by $A^{(\nu)}$, and see also [11], where the author uses the notation $\mathbf{A}_Q$ for this matrix. The quantum bilinear form of the braid arrangement and the formula for its determinant can be found in [1]. A decomposition of the matrix B_n^*, by matrix-level factorizations are given in [5]. Here we are motivated to simplify these algebraic manipulations. By labeling the regions of the braid arrangements by permutations from the symmetric group S_n (i.e., the set of all permutations of the first n natural numbers), we can simplify these algebraic manipulations by replacing these matrix-level factorizations by more appropriate and algebraically much simpler algebraic expressions in a twisted group algebra $\mathcal{A}(S_n)$ of the symmetric group S_n with coefficients in the commutative polynomial ring $R_n = \mathbb{C}[X_{a\,b} \mid 1 \leq a, b \leq n]$ with $1 \in R_n$ as unit element of R_n, where we studied the nontrivial factorization of certain canonically defined elements [13]. Furthermore, by using a natural representation of some factorizations of these elements of $\mathcal{A}(S_n)$ on the generic weight subspaces $\mathcal{B}_Q$ of the multiparametric quon algebra $\mathcal{B}$, which is equipped with a multiparametric q-differential structure, we then obtain the corresponding factorizations of the matrix (B_n^*) and hence of the matrix $(B_n^*)^{-1}$, c.f. [10, 11].

2. A twisted group algebra of the symmetric group

In [13] we obtained a factorization of certain canonically defined elements in the algebra $\mathcal{A}(S_n)$ first as a product of previously defined simpler elements and then as a product of still simpler elements. Now we briefly recall the algebra $\mathcal{A}(S_n)$ and some of its canonically defined elements. We use the standard notation S_n for the symmetric group on n letters, i.e., the set of all permutations of the first n natural numbers. Let $R_n = \mathbb{C}[X_{a\,b} \mid 1 \le a, b \le n]$ be the polynomial ring of all polynomials in n^2 variables $X_{a\,b}$ over the set of complex numbers. Then we define a twisted group algebra of the symmetric group S_n with coefficients in the commutative polynomial ring R_n, denoted by $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$, where $\rtimes$ denotes the semidirect product. The elements of $\mathcal{A}(S_n)$ are the linear combinations $\sum_{g_i \in S_n} p_i\, g_i$, with p_i belonging to R_n. Consider the action of S_n on R_n given by $g.p(.., X_{a\,b}, ..) = p(.., X_{g(a)\,g(b)}, ..)$ for each $g \in S_n$ and each $p \in R_n$, the multiplication in $\mathcal{A}(S_n)$ is then given by

$$(p_1(.., X_{a\,b}, ..)\, g_1) \cdot (p_2(.., X_{c\,d}, ..)\, g_2) = p_1(.., X_{a\,b}, ..) \cdot p_2(.., X_{g_1(c)\,g_1(d)}, ..)\, g_1 g_2 \tag{2.6}$$

where $g_1 g_2$ is the product (i.e., the composition $g_1 \circ g_2$) of g_1 and g_2 in S_n. Note that (2.6) is the consequence of the following two kinds of basic relations

$$X_{a\,b} \cdot X_{c\,d} = X_{c\,d} \cdot X_{a\,b}, \qquad g.X_{a\,b} = X_{g(a)\,g(b)}\, g. \tag{2.7}$$

The algebra $\mathcal{A}(S_n)$ is associative but not commutative.
To each $g \in S_n$ we first assign a unique element g^* in the algebra $\mathcal{A}(S_n)$ by

$$g^* = \prod_{(a,b) \in I(g^{-1})} X_{a\,b}\; g \tag{2.8}$$

where $I(g^{-1}) = \{(a,b) \mid 1 \le a < b \le n,\ g^{-1}(a) > g^{-1}(b)\}$ denotes the set of inversions of the permutation $g^{-1} \in S_n$ (i.e., the inverse of $g \in S_n$), and we then consider the following canonical element of the algebra $\mathcal{A}(S_n)$ as follows

$$\alpha_n^* = \sum_{g \in S_n} g^* \tag{2.9}$$

c.f. [13]. Of particular interest is its factorization into the product of the simpler elements of the algebra $\mathcal{A}(S_n)$. So before we perform the decomposition of $\alpha_n^* \in \mathcal{A}(S_n)$ and also of $g^* \in \mathcal{A}(S_n)$ for all $g \in S_n$, we first consider the cyclic permutation $t_{a,b} \in S_n$ which maps b to $b-1$ to $b-2$ $\cdots$ to a to b, and then its inverse $t_{b,a} \in S_n$ which maps a to $a+1$ to $a+2$ $\cdots$ to b to a for all $1 \le a \le b \le n$, where in both cyclic permutations all $1 \le k \le a-1$ and $b+1 \le k \le n$ are fixed. Thus in the algebra $\mathcal{A}(S_n)$ the corresponding elements are given by

$$t^*_{a,b} = \prod_{a \le i \le b-1} X_{i\,b}\; t_{a,b}, \qquad t^*_{b,a} = \prod_{a+1 \le j \le b} X_{a\,j}\; t_{b,a}$$

$1 \le a \le b \le n$, where $t^*_{k,k} = id$ for each $1 \le k \le n$.
Then a permutation $g \in S_n$ can be decomposed into cycles from the left as follows $g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1}$, where $k_j \ge j$ (see [13, Section 3] and compare with [5], where $g \in S_n$ is decomposed into cycles from

the right), so that the corresponding element of the algebra $\mathcal{A}(S_n)$ is given by

$$g^* = t^*_{k_n,n} \cdot t^*_{k_{n-1},n-1} \cdots t^*_{k_j,j} \cdots t^*_{k_2,2} \cdot t^*_{k_1,1}.$$

Moreover, in the algebra $\mathcal{A}(S_n)$ we define the following element

$$\beta^*_{n-k+1} = t^*_{n,k} + t^*_{n-1,k} + \cdots + t^*_{k+1,k} + t^*_{k,k} \tag{2.10}$$

for all $1 \le k \le n$ (c.f. [13, Definition 3.2]), where $t^*_{k,k} = id$. Note that $k = n$ implies $\beta^*_1 = id$, so for $1 \le k \le n-1$ we define the simpler elements γ^*_{n-k+1} and δ^*_{n-k+1} as follows

$$\gamma^*_{n-k+1} = \big(id - t^*_{n,k}\big) \cdot \big(id - t^*_{n-1,k}\big) \cdots \big(id - t^*_{k+1,k}\big) \tag{2.11}$$

$$\begin{aligned} \delta^*_{n-k+1} = {} & \big(id - (t^*_k)^2\, t^*_{n,k+1}\big) \cdot \big(id - (t^*_k)^2\, t^*_{n-1,k+1}\big) \\ & \cdots \big(id - (t^*_k)^2\, t^*_{k+2,k+1}\big) \cdot \big(id - (t^*_k)^2\big) \end{aligned} \tag{2.12}$$

with $(t^*_k)^2 = X_{\{k,\,k+1\}}\, id$, where $t^*_k := t^*_{k+1,k}$ and $t^*_{k+1,k+1} = id$, see [13, Definition 3.5, Corollary 2.7, and Remark 2.6]). Here we have applied the notation

$$X_{\{a,b\}} := X_{a\,b} \cdot X_{b\,a} \tag{2.13}$$

$1 \le a < b \le n$. In addition, we denote by

$$X_P := \prod_{\{a,b\} \subseteq P} X_{\{a,b\}} \tag{2.14}$$

for each $P \subseteq \{1, 2, \ldots, n\}$. Considering Theorem 3.4 and Proposition 3.6 of [13], we obtain that the canonical element (2.9) has the following nontrivial factorization

$$\alpha^*_n = \beta^*_2 \cdot \beta^*_3 \cdots \beta^*_n \tag{2.15}$$

of simpler elements (2.10) over all $1 \le k \le n-1$, where each β^*_i, $2 \le i \le n$ is given as a product

$$\beta^*_i = \delta^*_i \cdot (\gamma^*_i)^{-1} \tag{2.16}$$

in terms of even simpler elements γ^*_i and δ^*_i, given by (2.11) and (2.12).

Remark 2.1. We emphasize that the elements defined by (2.10), (2.11) and (2.12) can be written as follows

$$\begin{aligned} \beta^*_i &= t^*_{n,n-i+1} + t^*_{n-1,n-i+1} + \cdots + t^*_{n-i+2,n-i+1} + t^*_{n-i+1,n-i+1} \\ \gamma^*_i &= \big(id - t^*_{n,n-i+1}\big) \cdot \big(id - t^*_{n-1,n-i+1}\big) \cdots \big(id - t^*_{n-i+2,n-i+1}\big) \\ \delta^*_i &= \big(id - (t^*_{n-i+1})^2\, t^*_{n,n-i+2}\big) \cdot \big(id - (t^*_{n-i+1})^2\, t^*_{n-1,n-i+2}\big) \\ & \qquad \cdots \big(id - (t^*_{n-i+1})^2\, t^*_{n-i+2,n-i+3}\big) \cdot \big(id - (t^*_{n-i+1})^2\big) \end{aligned}$$

for all $2 \le i \le n$. In particular, $i = 1$ implies $\beta^*_1 = t^*_{n,n} = id$. However, comparing the corresponding right-hand sides of β^*_i, γ^*_i, δ^*_i, $2 \le k \le n$ with β^*_{n-k+1}, γ^*_{n-k+1}, δ^*_{n-k+1}, $1 \le k \le n-1$ (each written in reverse order), we see that (2.10), (2.11), (2.12) are better suited for further algebraic manipulations.

Thus, from the application of (2.15) and (2.16) it follows directly that $\alpha_n^* \in \mathcal{A}(S_n)$ has the following factorization

$$\alpha_n^* = \overleftarrow{\prod_{1\le k\le n-1}} \delta_{n-k+1}^* \cdot \left(\gamma_{n-k+1}^*\right)^{-1} \tag{2.17}$$

so that its inverse is given by

$$\left(\alpha_n^*\right)^{-1} = \prod_{1\le k\le n-1} \gamma_{n-k+1}^* \cdot \left(\delta_{n-k+1}^*\right)^{-1}. \tag{2.18}$$

Note that the product on the right-hand side of (2.17) is written from right to left for all $1 \le k \le n-1$. We reproduce here Proposition 3.10 of [13] because it is so important for the further calculation of the inverse matrix of the quantum bilinear form of the oriented braid arrangement. For simplicity, we shall omit the second index n in Proposition 3.10 of [13] when written as Proposition 2.2 bellow. Let $Des(\sigma) = \{1 \le i \le n-1 \mid \sigma(i) > \sigma(i+1)\}$ be the descent set of a permutation $\sigma \in S_n$.

