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Ema Jurkin

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EQUIDISTANT CURVE OF CONICS IN ISOTROPIC PLANE

Ema Jurkin
University of Zagreb, Croatia

Abstract. In this paper we introduce the concept of equidistant curve of two curves in an isotropic plane. We study the properties of equidistant curve of conics and classify them according to their type of circularity.

1. Motivation

In the Euclidean plane the bisector of an angle is defined as the line that divides the angle into two equal parts. Each point of an angle bisector is equidistant from the sides of the angle. That's why the bisector of two planar curves is usually defined as the set of all points in the plane that are at the same distance from two given curves.

It is well-known that the set of points equidistant from two intersecting lines is a pair of perpendicular lines as well as that a parabola is the set of points that are equidistant from a point and a line. It is less known that ellipses and single branches of hyperbolas admit similar definitions, [10]. In [7] it was showed that all conics can be realized as the equidistant set of two circles. The set of points equidistant from two nested circles is an ellipse with foci at their centers. The set of points equidistant from two disjoint, not nested, circles of different sizes, is that branch of a hyperbola with foci at their centers which opens around the smaller circle. If the radii of circles are equal, a straight line is obtained.

In [5] the authors studied the equidistant sets of a conic and a line. They showed that it is a part of a curve of degree 8 if the conic is an ellipse or a hyperbola, and it is a part of a curve of degree 6 in the case of parabola. The degree of obtained curve decreases if the line touches the conic.

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The authors of [1] consider $\mathcal{C}^1$-continuous plane rational curves, and present a method of elimination to obtain a representation of the bisector in terms of the parameters of the initial curves. In [2] a general theoretical treatment, from the perspective of the algebraic geometry, of the untrimmed bisector of two real algebraic plane curves is presented. The trimmed bisector of two curves is the locus of those points being at the same distance from the two curves. The untrimmed bisector is the locus of those points that, being on the normal lines to both curves, are at the same distance from the two foot-points in the intersection of each curve with the normal line. The points in the untrimmed bisector are the centers of the circles which are tangent to both curves. The trimmed bisector is a subset of the untrimmed bisector, [2].

Let us now consider the situation in an isotropic plane. In such a plane the bisector t of two lines a and b is defined as a line such that $\angle(a,t) = \angle(t,b)$. If an isotropic line intersects lines a, b and their bisector in points A, B and T respectively, then $s(A,T) = s(T,B)$, where s denotes a span whose definition is given in Section 2. For bisector defined in such a way does not hold that it is a set of points equidistant from two given lines. For this reason it makes sense to observe two types of curves, the bisectors and the equidistant curves. The bisector of the algebraic curves was defined and studied in [3]. In this paper we offer the definition of the equidistant curve of algebraic curves and we study the equidistant curves of two conics.

2. Introduction

Let us start by recalling some basic definitions in an isotropic plane. The isotropic plane is a real projective plane where the metric is induced by a real line f and a real point F incident with it. The lines incident with the absolute point F are called isotropic lines, and the points incident with the absolute line f are called isotropic points. Two lines are parallel if they are incident with the same isotropic point, and two points are parallel if they are incident with same isotropic line.

In the affine model of the isotropic plane where the coordinates of points are defined by $x = \dfrac{x_1}{x_0}$, $y = \dfrac{x_2}{x_0}$, the absolute line has the equation $x_0 = 0$ and the absolute point has the coordinates $(0,0,1)$. For two non-parallel points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ a distance is defined by $d(A,B) = x_B - x_A$, and for two parallel points $A = (x, y_A)$ and $B = (x, y_B)$ a span is defined by $s(A,B) = y_B - y_A$. Two non-parallel lines p and q, given by the equations $y = k_p x + l_p$ and $y = k_q x + l_q$, form an angle defined by $\angle(p,q) = k_q - k_p$, [6], [8]. The midpoint of points A and B has coordinates $\left(\frac{1}{2}(x_A + x_B), \frac{1}{2}(y_A + y_B)\right)$, while the bisector of lines p and q is a line given by the equation $y = \frac{1}{2}(k_p + k_q)x + \frac{1}{2}(l_p + l_q)$. If an isotropic line i intersects lines p, q and their bisector in points A, B and T respectively, then $s(A,T) = s(T,B)$.

This fact was used as a definition of the bisector of the algebraic curves of general degree, [3].

3. Equidistant curve of two curves

Let the curves $\mathcal{A}$ and $\mathcal{B}$ of degrees n and m, respectively, be given. Every (horizontal) line $y = t$ intersects $\mathcal{A}$ in n points A_i, $i = 1 \ldots n$, and $\mathcal{B}$ in m points B_j, $j = 1, \ldots m$. Let M_{ij} be midpoints of points A_i and B_j. All points M_{ij} belong to the *equidistant curve* $\mathcal{E}$ of curves $\mathcal{A}$ and $\mathcal{B}$. Since every isotropic line intersects $\mathcal{E}$ in $n \cdot m$ points, we can conclude that $\mathcal{E}$ is a curve of degree $n \cdot m$.

Example 3.1. The equidistant curve of lines p and q given by the equations $y = k_p x + l_p$ and $y = k_q x + l_q$, respectively, is the line given by the equation

$$2k_p k_q x - (k_p + k_l)y + k_p l_q + k_q l_p = 0.$$

4. Equidistant curve of conics

Let conics $\mathcal{A}$ and $\mathcal{B}$ be given by the equations of the form

$$\mathcal{A} \ldots a_{00} + a_{11}x^2 + a_{22}y^2 + 2a_{01}x + 2a_{02}y + 2a_{12}xy = 0$$
$$\mathcal{B} \ldots b_{00} + b_{11}x^2 + b_{22}y^2 + 2b_{01}x + 2b_{02}y + 2b_{12}xy = 0.$$

In general, $a_{11}, b_{11} \neq 0$, the equations of $\mathcal{A}$ and $\mathcal{B}$ can be rewritten as

$$\mathcal{A} \ldots x^2 + 2p_1(y)x + p_2(y) = 0, \tag{4.1}$$

$$\mathcal{B} \ldots x^2 + 2q_1(y)x + q_2(y) = 0, \tag{4.2}$$

where p_i, q_i are polinomials of degree i in y, i.e.

$$\begin{aligned} p_1(y) &= a_{12}y + a_{01}, \quad p_2(y) = a_{22}y^2 + 2a_{02}y + a_{00}, \\ q_1(y) &= b_{12}y + b_{01}, \quad q_2(y) = b_{22}y^2 + 2b_{02}y + b_{00}. \end{aligned} \tag{4.3}$$

After intersecting $\mathcal{A}$ and $\mathcal{B}$ with the line

$$y = t \tag{4.4}$$

we get the intersection points

$$A_{1,2} = \left(-p_1 \pm \sqrt{p_1^2 - p_2}, t\right), \quad B_{1,2} = \left(-q_1 \pm \sqrt{q_1^2 - q_2}, t\right)$$

The midpoints M_{ij}, $i, j \in \{1, 2\}$, have coordinates

$$\left(\frac{-p_1 \pm \sqrt{p_1^2 - p_2} - q_1 \pm \sqrt{q_1^2 - q_2}}{2}, t\right),$$

for four combinations of signs. For all four parametrizations above, the following implicit equation of unique quartic curve $\mathcal{E}$ is obtained:

$$\left[(2x + p_1 + q_1)^2 - p_1^2 - q_1^2 + p_2 + q_2\right]^2 = 4(p_1^2 - p_2)(q_1^2 - q_2), \tag{4.5}$$

where p_i, q_i are given by (4.3). The term of (4.5) of the highest degree equals (4.6)

$$\left[(2x + (a_{12} + b_{12})\, y)^2 + \left(a_{22} + b_{22} - a_{12}^2 - b_{12}^2\right) y^2\right]^2 - 4(a_{12}^2 - a_{22})(b_{12}^2 - b_{22})y^4,$$

which equals

(4.7)

$$\begin{aligned}
&16x^4 + 32(a_{12} + b_{12})x^3y + 8\left(2a_{12}^2 + 2b_{12}^2 + 3a_{12}b_{12} + a_{22} + b_{22}\right) x^2y^2 + \\
&+ 4\,(a_{12}+b_{12})(2a_{12}b_{12}+a_{22}+b_{22})\, xy^3 + \\
&+ \left((2a_{12}b_{12}+a_{22}+b_{22})^2 - 4(a_{12}^2-a_{22})(b_{12}^2-b_{22})\right) y^4.
\end{aligned}$$

Since the deqree of circularety of a curve is defined as a number of its intersection points with the absolute line that fall into the absolute point, [4], the degree of circularity of the obtained quartic obviously depends on the degree of circularity of initial conics.