Proposition 2.2. *For all $1 \le k \le n-1$ the inverse of δ_{n-k+1}^* is given by the following formula*

$$\left(\delta_{n-k+1}^*\right)^{-1} = \left(\Delta_{n-k+1}\right)^{-1} \cdot \varepsilon_{n-k+1}^* \tag{2.19}$$

where

$$\Delta_{n-k+1} := \prod_{k+1\le m\le n} \left(id - X_{\{k,k+1,\dots,m\}}\right) \tag{2.20}$$

$$\varepsilon_{n-k+1}^* := \sum_{g\in S_1^k\times S_{n-k}} \prod_{i\in Des(g^{-1})} X_{\{k,k+1,\dots,i\}} \cdot g^*. \tag{2.21}$$

We consider here that for each permutation $g \in S_1^k \times S_{n-k}$ the corresponding descent set of its inverse $g^{-1} \in S_1^k \times S_{n-k}$ is given by $Des(g^{-1}) = \{k+1 \le i \le n-1 \mid g^{-1}(i) > g^{-1}(i+1)\}$. On the other hand, from the fact that g^* is given by (2.8), it follows that (2.21) can be written in the following form

$$\varepsilon_{n-k+1}^* = \sum_{g\in S_1^k\times S_{n-k}} \prod_{i\in Des(g^{-1})} X_{\{k,k+1,\dots,i\}} \cdot \prod_{(a,b)\in I(g^{-1})} X_{a\,b}\ g \tag{2.22}$$

so it goes without saying that the corresponding set of inversions of $g^{-1} \in S_1^k \times S_{n-k}$ is given by $I(g^{-1}) = \{(a,b) \mid k+1 \le a < b \le n-1,\ g^{-1}(a) > g^{-1}(b)\}$. Note that for each $1 \le k \le n-1$, the factors $X_{\{k,k+1,\dots,m\}}$ for $k+1 \le m \le n$ on the right-hand side of (2.20) and also $X_{\{k,k+1,\dots,i\}}$ for $i \in Des(g^{-1})$ on the right-hand side of (2.21) are given by (2.14).

3. A twisted regular representation on the generic weight subspaces $\mathcal{B}_Q$ of the algebra $\mathcal{B}$

In what follows we use a natural representation of the twisted group algebra $\mathcal{A}(S_n)$ on the generic weight subspaces of the multiparametric quon algebra $\mathcal{B}$, so we first briefly recall the main notions of the algebra $\mathcal{B}$. A multiparametric quon algebra $\mathcal{B}$ is the free unital associative complex algebra $\mathcal{B} = \mathbb{C}\langle e_{i_1}, e_{i_2}, \ldots, e_{i_N}\rangle$ generated by N generators $\{e_i\}_{i\in\mathcal{N}}$ each of degree one, equipped with a multiparametric q-differential structure given by q-differential operators $\{\partial_i\}_{i\in\mathcal{N}}$ acting on $\mathcal{B}$ according to the twisted Leibniz rule

$$\partial_i(e_j x) = \delta_{ij} x + q_{ij} e_j \partial_i(x) \tag{3.23}$$

where $\partial_i(1) = 0$ and $\partial_i(e_j) = \delta_{ij}$. The algebra $\mathcal{B}$ is graded by the total degree, and more generally it is multigraded and has a finer decomposition into multigraded weight subspaces

$$\mathcal{B}_Q = span_{\mathbb{C}}\left\{e_{j_1\ldots j_n} = e_{j_1}\cdots e_{j_n} \mid j_1\ldots j_n \in \widehat{Q}\right\}, \tag{3.24}$$

for each $x \in \mathcal{B}$, $i, j \in \mathcal{N}$, where each weight subspace $\mathcal{B}_Q$ corresponds to a multiset $Q = \{l_1 \leq \cdots \leq l_n\}$ of cardinality n. Here $\widehat{Q}$ denotes the set of all distinct permutations of Q and hence $\dim \mathcal{B}_Q = \text{Card } \widehat{Q}$. We note that the algebra $\mathcal{B}$ can be written as the following direct sum $\mathcal{B} = \mathcal{B}^{\text{gen}} \oplus \mathcal{B}^{\text{deg}}$, where $\mathcal{B}^{\text{gen}}$ denotes the (generic) subspace of $\mathcal{B}$, spanned by all multilinear monomials, and $\mathcal{B}^{\text{deg}}$ denotes the (degenerate) subspace of $\mathcal{B}$ spanned by all monomials which are nonlinear in at least one variable. The weight subspace $\mathcal{B}_Q$ corresponding to the set $Q = \{l_1, \ldots, l_n\}$ ($l_i \neq l_j$, $1 \leq i < j \leq n$) is called generic, otherwise it is called degenerate. In what follows we consider only the generic weight subspaces $\mathcal{B}_Q$ of the algebra $\mathcal{B}$, so we give a special case of the action of ∂_i on a typical monomial $e_{j_1\ldots j_n}$ in the monomial basis of the generic weight subspace $\mathcal{B}_Q \subseteq \mathcal{B}$ given by

$$\partial_{j_k}(e_{j_1\ldots j_n}) = q_{j_k j_1}\cdots q_{j_k j_{k-1}} e_{j_1\ldots\widehat{j_k}\ldots j_n} \tag{3.25}$$

$1 \leq k \leq n$, $j_1\ldots j_n \in \widehat{Q}$, where $\widehat{j_k}$ denotes the omission of the corresponding index j_k (see Section 2 of [11] for more details). In this (generic) case, where $\text{Card } Q = n$, it follows that $\dim \mathcal{B}_Q = \text{Card } \widehat{Q} = n!$.

Before we define a representation $\varrho\colon \mathcal{A}(S_n) \to \text{End}(\mathcal{B}_Q)$ (see (3.32)) of the twisted group algebra $\mathcal{A}(S_n) = R_n \rtimes \mathbb{C}[S_n]$ on the generic weight subspace of the algebra $\mathcal{B}$, we recall that $R_n = \mathbb{C}[X_{a\,b} \mid 1 \leq a, b \leq n]$ denotes the polynomial ring with unit element $1 \in R_n$ and $\mathbb{C}[S_n] = \left\{\sum_{\sigma\in S_n} c_\sigma\sigma \mid c_\sigma \in \mathbb{C}\right\}$ denotes the usual group algebra in which multiplication is given by

$$\left(\sum_{\sigma\in S_n} c_\sigma\sigma\right)\cdot\left(\sum_{\tau\in S_n} d_\tau\tau\right) = \sum_{\sigma,\tau\in S_n} (c_\sigma d_\tau)\,\sigma\tau$$

where $\sigma\tau$ denotes the composition $\sigma \circ \tau$, i.e., the product of σ and τ in S_n. We first consider a representation $\varrho_1 \colon R_n \to \mathrm{End}(\mathcal{B}_Q)$ on the generators $X_{a\,b} \in R_n$ defined by

$$\varrho_1(X_{a\,b}) := Q_{a\,b} \tag{3.26}$$

$j_1 \dots j_n \in \widehat{Q}$, where $Q_{a\,b}$ denotes a diagonal operator on $\mathcal{B}_Q$ given by (c.f. [5], p6)

$$Q_{a\,b}\, e_{j_1 \dots j_n} := q_{j_a j_b}\, e_{j_1 \dots j_n}. \tag{3.27}$$

With reference to the notation (2.13) and also (3.26), (3.27), we obtain that $\varrho_1(X_{\{a,b\}}) = Q_{\{a,b\}}$, where $Q_{\{a,b\}} = Q_{a\,b} \cdot Q_{b\,a}$, $1 \le a < b \le n$ is a diagonal operator which can be written with the notation $\sigma_{j_a j_b} = q_{j_a j_b} q_{j_b j_a}$ as follows

$$Q_{\{a,b\}}\, e_{j_1 \dots j_n} = \sigma_{j_a j_b}\, e_{j_1 \dots j_n}. \tag{3.28}$$

Similarly, referring to the notation (2.14), for each subset P of the set of cardinality n we obtain $\varrho_1(X_P) = Q_P$, where $Q_P = \prod_{\{a,b\} \subseteq P} Q_{\{a,b\}}$ denotes the corresponding diagonal operator given by

$$Q_P\, e_{j_1 \dots j_n} = \prod_{\{a,b\} \subseteq P} \sigma_{j_a j_b}\, e_{j_1 \dots j_n}, \tag{3.29}$$

where we applied (3.28). We emphasize that Q_P on the right-hand side of (3.29) corresponds to $\sigma_{j_1 \dots j_k}$ if $P = \{1, 2, \dots, k\} \subseteq \{1, 2, \dots, n\}$, which is also consistent with (1.5). Therefore, we denote by

$$Q_{\{1,2,\dots,k\}}\, e_{j_1 \dots j_n} = \sigma_{j_1 j_2 \dots j_k}\, e_{j_1 \dots j_n}, \tag{3.30}$$

where $\sigma_{j_1 j_2 \dots j_k}\, e_{j_1 \dots j_n} = \prod_{\{a,b\} \subseteq \{1,2,\dots,k\}} \sigma_{j_a j_b}\, e_{j_1 \dots j_n}$.

If we define a linear operator $\varrho_2 \colon \mathbb{C}[S_n] \to \mathrm{End}(\mathcal{B}_Q)$ by

$$\varrho_2(g)\, e_{j_1 \dots j_n} := e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}} \tag{3.31}$$

for each $g \in \mathbb{C}[S_n]$, then ϱ_2 is a regular representation. Now if we define a map $\varrho \colon \mathcal{A}(S_n) \to \mathrm{End}(\mathcal{B}_Q)$ on decomposable elements

$$\varrho\,(pg) := \varrho_1(p) \cdot \varrho_2(g) \tag{3.32}$$

for each $p \in R_n$ and $g \in \mathbb{C}[S_n]$ and extended by additivity, then ϱ is a representation, see [10, Proposition 4.5], where it was shown that ϱ preserves the basic relations (2.7) of multiplication in the algebra $\mathcal{A}(S_n)$ given by (2.6). In other words, from the application of (2.7), (3.26), (3.27) and (3.31) it follows that

$$\varrho\,(X_{a\,b} \cdot X_{c\,d}) = Q_{a\,b} \cdot Q_{c\,d} = Q_{c\,d} \cdot Q_{a\,b} = \varrho\,(X_{c\,d} \cdot X_{a\,b})$$
$$\varrho\,(g.X_{a\,b})\, e_{j_1 \dots j_n} = \varrho\,(X_{g(a)\,g(b)}\, g)\, e_{j_1 \dots j_n} = q_{j_a j_b}\, e_{j_{g^{-1}(1)} \cdots j_{g^{-1}(n)}}$$

$j_1 \dots j_n \in \widehat{Q}$. In the generic case (i.e., when $\mathcal{B}_Q$ is the generic weight subspace of the algebra $\mathcal{B}$) a representation ϱ is called a twisted regular representation, so in what follows we consider only a twisted regular representation ϱ. We note that the trivial cases of a (twisted) representation ϱ are given by

$$\varrho(1\cdot g)\, e_{j_1\ldots j_n} = \varrho_1(1)\cdot\varrho_2(g)\, e_{j_1\ldots j_n} = 1\cdot e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}} = e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}$$

$$\varrho(X_{a\,b}\, id)\, e_{j_1\ldots j_n} = \varrho_1(X_{a\,b})\cdot\varrho_2(id)\, e_{j_1\ldots j_n} = Q_{a\,b}\, e_{j_1\ldots j_n} = q_{j_a j_b}\, e_{j_1\ldots j_n}.$$

Proposition 3.1. *Let $\varrho\colon \mathcal{A}(S_n)\to End(\mathcal{B}_Q)$ be the twisted regular representation on the generic weight subspace $\mathcal{B}_Q$ of the algebra $\mathcal{B}$. Then the multiplication of the operators $\varrho\,(p_1(..,X_{a\,b},..)\,g_1)$ and $\varrho\,(p_2(..,X_{c\,d},..)\,g_2)$ of $End(\mathcal{B}_Q)$ is given by the following formula*

(3.33)

$$\begin{aligned}&\varrho\,(p_1(..,X_{a\,b},..)\,g_1)\cdot\varrho\,(p_2(..,X_{c\,d},..)\,g_2)\, e_{j_1\ldots j_n}\\ &= p_1(..,q_{j_{g_2^{-1}g_1^{-1}(a)} j_{g_2^{-1}g_1^{-1}(b)}},..)\cdot p_2(..,q_{j_{g_2^{-1}(c)} j_{g_2^{-1}(d)}},..)\, e_{j_{g_2^{-1}g_1^{-1}(1)}\cdots j_{g_2^{-1}g_1^{-1}(n)}}.\end{aligned}$$

Proof. Applying the formula (3.32) to the multiplication of any two elements $p_1(..,X_{a\,b},..)\,g_1$ and $p_2(..,X_{c\,d},..)\,g_2$ of $\mathcal{A}(S_n)$, given by (2.6), yields

$$\begin{aligned}&\varrho\,((p_1(..,X_{a\,b},..)\,g_1)\cdot(p_2(..,X_{c\,d},..)\,g_2))\, e_{j_1\ldots j_n}\\ &= \varrho\left(p_1(..,X_{a\,b},..)\cdot p_2(..,X_{g_1(c)\,g_1(d)},..)\,g_1g_2\right) e_{j_1\ldots j_n}\\ &= \varrho_1\left(p_1(..,X_{a\,b},..)\cdot p_2(..,X_{g_1(c)\,g_1(d)},..)\right)\cdot\varrho_2\,(g_1g_2)\, e_{j_1\ldots j_n}\\ &= p_1(..,Q_{a\,b},..)\cdot p_2(..,Q_{g_1(c)\,g_1(d)},..)\, e_{j_{g_2^{-1}g_1^{-1}(1)}\cdots j_{g_2^{-1}g_1^{-1}(n)}}\\ &= p_1(..,q_{j_{g_2^{-1}g_1^{-1}(a)} j_{g_2^{-1}g_1^{-1}(b)}},..)\cdot p_2(..,q_{j_{g_2^{-1}(c)} j_{g_2^{-1}(d)}},..)\, e_{j_{g_2^{-1}g_1^{-1}(1)}\cdots j_{g_2^{-1}g_1^{-1}(n)}}.\end{aligned}$$

On the other hand, it holds that

$$\begin{aligned}&\varrho\,((p_1(..,X_{a\,b},..)\,g_1)\cdot(p_2(..,X_{c\,d},..)\,g_2))\, e_{j_1\ldots j_n}\\ &\quad= \varrho\,(p_1(..,X_{a\,b},..)\,g_1)\cdot\varrho\,(p_2(..,X_{c\,d},..)\,g_2)\, e_{j_1\ldots j_n}\end{aligned}$$

so the formula (3.33) follows directly. □

Lemma 3.2. *The twisted regular representation $\varrho\colon \mathcal{A}(S_n)\to End(\mathcal{B}_Q)$ applied to the element $g^* = \prod_{(a,b)\in I(g^{-1})} X_{a\,b}\ g$ of the algebra $\mathcal{A}(S_n)$ is given by*

$$\varrho\,(g^*)\, e_{j_1\ldots j_n} = \prod_{(a,b)\in I(g)} q_{j_b j_a}\, e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}} \tag{3.34}$$

where $I(g)=\{(a,b)\mid 1\le a<b\le n,\ g(a)>g(b)\}$.

Proof. If we first rewrite the element $g^*\in\mathcal{A}(S_n)$ into the following form $g^*=\prod_{(a',b')\in I(g^{-1})} X_{a'b'}\ g$, then by applying (3.32) with (3.26) and (3.31) we obtain

$$\begin{aligned}\varrho\,(g^*)\,e_{j_1\ldots j_n} &= \prod_{(a',b')\in I(g^{-1})} \varrho\,(X_{a'b'}\ g) = \prod_{(a',b')\in I(g^{-1})} \varrho_1\,(X_{a'b'})\cdot\varrho_2\,(g)\,e_{j_1\ldots j_n}\\ &= \prod_{(a',b')\in I(g^{-1})} Q_{a'b'}\,e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}\\ &= \prod_{(a',b')\in I(g^{-1})} q_{j_{g^{-1}(a')}j_{g^{-1}(b')}}\,e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}\\ &= \prod_{(b,a)\in I(g)} q_{j_a j_b}\,e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}} = \prod_{(a,b)\in I(g)} q_{j_b j_a}\,e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}\end{aligned}$$

with $a = g^{-1}(a')$, $b = g^{-1}(b')$. Note that $(a', b') \in I(g^{-1})$ implies $a' < b'$ and $g^{-1}(a') > g^{-1}(b')$. If we assume that $a = g^{-1}(a')$, $b = g^{-1}(b')$, then it follows directly $a > b$ and $g(a) < g(b)$, where $g(a) = a'$, $g(b) = b'$, which implies $(b, a) \in I(g)$. □

REMARK 3.3. By considering Lemma 3.2 and its proof, we obtain that the operator $\varrho\,(g^*) \in \mathrm{End}(\mathcal{B}_Q)$ corresponding to the element $g^* \in \mathcal{A}(S_n)$ of the form $g^* = \prod_{(a,b)\in I(g^{-1})} X_{a\,b}\ g$ can be written in two ways: first, as given in (3.34), and second, as follows

$$\varrho\,(g^*)\,e_{j_1\ldots j_n} = \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)}j_{g^{-1}(b)}}\,e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}} \tag{3.35}$$

which follows directly from the application of (3.32). We emphasize that the notation (3.34) of $\varrho\,(g^*) \in \mathrm{End}(\mathcal{B}_Q)$ is more appropriate here, but (3.35) is also used in what follows because it fits better with the other notations, see Proposition 3.5.

Moreover, by applying (3.34) we obtain

$$\begin{aligned}\varrho\,(t^*_{b,a})\,e_{j_1\ldots j_a j_{a+1}\ldots j_b\ldots j_n} &= \prod_{a\le i\le b-1} q_{j_b j_i}\,e_{j_1\ldots j_b j_a\ldots j_{b-1}\ldots j_n}\\ &= q_{j_b j_a} q_{j_b j_{a+1}}\cdots q_{j_b j_{b-1}}\,e_{j_1\ldots j_b j_a j_{a+1}\ldots j_{b-1}\ldots j_n}\end{aligned} \tag{3.36}$$

$1 \le a \le b \le n$ and in the special case

$$\varrho\,((t^*_a)^2)\,e_{j_1\ldots j_n} = \sigma_{j_a j_{a+1}}\,e_{j_1\ldots j_n} \tag{3.37}$$

$1 \le a \le n-1$.

REMARK 3.4. We now write the elements $\beta^*_{n-k+1}, \gamma^*_{n-k+1}, \delta^*_{n-k+1} \in \mathcal{A}(S_n)$ given by (2.10), (2.11) and (2.12) as follows:

$$\beta^*_{n-k+1} = \overset{\leftarrow}{\sum_{k\le m\le n}} t^*_{m,k} = \overset{\leftarrow}{\sum_{k+1\le m\le n}} t^*_{m,k} + id,$$

$$\gamma_{n-k+1}^* = \overleftarrow{\prod_{k+1\le m\le n}} \left(id - t_{m,k}^*\right), \qquad \delta_{n-k+1}^* = \overleftarrow{\prod_{k+1\le m\le n}} \left(id - (t_k^*)^2\, t_{m,k+1}^*\right)$$

for each $1 \le k \le n-1$. We note that the sum and products are written from right to left. Let us introduce the abbreviation $\underline{j} := j_1 \dots j_n \in \widehat{Q}$. Then it is easy to verify that by applying (3.32) and (3.26), (3.31) as well as (3.36), (3.37), the corresponding operators $\varrho\,(\beta_{n-k+1}^*)$, $\varrho\,(\gamma_{n-k+1}^*)$, $\varrho\,(\delta_{n-k+1}^*)$ of $\mathrm{End}(\mathcal{B}_Q)$, $1 \le k \le n-1$ are given by

$$\begin{aligned}
\varrho\,(\beta_{n-k+1}^*)\, e_{\underline{j}} &= \overleftarrow{\sum_{k\le m\le n}} \varrho\,(t_{m,k}^*)\, e_{\underline{j}} \\
&= \overleftarrow{\sum_{k\le m\le n}} q_{j_m j_k} q_{j_m j_{k+1}} \cdots q_{j_m j_{m-1}}\, e_{j_1\dots j_m j_k j_{k+1}\dots j_{m-1}\dots j_n} \\
\varrho\,(\gamma_{n-k+1}^*)\, e_{\underline{j}} &= \overleftarrow{\prod_{k+1\le m\le n}} \varrho\left(id - t_{m,k}^*\right) e_{\underline{j}} \\
\varrho\,(\delta_{n-k+1}^*)\, e_{\underline{j}} &= \overleftarrow{\prod_{k+1\le m\le n}} \varrho\left(id - (t_k^*)^2\, t_{m,k+1}^*\right) e_{\underline{j}}
\end{aligned}$$

for each $1 \le k \le n-1$, $\underline{j} = j_1 \dots j_n \in \widehat{Q}$. Recall that for $m = k$ we obtain that $\varrho\,(t_{k,k}^*)\, e_{\underline{j}} = \varrho\,(1\cdot id)\, e_{\underline{j}} = e_{\underline{j}}$, which means that in this case the product $q_{j_m j_k} \cdots q_{j_m j_{m-1}}$ is equal to one. Similarly, for $m = k+1$ we obtain that $\varrho\,(id - (t_k^*)^2\, t_{k+1,k+1}^*)\, e_{\underline{j}} = \varrho\,(id - (t_k^*)^2)\, e_{\underline{j}} = \sigma_{j_a j_{a+1}}\, e_{\underline{j}}$. We note that the products in $\varrho\,(\gamma_{n-k+1}^*)$ and $\varrho\,(\delta_{n-k+1}^*)$ should be computed below using the formula (3.33), which are not considered here because of the complexity of their notations, see Proposition 3.1.

Considering first that $\varrho\,(g^*) \in \mathrm{End}(\mathcal{B}_Q)$ is given by (3.35), see Remark 3.3, and then the canonical element α_n^* of the algebra $\mathcal{A}(S_n)$, given by (2.9), it follows that the operator $\varrho\,(\alpha_n^*) \in \mathrm{End}(\mathcal{B}_Q)$ can be written as follows

$$\varrho\,(\alpha_n^*)\, e_{\underline{j}} = \sum_{g\in S_n} \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)} j_{g^{-1}(b)}}\, e_{j_{g^{-1}(1)}\dots j_{g^{-1}(n)}}. \tag{3.38}$$

From the factorization of $\alpha_n^* \in \mathcal{A}(S_n)$ given by (2.15) with (2.16), we also obtain directly that $\varrho\,(\alpha_n^*)$ has the following factorization

$$\varrho\,(\alpha_n^*)\, e_{\underline{j}} = \overleftarrow{\prod_{1\le k\le n-1}} \varrho\,(\beta_{n-k+1}^*)\, e_{\underline{j}} \quad \Big(= \varrho\,(\beta_2^*)\cdot\varrho\,(\beta_3^*)\cdots\varrho\,(\beta_n^*)\, e_{\underline{j}}\Big) \tag{3.39}$$

with

$$\varrho\,(\beta_{n-k+1}^*)\, e_{\underline{j}} = \varrho\,(\delta_{n-k+1}^*)\cdot\varrho\,((\gamma_{n-k+1}^*)^{-1})\, e_{\underline{j}} \tag{3.40}$$