THEOREM 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be r- and s-circular conics, respectively. The equidistant curve $\mathcal{E}$ of $\mathcal{A}$ and $\mathcal{B}$ is an rs-circular quartic.*

PROOF. Let $\mathcal{A}$ and $\mathcal{B}$ be 1-circular conics, i.e. special hyperbolas. Then $a_{22} = b_{22} = 0$ and the coefficient of y^4 in (4.7) vanishes. Homogenizing ($x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$) equation (4.5) and setting $x_0 = 0$ yields the intersections of $\mathcal{E}$ with the absolute line f as

$$\begin{aligned}
(4.8) \qquad &2x_1^4 + 4(a_{12} + b_{12})x_1^3x_2 + \left(2a_{12}^2 + 2b_{12}^2 + 3a_{12}b_{12}\right) x_1^2x_2^2 + \\
&+ a_{12}b_{12}\,(a_{12} + b_{12})\, x_1x_2^3 = 0.
\end{aligned}$$

The solution $x_1 = 0$ corresponds to the absolute point $(0, 0, 1)$. Thus, quartic $\mathcal{E}$ is circular.

If $\mathcal{A}$ is 1-circular conic (special hyperbola) and $\mathcal{B}$ be 2-circular conic (circle), than additionally $b_{12} = 0$ holds. Thus, (4.8) turns into

$$(4.9) \qquad 2x_1^4 + 4a_{12}x_1^3x_2 + 2a_{12}^2x_1^2x_2^2 = 0,$$

which is equivalent to

$$(4.10) \qquad x_1^2(x_1 + a_{12}x_2)^2 = 0.$$

Now, $x_1 = 0$ is root with multiplicity 2, and the absolute point $(0, 0, 1)$ is the intersection of $\mathcal{E}$ and f with the intersection multiplicity 2. Similarly we conclude that the isotropic point $(0, -a_{12}, 1)$ is also the intersection point of $\mathcal{E}$ and f with the intersection multiplicity 2.

If both conics $\mathcal{A}$ and $\mathcal{B}$ are circles, than $a_{12} = 0$ also holds, and (4.9) becomes

$$(4.11) \qquad x_1^4 = 0.$$

Thus, all four intersection points of $\mathcal{E}$ with the absolute line coincide with the absolute point.

It can be check similarly that in the cases when at least one of the conic is non-circular, the obtained quartic is in general also non-circular. $\square$

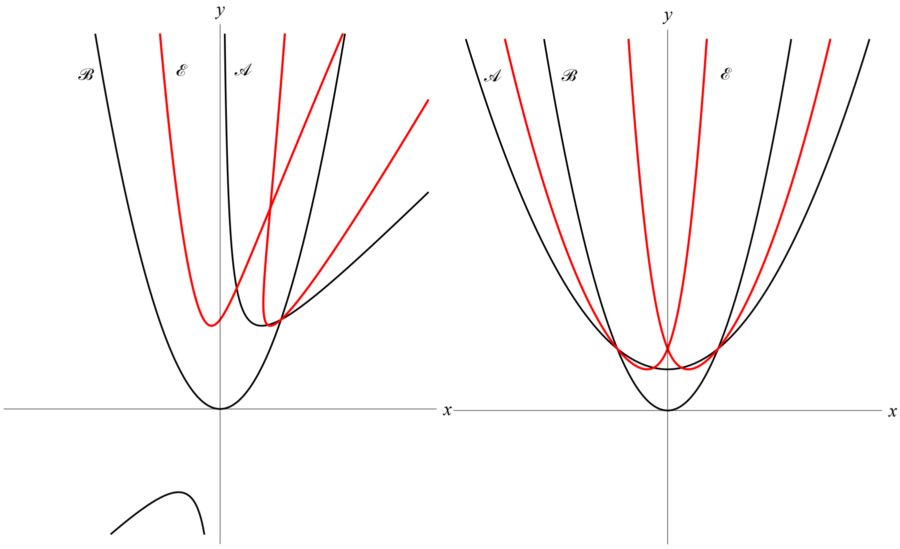

FIGURE 1. The equidistant curve $\mathcal{E}$ of the circle $\mathcal{A}$ and special hyperbola $\mathcal{B}$ (left). The equidistant curve $\mathcal{E}$ of the circles $\mathcal{A}$ and $\mathcal{B}$ (right).