$1 \leq k \leq n-1$. Thus, we obtain

$$\varrho\,((\alpha_n^*)^{-1})\,e_{j_1\cdots j_n} = \prod_{1\leq k\leq n-1} \varrho\,(\gamma_{n-k+1}^*)\cdot\varrho\,((\delta_{n-k+1}^*)^{-1})\,e_{j_1\cdots j_n} \tag{3.41}$$

see also (2.18). Thus, to determine the operator $\varrho\,((\alpha_n^*)^{-1})$, the operators $\varrho\,((\gamma_{n-k+1}^*)^{-1})$ are not involved in it, so they are not computed here. We recall that the operators $\varrho\,(\gamma_{n-k+1}^*)$, $1 \leq k \leq n-1$ are given in Remark 3.4. On the other hand, the computation of the operators $\varrho\,((\delta_{n-k+1}^*)^{-1})$ for all $1 \leq k \leq n-1$ is of special interest, see (2.19). If we consider previously the identity (2.20) and also (2.14), then for each $1 \leq k \leq n-1$ the element Δ_{n-k+1} of the algebra $\mathcal{A}(S_n)$ has the form of the product of the invertible elements $\big(id - X_{\{k,k+1,\ldots,m\}}\big)$ of the algebra $\mathcal{A}(S_n)$ for all $k+1 \leq m \leq n$, so that Δ_{n-k+1} is also invertible for all $1 \leq k \leq n-1$, see also [13, Proposition 3.10]. In this way the identity (2.19) can be written in accordance with (2.20) and (2.22) in the following form

$$\begin{aligned}
(\delta_{n-k+1}^*)^{-1} &= (\Delta_{n-k+1}^*)^{-1}\cdot\varepsilon_{n-k+1}^* \\
&= \frac{\displaystyle\sum_{g\in S_1^k\times S_{n-k}} \prod_{i\in Des(g^{-1})} X_{\{k,k+1,\ldots,i\}}\cdot \prod_{(a,b)\in I(g^{-1})} X_{ab}\; g}{\displaystyle\prod_{k+1\leq m\leq n} \big(id - X_{\{k,k+1,\ldots,m\}}\big)} \\
&= \sum_{g\in S_1^k\times S_{n-k}} \frac{\displaystyle\prod_{i\in Des(g^{-1})} X_{\{k,k+1,\ldots,i\}}\; id}{\displaystyle\prod_{k+1\leq m\leq n} \big(1 - X_{\{k,k+1,\ldots,m\}}\big)\, id}\cdot \prod_{(a,b)\in I(g^{-1})} X_{ab}\; g \\
&= \sum_{g\in S_1^k\times S_{n-k}} \frac{\displaystyle\prod_{i\in Des(g^{-1})} X_{\{k,k+1,\ldots,i\}}}{\displaystyle\prod_{k+1\leq m\leq n} \big(1 - X_{\{k,k+1,\ldots,m\}}\big)}\, id\cdot \prod_{(a,b)\in I(g^{-1})} X_{ab}\; g
\end{aligned}$$

where by applying the formula (2.6) for multiplication in the algebra $\mathcal{A}(S_n)$ we obtain

$$(\delta_{n-k+1}^*)^{-1} = \sum_{g\in S_1^k\times S_{n-k}} \left(\frac{\displaystyle\prod_{i\in Des(g^{-1})} X_{\{k,k+1,\ldots,i\}}}{\displaystyle\prod_{k+1\leq m\leq n} \big(1 - X_{\{k,k+1,\ldots,m\}}\big)}\cdot \prod_{(a,b)\in I(g^{-1})} X_{ab} \right) g. \tag{3.42}$$

Then the formula for determining the operator $\varrho\,((\delta_{n-k+1})^{-1}) \in \mathrm{End}(\mathcal{B}_Q)$ for each $1 \leq k \leq n-1$ is given in the following proposition.

Proposition 3.5. *Let $\varrho\colon \mathcal{A}(S_n) \to End(\mathcal{B}_Q)$ be the twisted regular representation on the generic weight subspace $\mathcal{B}_Q$ of the algebra $\mathcal{B}$. Suppose that* *for every* $g \in S_1^k \times S_{n-k}$ *the conditions* $1-\sigma_{j_{g^{-1}(k)}j_{g^{-1}(k+1)}\cdots j_{g^{-1}(m)}} \neq 0$ *hold true for all* $k+1 \le m \le n$. *Then the operator* $\varrho\left((\delta^*_{n-k+1})^{-1}\right) \in End(\mathcal{B}_Q)$, $1 \le k \le n-1$ *is given as follows*

$$\varrho\left((\delta^*_{n-k+1})^{-1}\right) e_{j_1\dots j_n}$$
$$= \sum_{g\in S_1^k\times S_{n-k}} \frac{\prod_{i\in Des(g^{-1})} \sigma_{j_{g^{-1}(k)}\cdots j_{g^{-1}(i)}} \cdot \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)}j_{g^{-1}(b)}}}{\prod_{k+1\le m\le n}\left(1-\sigma_{j_{g^{-1}(k)}j_{g^{-1}(k+1)}\cdots j_{g^{-1}(m)}}\right)} e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}.$$

Proof. Considering that $(\delta^*_{n-k+1})^{-1} \in \mathcal{A}(S_n)$ is given by (3.42) for each $1 \le k \le n-1$, we obtain by applying (3.32) and also (3.26), (3.31) that

$$\varrho\left((\delta^*_{n-k+1})^{-1}\right) e_{\underline{j}} = \sum_{g\in S_1^k\times S_{n-k}} \varrho\left(\frac{\prod_{i\in Des(g^{-1})} X_{\{k,k+1,\dots,i\}} \cdot \prod_{(a,b)\in I(g^{-1})} X_{ab}}{\prod_{k+1\le m\le n} (1-X_{\{k,k+1,\dots,m\}})} g\right) e_{\underline{j}}$$
$$= \sum_{g\in S_1^k\times S_{n-k}} \varrho_1\left(\frac{\prod_{i\in Des(g^{-1})} X_{\{k,k+1,\dots,i\}} \cdot \prod_{(a,b)\in I(g^{-1})} X_{ab}}{\prod_{k+1\le m\le n} (1-X_{\{k,k+1,\dots,m\}})}\right) \cdot \varrho_2(g)\, e_{\underline{j}}$$
$$= \sum_{g\in S_1^k\times S_{n-k}} \left(\frac{\prod_{i\in Des(g^{-1})} Q_{\{k,k+1,\dots,i\}} \cdot \prod_{(a,b)\in I(g^{-1})} Q_{ab}}{\prod_{k+1\le m\le n} (1-Q_{\{k,k+1,\dots,m\}})}\right) e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}$$
$$= \sum_{g\in S_1^k\times S_{n-k}} \frac{\prod_{i\in Des(g^{-1})} \sigma_{j_{g^{-1}(k)}\cdots j_{g^{-1}(i)}} \cdot \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)}j_{g^{-1}(b)}}}{\prod_{k+1\le m\le n}\left(1-\sigma_{j_{g^{-1}(k)}j_{g^{-1}(k+1)}\cdots j_{g^{-1}(m)}}\right)} e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}$$

$\underline{j} = j_1 \dots j_n \in \widehat{Q}$, where the operator $\varrho\left((\delta^*_{n-k+1})^{-1}\right)$, $1 \le k \le n-1$ is invertible if for every $g \in S_1^k \times S_{n-k}$ it holds that

$$1-\sigma_{j_{g^{-1}(k)}j_{g^{-1}(k+1)}\cdots j_{g^{-1}(m)}} \neq 0$$

for all $k+1 \le m \le n$. □

We recall that $Des(g^{-1}) = \{k+1 \le i \le n-1 \mid g^{-1}(i) > g^{-1}(i+1)\}$ denotes a descent set of $g^{-1} \in S_1^k \times S_{n-k}$ and $I(g^{-1}) = \{(a,b) \mid a<b,\ g^{-1}(a) > g^{-1}(b)\}$ denotes a set of inversions of the permutation $g^{-1} \in S_1^k \times S_{n-k}$. Note that $g \in S_1^k \times S_{n-k}$ implies $g^{-1} \in S_1^k \times S_{n-k}$. We also note that in the special case $Des(g^{-1}) = \emptyset$ if and only if $I(g^{-1}) = \emptyset$, which implies that in this case the product over $Des(g^{-1})$ and likewise the product over $I(g^{-1})$ is equal to one. Moreover, the following theorem follows from the above.

Theorem 3.6. *Let* $\varrho\colon \mathcal{A}(S_n) \to End(\mathcal{B}_Q)$ *be the twisted regular representation on the generic weight subspace* $\mathcal{B}_Q$ *of the algebra* $\mathcal{B}$. *Then the inverse* *of the operator* $\varrho\,(\alpha_n^*) \in End(\mathcal{B}_Q)$, $n \geq 2$ *given by*

$$\varrho\,(\alpha_n^*)\, e_{\underline{j}} = \sum_{g\in S_n} \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)} j_{g^{-1}(b)}}\, e_{j_{g^{-1}(1)}\cdots j_{g^{-1}(n)}}$$

has the following factorization

$$\varrho\,((\alpha_n^*)^{-1})\, e_{\underline{j}} = \prod_{1\leq k\leq n-1} \varrho\,((\beta_{n-k+1}^*)^{-1})\, e_{\underline{j}}$$

with $\varrho\,((\beta_{n-k+1}^*)^{-1})\, e_{\underline{j}} = \varrho\,(\gamma_{n-k+1}^*)\cdot\varrho\,((\delta_{n-k+1}^*)^{-1})\, e_{\underline{j}}$.

We recall that the operators $\varrho\,(\gamma_{n-k+1}^*) \in \mathrm{End}(\mathcal{B}_Q)$, $1\leq k\leq n-1$ are given in Remark 3.4 and $\varrho\,((\delta_{n-k+1}^*)^{-1}) \in \mathrm{End}(\mathcal{B}_Q)$, $1 \leq k \leq n-1$ are given in Proposition 3.5.

Example 3.7. Let us take $n = 3$. Then, considering Remark 3.4 for $k = 1, 2$, we obtain the following operators $\varrho\,(\gamma_3^*), \varrho\,(\gamma_2^*) \in \mathrm{End}(\mathcal{B}_Q)$ given by

$$\begin{aligned}\varrho\,(\gamma_3^*)\, e_{j_1j_2j_3} &= \varrho\left(id - t_{3,1}^*\right)\cdot\varrho\left(id - t_{2,1}^*\right) e_{j_1j_2j_3}\\ &= e_{j_1j_2j_3} - q_{j_2j_1}\, e_{j_2j_1j_3} - q_{j_3j_1}q_{j_3j_2}\, e_{j_3j_1j_2} + q_{j_3j_2}q_{j_3j_1}q_{j_2j_1}\, e_{j_3j_2j_1}\end{aligned}$$

where we used the formula (3.33) for multiplying the operators of $\mathrm{End}(\mathcal{B}_Q)$. If we apply the Johnson-Trotter order of permutations in S_3 given in the monomial basis of $\mathcal{B}_Q$ with $e_{j_1j_2j_3}$, $e_{j_1j_3j_2}$, $e_{j_3j_1j_2}$, $e_{j_3j_2j_1}$, $e_{j_2j_3j_1}$, $e_{j_2j_1j_3}$, then we obtain

$$\varrho\,(\gamma_3^*)\, e_{j_1j_2j_3} = e_{j_1j_2j_3} - q_{j_3j_1}q_{j_3j_2}\, e_{j_3j_1j_2} + q_{j_3j_2}q_{j_3j_1}q_{j_2j_1}\, e_{j_3j_2j_1} - q_{j_2j_1}\, e_{j_2j_1j_3}.$$

Similarly, we get

$$\varrho\,(\gamma_2^*)\, e_{j_1j_2j_3} = \varrho\left(id - t_{3,2}^*\right)\, e_{j_1j_2j_3} = e_{j_1j_2j_3} - q_{j_3j_2}\, e_{j_1j_3j_2}.$$