The equidistant curve $\mathcal{E}$ of the circle $\mathcal{B}$ with equation $x^2-y=0$ and special hyperbola $\mathcal{A}$ with equation $x^2-xy+1=0$ is depicted in Fig. 1 (left). It is given by $16x^4-16x^3y+4x^2y^2+16x^3-16x^2y+4xy^2-y^3+12x^2-8xy+3y^2+4x+y+1=0$ and touches the absolute line at the absolute point and at the isotropic point of the line $y=2x$.
The equidistant curve $\mathcal{E}$ of the circles $\mathcal{A}$ and $\mathcal{B}$ with equations $x^2-4y+3=0$ and $x^2-y=0$, respectively, is shown in Fig. 1 (right). It is given by the equation $16x^4-40x^2y+24x^2+9y^2-18y+9=0$. The absolute point is its double point at which the absolute line touches its both branches. Such a double point is classified as a singular point of a higher order, [9].

In our study we assumed that $a_{11}, b_{11} \neq 0$, i.e. we omitted the cases when conics $\mathcal{A}$ and $\mathcal{B}$ pass through the isotropic point $(0,1,0)$. Since it is the isotropic point all horizontal lines pass through, exactly in that cases equidistant curve degenerates into a cubic or conic.

THEOREM 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two conics and $\mathcal{E}$ their equidistant curve.*

- *If $\mathcal{A}$ and $\mathcal{B}$ pass through the isotropic point $(0,1,0)$, then $\mathcal{E}$ is a cubic with a node at $(0,1,0)$. Particularly, if $\mathcal{A}$ and $\mathcal{B}$ touch each other at $(0,1,0)$, $\mathcal{E}$ splits onto a conic and the common tangent line.*
- *If $\mathcal{A}$ and $\mathcal{B}$ pass through the isotropic point $(0,1,0)$ and one of them touches the absolute line at $(0,1,0)$, then $\mathcal{E}$ is a cubic with a node at $(0,1,0)$ at which one of the tangent lines coincides with the absolute line.*
- *If $\mathcal{A}$ and $\mathcal{B}$ touch the absolute line at $(0,1,0)$, then $\mathcal{E}$ is a parabola touching the absolute line at the same point.*

PROOF. If $\mathcal{A}$ and $\mathcal{B}$ pass through the isotropic point $I=(0,1,0)$ ($a_{11}=b_{11}=0$), then they have equations of the form

$$\begin{aligned}\mathcal{A}\ldots 2p_1(y)x+p_2(y)=0,\\ \mathcal{B}\ldots 2q_1(y)x+q_2(y)=0,\end{aligned}$$

where p_i, q_i are given by (4.3). The intersection points A, B of $\mathcal{A}$, $\mathcal{B}$ with the line $y=t$ are

$$A=\left(-\frac{p_2}{2p_1},t\right),\quad B=\left(-\frac{q_2}{2q_1},t\right).$$

There midpoint M has coordinates

$$M=\left(-\frac{p_1q_2+p_2q_1}{4p_1q_1},t\right)$$

and lies on the cubic $\mathcal{E}$ with equation

$$4p_1q_1x+p_1q_2+p_2q_1=0. \tag{4.12}$$

If $\mathcal{A}$ and $\mathcal{B}$ have the common tangent line $y=l$ at I, then $a_{01}=-a_{12}l$ and $b_{01}=-b_{12}l$, and therefore $p_1(y)=a_{12}(y-l)$ and $q_1(y)=b_{12}(y-l)$. Now (4.12) takes the form

$$(y-l)\left[4a_{12}b_{12}(y-l)x+a_{12}q_2+b_{12}p_2\right]=0. \tag{4.13}$$

Thus, we have proved the first part of Theorem 2.

Let us now assume that $\mathcal{A}$ touches absolute line f at I, i.e. $a_{12}=0$. From (4.3) and (4.12), the following equation of $\mathcal{E}$ is obtained:

(4.14)

$$4a_{01}(b_{12}y+b_{01})x+a_{01}(b_{22}y^2+2b_{02}y+b_{00})+(b_{12}y+b_{01})(a_{22}y^2+2a_{02}y+a_{00})=0.$$

Homogenizing equation (4.16) and setting $x_0=0$ yields the intersections of $\mathcal{E}$ with the absolute line f as

$$b_{12}a_{22}x_2^3=0. \tag{4.15}$$

Thus, I is the intersection point of $\mathcal{E}$ and f with intersection multiplicity 3. Obviously every line $y=m$ through I intersects $\mathcal{E}$ at a unique proper point, while two other intersection points fall into the isotropic point I. I is therefore a node of $\mathcal{E}$, Fig. 2.

It is left to study the case when both $\mathcal{A}$ and $\mathcal{B}$ touch the absolute line f at I, i.e. $a_{12} = b_{12} = 0$. The equation (4.16) becomes

$$4a_{01}b_{01}x + a_{01}(b_{22}y^2 + 2b_{02}y + b_{00}) + b_{01}(a_{22}y^2 + 2a_{02}y + a_{00}) = 0. \tag{4.16}$$

The equation above represents a parabola touching the absolute line at I. $\square$

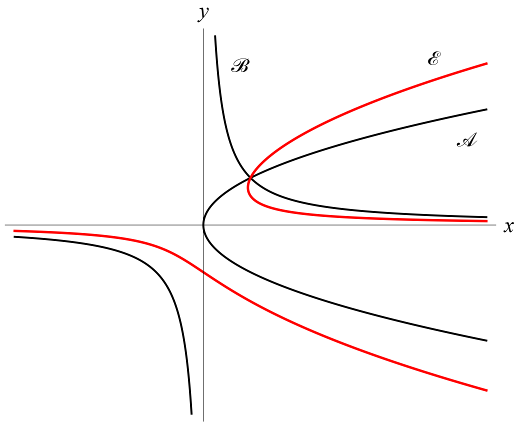

FIGURE 2. The equidistant curve $\mathcal{E}$ of conics $\mathcal{A}$ and $\mathcal{B}$ through isotropic point $(0, 1, 0)$.

In Fig. 2 the equidistant curve $\mathcal{E}$ of the parabola $\mathcal{A}$ with equation $x = y^2$ and special hyperbola $\mathcal{B}$ with equation $xy = 1$ is depicted. It is a cubic curve given by equation $y^3 - 2xy + 1 = 0$.

5. Equidistant curve of conic and line

Let us for the end study the equidistant curve of a line and a conic. It should be noticed that such a curve is not well defined for the horizontal line.

THEOREM 5.1. *Let $\mathcal{A}$ be a conic and $\mathcal{B}$ a non-horizontal line. For the equidistant curve $\mathcal{E}$ of $\mathcal{A}$ and $\mathcal{B}$ the following statements hold:*

- *If $\mathcal{B}$ is a non-isotropic line, then $\mathcal{E}$ is in general a 0-circular conic.*
- *If $\mathcal{B}$ is an isotropic line, then $\mathcal{E}$ is a conic of the same degree of circularity as $\mathcal{A}$.*

PROOF. Let $\mathcal{A}$ be given by equation (4.1) and $\mathcal{B}$ by

$$y = kx + l.$$

The line $y = t$ intersects $\mathcal{A}$ in the points $\left(-p_1 \pm \sqrt{p_1^2 - p_2}, t\right)$ and $\mathcal{B}$ in the point $\left(\frac{y-l}{k}, t\right)$. Using the same method as in the previous proofs we get the equation of $\mathcal{E}$ as

$$[k(2x + p_1) + (l - y)]^2 = k^2(p_1^2 - p_2). \tag{5.17}$$

Since the term of the second degree equals

$$4k^2x^2 + 4k(ka_{12} - 1)xy + (k^2a_{22} - 2ka_{12} + 1)y^2,$$

conic $\mathcal{E}$ is in general non-circular, see Fig. 3.