On the other hand, considering Proposition 3.5, we obtain that the operators $\varrho\,((\delta_3^*)^{-1}), \varrho\,((\delta_2^*)^{-1}) \in \mathrm{End}(\mathcal{B}_Q)$ are given as follows.
We note that for $k = 1$ there are two permutations $g_1 = 123 = id$ and $g_2 = 132$ in $S_1 \times S_2$, therefore we obtain

$$\begin{aligned}&\varrho\,((\delta_3^*)^{-1})\, e_{j_1j_2j_3}\\ &= \frac{1}{\left(1-\sigma_{j_{g_1^{-1}(1)}j_{g_1^{-1}(2)}}\right)\left(1-\sigma_{j_{g_1^{-1}(1)}j_{g_1^{-1}(2)}j_{g_1^{-1}(3)}}\right)}\, e_{j_{g_1^{-1}(1)}j_{g_1^{-1}(2)}j_{g_1^{-1}(3)}}\\ &\quad + \frac{\sigma_{j_{g_2^{-1}(1)}j_{g_2^{-1}(2)}}\; q_{j_{g_2^{-1}(2)}j_{g_2^{-1}(3)}}}{\left(1-\sigma_{j_{g_2^{-1}(1)}j_{g_2^{-1}(2)}}\right)\left(1-\sigma_{j_{g_2^{-1}(1)}j_{g_2^{-1}(2)}j_{g_2^{-1}(3)}}\right)}\, e_{j_{g_2^{-1}(1)}j_{g_2^{-1}(2)}j_{g_2^{-1}(3)}}\\ &= \frac{1}{(1-\sigma_{j_1j_2})\,(1-\sigma_{j_1j_2j_3})}\, e_{j_1j_2j_3} + \frac{\sigma_{j_1j_3}\, q_{j_3j_2}}{(1-\sigma_{j_1j_3})\,(1-\sigma_{j_1j_2j_3})}\, e_{j_1j_3j_2}\\ &= \frac{1}{1-\sigma_{j_1j_2j_3}}\left(\frac{1}{1-\sigma_{j_1j_2}}\, e_{j_1j_2j_3} + \frac{q_{j_3j_2}\sigma_{j_1j_3}}{1-\sigma_{j_1j_3}}\, e_{j_1j_3j_2}\right)\end{aligned}$$

where we used that $Des(g_1^{-1}) = I(g_1^{-1}) = \emptyset$ and $Des(g_2^{-1}) = \{2\}$, $I(g_2^{-1}) = \{(2,3)\}$. Note that $g_2^{-1} = g_2 = 132$ and also that $\sigma_{j_1j_3j_2} = \sigma_{j_1j_2j_3}$.
On the other hand, considering that only the permutation $g = 123 = id \in S_1^2 \times S_1$ fixes the first two indices and that $Des(g^{-1}) = I(g^{-1}) = \emptyset$, we obtain

$$\varrho\left((\delta_2^*)^{-1}\right) e_{j_1j_2j_3} = \frac{1}{1-\sigma_{j_{g^{-1}(2)}j_{g^{-1}(3)}}}\, e_{j_{g^{-1}(1)}j_{g^{-1}(2)}j_{g^{-1}(3)}} = \frac{1}{1-\sigma_{j_2j_3}}\, e_{j_1j_2j_3}.$$

From the application of Theorem 3.6 we then first obtain

$$\begin{aligned}
&\varrho\left((\beta_3^*)^{-1}\right) e_{j_1j_2j_3} = \varrho\left(\gamma_3^*\right)\cdot\varrho\left((\delta_3^*)^{-1}\right) e_{j_1j_2j_3}\\
&\quad = \left(e_{j_1j_2j_3} - q_{j_3j_1}q_{j_3j_2}\, e_{j_3j_1j_2} + q_{j_3j_2}q_{j_3j_1}q_{j_2j_1}\, e_{j_3j_2j_1} - q_{j_2j_1}\, e_{j_2j_1j_3}\right)\\
&\quad\quad \cdot \frac{1}{1-\sigma_{j_1j_2j_3}}\left(\frac{1}{1-\sigma_{j_1j_2}}\, e_{j_1j_2j_3} + \frac{q_{j_3j_2}\sigma_{j_1j_3}}{1-\sigma_{j_1j_3}}\, e_{j_1j_3j_2}\right)\\
&\quad = \frac{1}{1-\sigma_{j_1j_2j_3}}\left(\frac{1}{1-\sigma_{j_1j_2}}\, e_{j_1j_2j_3} + \frac{q_{j_3j_2}\sigma_{j_1j_3}}{1-\sigma_{j_1j_3}}\, e_{j_1j_3j_2} - \frac{q_{j_3j_1}q_{j_3j_2}}{1-\sigma_{j_1j_2}}\, e_{j_3j_1j_2}\right.\\
&\quad\quad - \frac{q_{j_2j_1}q_{j_2j_3}q_{j_3j_2}\sigma_{j_1j_3}}{1-\sigma_{j_1j_3}}\, e_{j_2j_1j_3} + \frac{q_{j_3j_2}q_{j_3j_1}q_{j_2j_1}}{1-\sigma_{j_1j_2}}\, e_{j_3j_2j_1}\\
&\quad\quad \left. + \frac{q_{j_2j_3}q_{j_2j_1}q_{j_3j_1}q_{j_3j_2}\sigma_{j_1j_3}}{1-\sigma_{j_1j_3}}\, e_{j_2j_3j_1} - \frac{q_{j_2j_1}}{1-\sigma_{j_1j_2}}\, e_{j_2j_1j_3} - \frac{q_{j_3j_1}q_{j_3j_2}\sigma_{j_1j_3}}{1-\sigma_{j_1j_3}}\, e_{j_3j_1j_2}\right)
\end{aligned}$$

where we used the formula (3.33) for multiplying the operators of $\mathrm{End}(\mathcal{B}_Q)$. After sorting the expression (by summing the same elements of the monomial basis of $\mathcal{B}_Q$) and applying the Johnson-Trotter order of permutations in S_3, we obtain:

(3.43)
$$\begin{aligned}
&\varrho\left((\beta_3^*)^{-1}\right) e_{j_1j_2j_3} = \frac{1}{1-\sigma_{j_1j_2j_3}}\left(\frac{1}{1-\sigma_{j_1j_2}}\, e_{j_1j_2j_3} + \frac{q_{j_3j_2}\sigma_{j_1j_3}}{1-\sigma_{j_1j_3}}\, e_{j_1j_3j_2}\right.\\
&\quad - \frac{q_{j_3j_1}q_{j_3j_2}(1-\sigma_{j_1j_2}\sigma_{j_1j_3})}{(1-\sigma_{j_1j_2})(1-\sigma_{j_1j_3})}\, e_{j_3j_1j_2} + \frac{q_{j_3j_2}q_{j_3j_1}q_{j_2j_1}}{1-\sigma_{j_1j_2}}\, e_{j_3j_2j_1}\\
&\quad \left. + \frac{q_{j_2j_1}q_{j_3j_1}\sigma_{j_1j_3}\sigma_{j_2j_3}}{1-\sigma_{j_1j_3}}\, e_{j_2j_3j_1} - \frac{q_{j_2j_1}(1-\sigma_{j_1j_3}+\sigma_{j_1j_3}\sigma_{j_2j_3}-\sigma_{j_1j_2j_3})}{(1-\sigma_{j_1j_2})(1-\sigma_{j_1j_3})}\, e_{j_2j_1j_3}\right).
\end{aligned}$$

Similarly, from $\varrho\left((\beta_2^*)^{-1}\right) e_{j_1j_2j_3} = \varrho\left(\gamma_2^*\right)\cdot\varrho\left((\delta_2^*)^{-1}\right) e_{j_1j_2j_3}$, i.e.,
$\varrho\left((\beta_2^*)^{-1}\right) e_{j_1j_2j_3} = \left(e_{j_1j_2j_3} - q_{j_3j_2}\, e_{j_1j_3j_2}\right)\cdot \frac{1}{1-\sigma_{j_2j_3}}\, e_{j_1j_2j_3}$ it follows

$$\varrho\left((\beta_2^*)^{-1}\right) e_{j_1j_2j_3} = \frac{1}{1-\sigma_{j_2j_3}}\, e_{j_1j_2j_3} - \frac{q_{j_3j_2}}{1-\sigma_{j_2j_3}}\, e_{j_1j_3j_2}. \tag{3.44}$$

Finally, by applying Theorem 3.6, we obtain that the inverse of the operator $\varrho(\alpha_n^*) \in \mathrm{End}(\mathcal{B}_Q)$, $n \geq 2$ is given by

$$\begin{aligned}
&\varrho((\alpha_3^*)^{-1})\, e_{j_1 j_2 j_3} = \varrho((\beta_3^*)^{-1}) \cdot \varrho((\beta_2^*)^{-1})\, e_{j_1 j_2 j_3} \\
&= \frac{1}{1-\sigma_{j_1 j_2 j_3}} \Bigg(\frac{1}{1-\sigma_{j_1 j_2}}\, e_{j_1 j_2 j_3} + \frac{q_{j_3 j_2}\sigma_{j_1 j_3}}{1-\sigma_{j_1 j_3}}\, e_{j_1 j_3 j_2} \\
&\quad - \frac{q_{j_3 j_1} q_{j_3 j_2}(1-\sigma_{j_1 j_2}\sigma_{j_1 j_3})}{(1-\sigma_{j_1 j_2})(1-\sigma_{j_1 j_3})}\, e_{j_3 j_1 j_2} + \frac{q_{j_3 j_2} q_{j_3 j_1} q_{j_2 j_1}}{1-\sigma_{j_1 j_2}}\, e_{j_3 j_2 j_1} \\
&\quad + \frac{q_{j_2 j_1} q_{j_3 j_1}\sigma_{j_1 j_3}\sigma_{j_2 j_3}}{1-\sigma_{j_1 j_3}}\, e_{j_2 j_3 j_1} - \frac{q_{j_2 j_1}(1-\sigma_{j_1 j_3}+\sigma_{j_1 j_3}\sigma_{j_2 j_3}-\sigma_{j_1 j_2 j_3})}{(1-\sigma_{j_1 j_2})(1-\sigma_{j_1 j_3})}\, e_{j_2 j_1 j_3} \Bigg) \\
&\quad \cdot \left(\frac{1}{1-\sigma_{j_2 j_3}}\, e_{j_1 j_2 j_3} - \frac{q_{j_3 j_2}}{1-\sigma_{j_2 j_3}}\, e_{j_1 j_3 j_2} \right)
\end{aligned}$$

where from the application of the formula (3.33) and the addition of the same elements of the monomial basis of $\mathcal{B}_Q$) it follows that

$$\begin{aligned}
\varrho((\alpha_3^*)^{-1})\, e_{j_1 j_2 j_3} &= \frac{1}{(1-\sigma_{j_1 j_2})(1-\sigma_{j_1 j_3})(1-\sigma_{j_2 j_3})(1-\sigma_{j_1 j_2 j_3})} \cdot \\
&[(1-\sigma_{j_1 j_3})(1-\sigma_{j_1 j_2}\sigma_{j_2 j_3})\, e_{j_1 j_2 j_3} - q_{j_3 j_2}(1-\sigma_{j_1 j_2})(1-\sigma_{j_1 j_3})\, e_{j_1 j_3 j_2} \\
&- q_{j_3 j_1} q_{j_3 j_2}\sigma_{j_1 j_2}(1-\sigma_{j_1 j_3})(1-\sigma_{j_2 j_3})\, e_{j_3 j_1 j_2} \\
&+ q_{j_3 j_2} q_{j_3 j_1} q_{j_2 j_1}(1-\sigma_{j_1 j_3})(1-\sigma_{j_1 j_2}\sigma_{j_2 j_3})\, e_{j_3 j_2 j_1} \\
&- q_{j_2 j_1} q_{j_3 j_1}\sigma_{j_2 j_3}(1-\sigma_{j_1 j_2})(1-\sigma_{j_1 j_3})\, e_{j_2 j_3 j_1} \\
&- q_{j_2 j_1}(1-\sigma_{j_1 j_3})(1-\sigma_{j_2 j_3})\, e_{j_2 j_1 j_3}]\,.
\end{aligned} \tag{3.45}$$

4. A decomposition of the matrix $(B_n^*)^{-1}$

We first introduce the appropriate matrix notations for the operators discussed above. Then, with respect to the monomial basis of a generic weight subspace $\mathcal{B}_Q$ of the algebra $\mathcal{B}$ (considered with Johnson-Trotter order of permutations, see [14]), we denote the matrix of the operator $\varrho(\alpha_n^*)$ with $\mathtt{A}_n$ and with $\mathtt{B}_{n-k+1}$, $\mathtt{C}_{n-k+1}$, $\mathtt{D}_{n-k+1}$, $1 \leq k \leq n-1$ respectively the matrix of the operator $\varrho(\beta_{n-k+1}^*)$, $\varrho(\gamma_{n-k+1}^*)$, $\varrho(\delta_{n-k+1}^*)$. Similarly, we denote by $\mathtt{T}_{m,k}$, $\mathtt{T}_k^2\, \mathtt{T}_{m,k+1}$, $1 \leq k \leq n-1$, $k+1 \leq m \leq n$ respectively the matrix of the operators $\varrho(t_{m,k})$, $\varrho((t_k^*)^2\, t_{m,k+1}^*)$. In particular, we denote the unit matrix corresponding to the operator $\varrho(id)$ by $\mathtt{I}$. Then the rows and columns of all introduced matrices are indexed by the elements $e_{\underline{j}}$ of the monomial basis of $\mathcal{B}_Q \subseteq \mathcal{B}$ for each $\underline{j} \in \widehat{Q}$. So these matrices are square matrices whose order is equal to $\dim \mathcal{B}_Q = \mathrm{Card}\, \widehat{Q} = n!$, where we assume that $\mathrm{Card}\, Q = n$.

Remark 4.1. Let Q be a set of cardinality n and let $\underline{j} = j_1 \ldots j_n \in \widehat{Q}$ and $\underline{k} = k_1 \ldots k_n \in \widehat{Q}$ be arbitrary permutations in the set $\widehat{Q}$ of all (distinct) permutations of the set Q. Then it is easy to verify that there exists a permutation $g \in S_n$ such that g satisfies the condition $\underline{k} = g.\underline{j}$, that is,

$$k_1 \ldots k_n = j_{g^{-1}(1)} \cdots j_{g^{-1}(n)} \tag{4.46}$$

or in the shorter form $k_p = j_{g^{-1}(p)}$ for all $1 \leq p \leq n$.

PROPOSITION 4.2. *The $(\underline{k}, \underline{j})$-entry of the matrix $\mathtt{A}_n$ is a monomial given by*

$$(\mathtt{A}_n)_{\underline{k},\underline{j}} = \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)} j_{g^{-1}(b)}} \tag{4.47}$$

where $\underline{k} = g.\underline{j}$ $(g \in S_n,\ \underline{j} = j_1 \dots j_n \in \widehat{Q},\ \underline{k} = k_1 \dots k_n \in \widehat{Q})$.

PROOF. By considering that $\mathtt{A}_n$ denotes the matrix of the operator $\varrho(\alpha_n^*)$ given by (3.38) and applying (4.46), we obtain

$$\varrho(\alpha_n^*)\, e_{\underline{j}} = \sum_{g\in S_n} \prod_{(a,b)\in I(g^{-1})} q_{j_{g^{-1}(a)} j_{g^{-1}(b)}}\, e_{g.\underline{j}}$$

from which it follows directly that the $(\underline{k}, \underline{j})$-entry of the matrix $\mathtt{A}_n$ is given by (4.47). □

We note that the operators $\varrho(\alpha_n^*) \in \mathrm{End}(\mathcal{B}_Q)$ and $\varrho((\alpha_n^*)^{-1}) \in \mathrm{End}(\mathcal{B}_Q)$ play an important role in determining the inverse of a matrix of the quantum bilinear form of the oriented braid arrangement in $\mathbb{R}^n$. Furthermore, if we assume that $\mathtt{A}_n$ denotes the matrix of the operator $\varrho(\alpha_n^*)$, then by comparing (4.47) with (1.3), we find that the matrices $\mathtt{A}_n$ and B_n^* are equal, i.e., have the same inverse matrix. In other words, computing the inverse of the matrix B_n^* leads to computing the inverse of the matrix $\mathtt{A}_n$. In this way we can write the matrix B_n^* instead of the matrix $\mathtt{A}_n$. Thus, from Proposition 4.2 it follows that the $(\underline{k}, \underline{j})$-entry of the matrix B_n^* is a monomial given by (4.47). Moreover, by applying Theorem 3.6 in a matrix notation, we obtain that the inverse $(B_n^*)^{-1}$ of B_n^* can be factorized in the following form

$$(B_n^*)^{-1} = \mathtt{B}_n^{-1} \cdot \mathtt{B}_{n-1}^{-1} \cdots \mathtt{B}_2^{-1} \left(= \prod_{1\leq k\leq n-1} \mathtt{B}_{n-k+1}^{-1} \right) \tag{4.48}$$

with $\mathtt{B}_i^{-1} = \mathtt{C}_i \cdot \mathtt{D}_i^{-1}$ for all $2 \leq i \leq n$.
Before we determine the matrix $\mathtt{B}_i^{-1}$, $2 \leq i \leq n$, we should note that the $(\underline{k}, \underline{j})$-entry of the matrix $\mathtt{T}_{m,k}$ and $\mathtt{T}_k^2\, \mathtt{T}_{m,k+1}$, $1 \leq k \leq n-1$, $k+1 \leq m \leq n$ is respectively given by

$$(\mathtt{T}_{m,k})_{\underline{k},\underline{j}} = \begin{cases} q_{j_m j_k} q_{j_m j_{k+1}} \cdots q_{j_m j_{m-1}} & \text{if } \underline{k} = t_{m,k}.\underline{j} \\ 0 & \text{otherwise} \end{cases} \tag{4.49}$$

$$\left(\mathtt{T}_k^2\, \mathtt{T}_{m,k+1}\right)_{\underline{k},\underline{j}} = \begin{cases} \sigma_{j_k j_m} q_{j_m j_{k+1}} \cdots q_{j_m j_{m-1}} & \text{if } \underline{k} = t_{m,k+1}.\underline{j} \\ 0 & \text{otherwise} \end{cases} \tag{4.50}$$

where $t_{m,k}.\underline{j} = j_{t_{k,m}(1)} \cdots j_{t_{k,m}(n)} = j_1 \dots j_m j_k j_{k+1} \dots j_{m-1} \dots j_n$

$t_{m,k+1}.\underline{j} = j_{t_{k+1,m}(1)} \cdots j_{t_{k+1,m}(n)} = j_1 \cdots j_k j_m j_{k+1} \cdots j_{m-1} \cdots j_n.$

We recall that $t_{k,m} = t^{-1}_{m,k}$, $t_{k+1,m} = t^{-1}_{m,k+1}$ and that $\mathtt{T}_k^2\,\mathtt{T}_{k+1,k+1} = \mathtt{T}_k^2$ is the diagonal matrix with $\sigma_{j_k j_{k+1}}$ as its $\underline{j}$-th diagonal entry, see (3.36), (3.37) and also Remark 3.4. Then each matrix $\mathtt{B}_{n-k+1}$, $1 \le k \le n-1$ can be written as the following sum of matrices

$$\mathtt{B}_{n-k+1} = \sum_{k \le m \le n} \mathtt{T}_{m,k}$$

where $\mathtt{T}_{k,k} = \mathtt{I}$ is the unit matrix. Thus, by applying (4.49), we obtain that the $(\underline{k}, \underline{j})$-entry of the matrix $\mathtt{B}_{n-k+1}$ is given by

$$(\mathtt{B}_{n-k+1})_{\underline{k},\underline{j}} = \begin{cases} q_{j_m j_k} q_{j_m j_{k+1}} \cdots q_{j_m j_{m-1}} & \text{if } \underline{k} = t_{m,k} \cdot \underline{j} \text{ for all } k \le m \le n \\ 0 & \text{otherwise.} \end{cases}$$

In accordance with the above, the following theorem follows.

Theorem 4.3. *The $(\underline{k}, \underline{j})$-entry of the quantum bilinear form B_n^* of the oriented braid arrangement is given by*

$$(B_n^*)_{\underline{k},\underline{j}} = \prod_{(a,b) \in I(g^{-1})} q_{j_{g^{-1}(a)} j_{g^{-1}(b)}} \tag{4.51}$$

where $\underline{j} = j_1 \ldots j_n \in \widehat{Q}$, $\underline{k} = k_1 \ldots k_n \in \widehat{Q}$ and $g \in S_n$ satisfies the condition that $k_p = j_{g^{-1}(p)}$ for all $1 \le p \le n$. Then the inverse $(B_n^*)^{-1}$ of B_n^* is given as follows

$$(B_n^*)^{-1} = \mathtt{B}_n^{-1} \cdot \mathtt{B}_{n-1}^{-1} \cdots \mathtt{B}_2^{-1}$$

with $\mathtt{B}_i^{-1} = \mathtt{C}_i \cdot \mathtt{D}_i^{-1}$, $2 \le i \le n$ *and*

$$\mathtt{C}_{n-k+1} = (\mathtt{I} - \mathtt{T}_{n,k}) \cdot (\mathtt{I} - \mathtt{T}_{n-1,k}) \cdots (\mathtt{I} - \mathtt{T}_{k+1,k})$$

$$\mathtt{D}_{n-k+1}^{-1} = \left(\mathtt{I} - \mathtt{T}_k^2\,\mathtt{T}_{k+1,k+1}\right)^{-1} \cdot \left(\mathtt{I} - \mathtt{T}_k^2\,\mathtt{T}_{k+2,k+1}\right)^{-1} \cdots \left(\mathtt{I} - \mathtt{T}_k^2\,\mathtt{T}_{n,k+1}\right)^{-1}$$

$1 \le k \le n-1$, *where the $(\underline{r}, \underline{s})$-entry of the matrix $\mathtt{D}_{n-k+1}^{-1}$, $1 \le k \le n-1$ is given by*

$$\left(\mathtt{D}_{n-k+1}^{-1}\right)_{\underline{r},\underline{s}} = \sum_{g \in S_1^k \times S_{n-k}} \frac{\prod_{i \in Des(g^{-1})} \sigma_{j_{g^{-1}(k)} \cdots j_{g^{-1}(i)}} \cdot \prod_{(a,b) \in I(g^{-1})} q_{j_{g^{-1}(a)} j_{g^{-1}(b)}}}{\prod_{k+1 \le m \le n} \left(1 - \sigma_{j_{g^{-1}(k)} j_{g^{-1}(k+1)} \cdots j_{g^{-1}(m)}}\right)}.$$

Here $\underline{r} = r_1 \ldots r_n \in \widehat{Q}$, $\underline{s} = s_1 \ldots s_n \in \widehat{Q}$ and $g \in S_1^k \times S_{n-k}$ satisfies the condition that $r_p = s_{g^{-1}(p)}$ for all $1 \le p \le n$ and $1 - \sigma_{j_{g^{-1}(k)} j_{g^{-1}(k+1)} \cdots j_{g^{-1}(m)}} \ne 0$ for all $k+1 \le m \le n$.

Remark 4.4. Taking into account [10, Lemma 4.11], where the author found the formulas for determining $\det(\mathtt{I} - \mathtt{T}_{b,a})$, $1 \le a < b \le n$ and $\det(\mathtt{I} - (\mathtt{T}_{a-1})^2\,\mathtt{T}_{b,a})$, $1 < a \le b \le n$, we get that

$$\det(\mathtt{I} - \mathtt{T}_{m,k}) = \prod_{T \in (Q; m-k+1)} (1 - \sigma_T)^{(m-k)! \cdot (n-m+k-1)!}$$

$$\det(\mathtt{I} - \mathtt{T}_k^2\,\mathtt{T}_{m,k+1}) = \prod_{T\in(Q;m-k+2)} (1-\sigma_T)^{(m-k)!\cdot(m-k+2)\cdot(n-m+k-2)!}$$

$1 \le k \le n-1$, $k+1 \le m \le n$, where we denote by

$$(Q;m) = \{T \subseteq Q \mid \mathrm{Card}\, T = m\} \quad \text{with} \quad \sigma_T = \prod_{i\neq j\in T} q_{ij}.$$

Then we get the following formulas

$$\det \mathtt{C}_{n-k+1} = \prod_{2\le m\le n-k+1}\ \prod_{T\in(Q;m)} (1-\sigma_T)^{(m-1)!\cdot(n-m)!}$$

$$\det \mathtt{D}_{n-k+1} = \prod_{2\le m\le n-k+1}\ \prod_{T\in(Q;m)} (1-\sigma_T)^{(m-2)!\cdot m\cdot(n-m)!}$$

$$\det \mathtt{B}_{n-k+1} = \prod_{2\le m\le n-k+1}\ \prod_{T\in(Q;m)} (1-\sigma_T)^{(m-2)!\cdot(n-m)!}$$

$1 \le k \le n-1$, where we used that $\det(\mathtt{B}_{n-k+1}) = \frac{\det(\mathtt{D}_{n-k+1})}{\det(\mathtt{C}_{n-k+1})}$, c.f. (3.40). Then considering (3.39), we obtain that the determinant of the quantum bilinear form B_n^* of the oriented braid arrangement is given by

$$\det B_n^* = \prod_{2\le m\le n}\ \prod_{T\in(Q;m)} (1-\sigma_T)^{(m-2)!\cdot(n-m+1)!} \tag{4.52}$$

c.f. [10, Theorem 4.12]. We recall that B_n^* and $\mathtt{A}_n$ are the same matrices, from which it follows that their determinants are the same.

Example 4.5. For $n=2$ the quantum bilinear form and its determinant of the braid arrangement B_2 are given by

$$B_2^* = \begin{matrix} e_{12} \\ e_{21} \end{matrix} \begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix}, \qquad \det B_2^* = 1 - \sigma_{12}$$

with $\sigma_{12} = q_{12}q_{21}$. If we assume that $1-\sigma_{12} \neq 0$, then B_2^* is an invertible matrix. In this trivial case it is easy to verify that

$$(B_2^*)^{-1} = \frac{1}{1-\sigma_{12}} \begin{pmatrix} 1 & -q_{12} \\ -q_{21} & 1 \end{pmatrix}$$

Example 4.6. For $n=3$ the matrix B_3^* (i.e., the quantum bilinear form of the oriented braid arrangement) has the following form

$$B_3^* = \begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix} \begin{bmatrix} 1 & q_{23} & q_{13}q_{23} & q_{12}q_{13}q_{23} & q_{12}q_{13} & q_{12} \\ q_{32} & 1 & q_{13} & q_{13}q_{12} & q_{13}q_{12}q_{32} & q_{12}q_{32} \\ q_{31}q_{32} & q_{31} & 1 & q_{12} & q_{32}q_{12} & q_{31}q_{32}q_{12} \\ q_{32}q_{31}q_{21} & q_{31}q_{21} & q_{21} & 1 & q_{32} & q_{32}q_{31} \\ q_{21}q_{31} & q_{23}q_{21}q_{31} & q_{23}q_{21} & q_{23} & 1 & q_{31} \\ q_{21} & q_{21}q_{23} & q_{21}q_{23}q_{13} & q_{23}q_{13} & q_{13} & 1 \end{bmatrix}$$

where $\det B_3^* = (1-\sigma_{12})^2 \cdot (1-\sigma_{13})^2 \cdot (1-\sigma_{23})^2 \cdot (1-\sigma_{123})$
with $\sigma_{ij} = q_{ij}q_{ji}$ and $\sigma_{123} = \sigma_{12}\sigma_{13}\sigma_{23}$, see (4.52). We have used here the

Johnson-Trotter order of permutations in S_3 given by 123, 132, 312, 321, 231, 213. Let $\det B_3^* \neq 0$, i.e., $1-\sigma_{12} \neq 0$ and $1-\sigma_{13} \neq 0$ and $1-\sigma_{23} \neq 0$ and $1-\sigma_{123} \neq 0$. Then B_3^* is an invertible matrix, so from Theorem 4.3 we get the following:

$$(B_3^*)^{-1} = \mathtt{B}_3^{-1} \cdot \mathtt{B}_2^{-1} = \left(\mathtt{C}_3 \cdot \mathtt{D}_3^{-1}\right) \cdot \left(\mathtt{C}_2 \cdot \mathtt{D}_2^{-1}\right).$$

with $\mathtt{C}_2 = \mathtt{I} - \mathtt{T}_{3,2}$, $\quad \mathtt{D}_2^{-1} = \left(\mathtt{I} - \mathtt{T}_2^2\right)^{-1}$, $\quad \mathtt{C}_3 = (\mathtt{I} - \mathtt{T}_{3,1}) \cdot (\mathtt{I} - \mathtt{T}_{2,1})$,

$$\mathtt{D}_3^{-1} = \left(\left(\mathtt{I} - \mathtt{T}_1^2\,\mathtt{T}_{3,2}\right) \cdot \left(\mathtt{I} - \mathtt{T}_1^2\right)\right)^{-1} = \left(\mathtt{I} - \mathtt{T}_1^2\right)^{-1} \cdot \left(\mathtt{I} - \mathtt{T}_1^2\,\mathtt{T}_{3,2}\right)^{-1},$$

where $\mathtt{T}_2^2 = \mathtt{T}_2^2\,\mathtt{T}_{3,3}$ and $\mathtt{T}_1^2 = \mathtt{T}_1^2\,\mathtt{T}_{2,2}$. We first calculate $\mathtt{B}_3^{-1}$ and then $\mathtt{B}_2^{-1}$, see also Example 3.7. Thus we obtain:

$$\mathtt{I} - \mathtt{T}_{3,1} = \begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & -q_{12}q_{13} & 0 \\ 0 & 1 & 0 & -q_{13}q_{12} & 0 & 0 \\ -q_{31}q_{32} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -q_{32}q_{31} \\ 0 & 0 & -q_{23}q_{21} & 0 & 1 & 0 \\ 0 & -q_{21}q_{23} & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{I} - \mathtt{T}_{2,1} = \begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -q_{12} \\ 0 & 1 & -q_{13} & 0 & 0 & 0 \\ 0 & -q_{31} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -q_{32} & 0 \\ 0 & 0 & 0 & -q_{23} & 1 & 0 \\ -q_{21} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{C}_3 = \begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & q_{12}q_{13}q_{23} & -q_{12}q_{13} & -q_{12} \\ 0 & 1 & -q_{13} & -q_{13}q_{12} & q_{13}q_{12}q_{32} & 0 \\ -q_{31}q_{32} & -q_{31} & 1 & 0 & 0 & q_{31}q_{32}q_{12} \\ q_{32}q_{31}q_{21} & 0 & 0 & 1 & -q_{32} & -q_{32}q_{31} \\ 0 & q_{23}q_{21}q_{31} & -q_{23}q_{21} & -q_{23} & 1 & 0 \\ -q_{21} & -q_{21}q_{23} & q_{21}q_{23}q_{13} & 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{I} - \mathtt{T}_1^2\,\mathtt{T}_{3,2} = \begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix} \begin{bmatrix} 1 & -\sigma_{12}q_{23} & 0 & 0 & 0 & 0 \\ -\sigma_{13}q_{32} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sigma_{13}q_{12} & 0 & 0 \\ 0 & 0 & -\sigma_{23}q_{21} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\sigma_{23}q_{31} \\ 0 & 0 & 0 & 0 & -\sigma_{12}q_{13} & 1 \end{bmatrix}$$

$$\mathtt{I} - \mathtt{T}_1^2 = \begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix} \begin{bmatrix} 1-\sigma_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\sigma_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\sigma_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\sigma_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\sigma_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\sigma_{12} \end{bmatrix}$$

$$\left(\mathtt{I}-\mathtt{T}_1^2\,\mathtt{T}_{3,2}\right)^{-1}=\frac{1}{1-\sigma_{123}}\begin{bmatrix} 1 & \sigma_{12}q_{23} & 0 & 0 & 0 & 0 \\ \sigma_{13}q_{32} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma_{13}q_{12} & 0 & 0 \\ 0 & 0 & \sigma_{23}q_{21} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \sigma_{23}q_{31} \\ 0 & 0 & 0 & 0 & \sigma_{12}q_{13} & 1 \end{bmatrix}$$

$$\left(\mathtt{I}-\mathtt{T}_1^2\right)^{-1}=\begin{bmatrix} \frac{1}{1-\sigma_{12}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-\sigma_{13}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\sigma_{13}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\sigma_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\sigma_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\sigma_{12}} \end{bmatrix}$$

$$\mathtt{D}_3^{-1}=\begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix}\begin{bmatrix} \frac{1}{1-\sigma_{12}} & \frac{\sigma_{12}q_{23}}{1-\sigma_{12}} & 0 & 0 & 0 & 0 \\ \frac{\sigma_{13}q_{32}}{1-\sigma_{13}} & \frac{1}{1-\sigma_{13}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\sigma_{13}} & \frac{\sigma_{13}q_{12}}{1-\sigma_{13}} & 0 & 0 \\ 0 & 0 & \frac{\sigma_{23}q_{21}}{1-\sigma_{23}} & \frac{1}{1-\sigma_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\sigma_{23}} & \frac{\sigma_{23}q_{31}}{1-\sigma_{23}} \\ 0 & 0 & 0 & 0 & \frac{\sigma_{12}q_{13}}{1-\sigma_{12}} & \frac{1}{1-\sigma_{12}} \end{bmatrix}\cdot\frac{1}{1-\sigma_{123}}$$

The multiplication of the obtained matrices $\mathtt{C}_3$ and $\mathtt{D}_3^{-1}$ results in $\mathtt{B}_3^{-1}=\mathtt{C}_3\cdot\mathtt{D}_3^{-1}=\frac{1}{1-\sigma_{123}}\cdot$

$$\left[\begin{matrix} \frac{1}{1-\sigma_{12}} & \frac{q_{23}\sigma_{12}}{1-\sigma_{12}} & \frac{q_{13}q_{23}\sigma_{23}\sigma_{12}}{1-\sigma_{23}} \\ \frac{q_{32}\sigma_{13}}{1-\sigma_{13}} & \frac{1}{1-\sigma_{13}} & -\frac{q_{13}(1-\sigma_{23}+\sigma_{23}\sigma_{12}-\sigma_{123})}{(1-\sigma_{13})(1-\sigma_{23})} \\ -\frac{q_{31}q_{32}(1-\sigma_{12}\sigma_{13})}{(1-\sigma_{12})(1-\sigma_{13})} & -\frac{q_{31}(1-\sigma_{12}+\sigma_{12}\sigma_{23}-\sigma_{123})}{(1-\sigma_{13})(1-\sigma_{12})} & \frac{1}{1-\sigma_{13}} \\ \frac{q_{32}q_{31}q_{21}}{1-\sigma_{12}} & \frac{q_{31}q_{21}\sigma_{12}\sigma_{23}}{1-\sigma_{12}} & \frac{q_{21}\sigma_{23}}{1-\sigma_{23}} \\ \frac{q_{21}q_{31}\sigma_{13}\sigma_{23}}{1-\sigma_{13}} & \frac{q_{23}q_{21}q_{31}}{1-\sigma_{13}} & -\frac{q_{23}q_{21}(1-\sigma_{13}\sigma_{23})}{(1-\sigma_{13})(1-\sigma_{23})} \\ -\frac{q_{21}(1-\sigma_{13}+\sigma_{13}\sigma_{23}-\sigma_{123})}{(1-\sigma_{12})(1-\sigma_{13})} & -\frac{q_{21}q_{23}(1-\sigma_{12}\sigma_{13})}{(1-\sigma_{12})(1-\sigma_{13})} & \frac{q_{21}q_{23}q_{13}}{1-\sigma_{13}} \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{q_{12}q_{13}q_{23}}{1-\sigma_{23}} & -\frac{q_{12}q_{13}(1-\sigma_{12}\sigma_{23})}{(1-\sigma_{12})(1-\sigma_{23})} & -\frac{q_{12}(1-\sigma_{23}+\sigma_{23}\sigma_{13}-\sigma_{123})}{(1-\sigma_{12})(1-\sigma_{23})} \\ -\frac{q_{13}q_{12}(1-\sigma_{13}\sigma_{23})}{(1-\sigma_{13})(1-\sigma_{23})} & \frac{q_{13}q_{12}q_{32}}{1-\sigma_{23}} & \frac{q_{12}q_{32}\sigma_{23}\sigma_{13}}{1-\sigma_{23}} \\ \frac{q_{12}\sigma_{13}}{1-\sigma_{13}} & \frac{q_{32}q_{12}\sigma_{12}\sigma_{13}}{1-\sigma_{12}} & \frac{q_{31}q_{32}q_{12}}{1-\sigma_{12}} \\ \frac{1}{1-\sigma_{23}} & -\frac{q_{32}(1-\sigma_{12}+\sigma_{12}\sigma_{13}-\sigma_{123})}{(1-\sigma_{23})(1-\sigma_{12})} & -\frac{q_{32}q_{31}(1-\sigma_{12}\sigma_{23})}{(1-\sigma_{12})(1-\sigma_{23})} \\ -\frac{q_{23}(1-\sigma_{13}+\sigma_{13}\sigma_{12}-\sigma_{123})}{(1-\sigma_{23})(1-\sigma_{13})} & \frac{1}{1-\sigma_{23}} & \frac{q_{31}\sigma_{23}}{1-\sigma_{23}} \\ \frac{q_{23}q_{13}\sigma_{13}\sigma_{12}}{1-\sigma_{13}} & \frac{q_{13}\sigma_{12}}{1-\sigma_{12}} & \frac{1}{1-\sigma_{12}} \end{matrix}\right]$$

(c.f. (3.43)), where we assume that the rows and columns are indexed in the order 123, 132, 312, 321, 231, 213. We now calculate $\mathtt{B}_2^{-1}$, taking into account that $\mathtt{B}_2^{-1}=\mathtt{C}_2\cdot\mathtt{D}_2^{-1}$ with $\mathtt{C}_2=\mathtt{I}-\mathtt{T}_{3,2}$ and $\mathtt{D}_2^{-1}=\left(\mathtt{I}-\mathtt{T}_2^2\right)^{-1}$. Thus, we obtain:

$$\mathtt{C}_2 = \mathtt{I} - \mathtt{T}_{3,2} = \begin{array}{c} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{array} \begin{bmatrix} 1 & -q_{23} & 0 & 0 & 0 & 0 \\ -q_{32} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -q_{12} & 0 & 0 \\ 0 & 0 & -q_{21} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -q_{31} \\ 0 & 0 & 0 & 0 & -q_{13} & 1 \end{bmatrix}.$$

We note that $\mathtt{D}_2 = \mathtt{I} - \mathtt{T}_2^2$ is a diagonal matrix, therefore its inverse $\mathtt{D}_2^{-1} = \left(\mathtt{I} - \mathtt{T}_2^2\right)^{-1}$ is also a diagonal matrix, so that:

$$\mathtt{D}_2^{-1} = \left(\mathtt{I} - \mathtt{T}_2^2\right)^{-1} = \begin{array}{c} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{array} \begin{bmatrix} \frac{1}{1-\sigma_{23}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{1-\sigma_{23}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\sigma_{12}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\sigma_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\sigma_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\sigma_{13}} \end{bmatrix}.$$

The multiplication of the obtained matrices $\mathtt{C}_2$ and $\mathtt{D}_2^{-1}$ gives then

$$\mathtt{B}_2^{-1} = \begin{bmatrix} \frac{1}{1-\sigma_{23}} & -\frac{q_{23}}{1-\sigma_{23}} & 0 & 0 & 0 & 0 \\ -\frac{q_{32}}{1-\sigma_{23}} & \frac{1}{1-\sigma_{23}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1-\sigma_{12}} & -\frac{q_{12}}{1-\sigma_{12}} & 0 & 0 \\ 0 & 0 & -\frac{q_{21}}{1-\sigma_{12}} & \frac{1}{1-\sigma_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{1-\sigma_{13}} & -\frac{q_{31}}{1-\sigma_{13}} \\ 0 & 0 & 0 & 0 & -\frac{q_{13}}{1-\sigma_{13}} & \frac{1}{1-\sigma_{13}} \end{bmatrix}$$

c.f. (3.44). In agreement with the obtained matrices $\mathtt{B}_3^{-1}$ and $\mathtt{B}_2^{-1}$, it follows that the inverse $(B_3^*)^{-1} = \mathtt{B}_3^{-1} \cdot \mathtt{B}_2^{-1}$ of the quantum bilinear form of the oriented braid arrangement in $\mathbb{R}^3$ is given in the following form
$(B_3^*)^{-1} = \frac{1}{(1-\sigma_{12})(1-\sigma_{13})(1-\sigma_{23})(1-\sigma_{123})}$.

$$\left[\begin{array}{ccc} (1-\sigma_{13})(1-\sigma_{12}\sigma_{23}) & -q_{23}(1-\sigma_{12})(1-\sigma_{13}) & -q_{13}q_{23}\sigma_{12}(1-\sigma_{13})(1-\sigma_{23}) \\ -q_{32}(1-\sigma_{12})(1-\sigma_{13}) & (1-\sigma_{12})(1-\sigma_{13}\sigma_{23}) & -q_{13}(1-\sigma_{12})(1-\sigma_{23}) \\ -q_{31}q_{32}\sigma_{12}(1-\sigma_{13})(1-\sigma_{23}) & -q_{31}(1-\sigma_{12})(1-\sigma_{23}) & (1-\sigma_{23})(1-\sigma_{12}\sigma_{13}) \\ q_{32}q_{31}q_{21}(1-\sigma_{13})(1-\sigma_{12}\sigma_{23}) & -q_{31}q_{21}\sigma_{23}(1-\sigma_{12})(1-\sigma_{13}) & -q_{21}(1-\sigma_{13})(1-\sigma_{23}) \\ -q_{21}q_{31}\sigma_{23}(1-\sigma_{12})(1-\sigma_{13}) & q_{23}q_{21}q_{31}(1-\sigma_{12})(1-\sigma_{13}\sigma_{23}) & -q_{23}q_{21}\sigma_{13}(1-\sigma_{12})(1-\sigma_{23}) \\ -q_{21}(1-\sigma_{13})(1-\sigma_{23}) & -q_{21}q_{23}\sigma_{13}(1-\sigma_{12})(1-\sigma_{23}) & q_{21}q_{23}q_{13}(1-\sigma_{23})(1-\sigma_{12}\sigma_{13}) \end{array}\right.$$

$$\left.\begin{array}{ccc} q_{12}q_{13}q_{23}(1-\sigma_{13})(1-\sigma_{12}\sigma_{23}) & -q_{12}q_{13}\sigma_{23}(1-\sigma_{12})(1-\sigma_{13}) & -q_{12}(1-\sigma_{13})(1-\sigma_{23}) \\ -q_{13}q_{12}\sigma_{23}(1-\sigma_{12})(1-\sigma_{13}) & q_{13}q_{12}q_{32}(1-\sigma_{12})(1-\sigma_{13}\sigma_{23}) & -q_{12}q_{32}\sigma_{13}(1-\sigma_{12})(1-\sigma_{23}) \\ -q_{12}(1-\sigma_{13})(1-\sigma_{23}) & -q_{32}q_{12}\sigma_{13}(1-\sigma_{12})(1-\sigma_{23}) & q_{31}q_{32}q_{12}(1-\sigma_{23})(1-\sigma_{12}\sigma_{13}) \\ (1-\sigma_{13})(1-\sigma_{12}\sigma_{23}) & -q_{32}(1-\sigma_{12})(1-\sigma_{13}) & -q_{32}q_{31}\sigma_{12}(1-\sigma_{13})(1-\sigma_{23}) \\ -q_{23}(1-\sigma_{12})(1-\sigma_{13}) & (1-\sigma_{12})(1-\sigma_{13}\sigma_{23}) & -q_{31}(1-\sigma_{12})(1-\sigma_{23}) \\ -q_{23}q_{13}\sigma_{12}(1-\sigma_{13})(1-\sigma_{23}) & -q_{13}(1-\sigma_{12})(1-\sigma_{23}) & (1-\sigma_{23})(1-\sigma_{12}\sigma_{13}) \end{array}\right]$$

where the rows and columns are indexed in the order 123, 132, 312, 321, 231, 213; compare with (3.45).

REMARK 4.7. We note that the matrices $\mathtt{B}_3$ and $\mathtt{B}_2$ are given by

$$\mathtt{B}_3 = \begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & q_{12}q_{13} & q_{12} \\ 0 & 1 & q_{13} & q_{13}q_{12} & 0 & 0 \\ q_{31}q_{32} & q_{31} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & q_{32} & q_{32}q_{31} \\ 0 & 0 & q_{23}q_{21} & q_{23} & 1 & 0 \\ q_{21} & q_{21}q_{23} & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathtt{B}_2 = \begin{matrix} e_{123} \\ e_{132} \\ e_{312} \\ e_{321} \\ e_{231} \\ e_{213} \end{matrix} \begin{bmatrix} 1 & q_{23} & 0 & 0 & 0 & 0 \\ q_{32} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & q_{12} & 0 & 0 \\ 0 & 0 & q_{21} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & q_{31} \\ 0 & 0 & 0 & 0 & q_{13} & 1 \end{bmatrix}$$

where $\det \mathtt{B}_3 = (1-\sigma_{12}) \cdot (1-\sigma_{13}) \cdot (1-\sigma_{23}) \cdot (1-\sigma_{123})$,
$\det \mathtt{B}_2 = (1-\sigma_{12}) \cdot (1-\sigma_{13}) \cdot (1-\sigma_{23})$.

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First author
Faculty of Mathematics
University of Rijeka
51 000 Rijeka
Croatia
E-mail: msosic@uniri.hr