If $\mathcal{B}$ is an isotropic line given by

$$x = c,$$

the equation of $\mathcal{E}$ is

$$(2x + p_1 - c)^2 = p_1^2 - p_2. \tag{5.18}$$

The term of the second degree of (5.18) equals

$$4x^2 + 4a_{12}xy + a_{22}y^2. \tag{5.19}$$

Therefore, after switching to homogeneous coordinates and inserting $x_0 = 0$ into (5.18), we get the intersections of $\mathcal{E}$ and the absolute line f as

$$4x_1^2 + 4a_{12}x_1x_2 + a_{22}x_2^2 = 0. \tag{5.20}$$

If $\mathcal{A}$ is a circular conic, $a_{22} = 0$, (5.20) becomes

$$4x_1(x_1 + a_{12}x_2) = 0.$$

The solution $x_1 = 0$ corresponds to the absolute point $(0, 0, 1)$. Therefore, $\mathcal{E}$ is a circular conic as well. If $\mathcal{A}$ is 2-circular conic, $a_{22} = a_{12} = 0$, (5.20) turns to

$$4x_1^2 = 0.$$

Thus, the absolute point is the intersection of $\mathcal{E}$ and f with intersection multiplicity 2, and $\mathcal{E}$ is 2-circular conic. □

Fig. 3 shows the circle $\mathcal{A}$ with equation $y = x^2$, line $\mathcal{B}$ with equation $y = x$ and their equidistant curve $\mathcal{E}$ with equation $4x^2 - 4xy + y^2 - y = 0$. $\mathcal{E}$ is a parabola touching the absolute line at the point $(0, 1, 2)$, the isotropic point of the line $y = 2x$.

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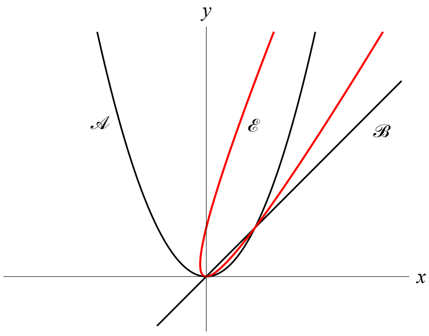

FIGURE 3. The equidistant curve $\mathcal{E}$ of circle $\mathcal{A}$ and line $\mathcal{B}$.

REFERENCES

[1] G. Elbert, M.-S. Kim, *Bisector curves of planar rational curves*, Computer-Aided Design **30(14)** (1998), 1089–1096.

[2] M. Fioravanti, J. Rafael Sendra, *Algebro-geometric analysis of bisectors of two algebraic plane curves*, Computer Aided Geometric Design **47** (2016), 189–203.

[3] E. Jurkin, *Bisector of Conics in Isotropic Plane*, manuscript submitted to Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.

[4] E. Jurkin, *Circular quartics in the isotropic plane generated by projectively linked pencils of conics*, Acta Math. Hung. **130(1-2)** (2011), 35–49.

[5] M. Katić Žlepalo, E. Jurkin, *Equidistant Sets of Conic and Line*, ICGG 2018 - Proceedings of the 18th International Conference on Geometry and Graphics, Cocchiarella, L. (ed.), Milan, Italy, Springer International Publishing, 2019.

[6] R. Kolar-Šuper, Z. Kolar-Begović, V. Volenec, J. Beban-Brkić, *Metrical relationships in a standard triangle in an isotropic plane*, Math. Comm. **10** (2005), 149–157.

[7] M. Ponce, P. Santibáñez, *On Equidistant Sets and Generalized Conics: The Old and the New*, Amer. Math. Monthly **121(1)** (2014), 18–32.

[8] H. Sachs, *Ebene Isotrope Geometrie*, Wieweg, Braunschweig/Wiesbaden, 1987.

[9] G. Salmon, *A Treatise on the Higher Plane Curves: Intended as a Sequel to a Treatise on Conic Sections*, 3rd edition, Chelsea Publishing Company, New York, 1879.

[10] J. B. Wilker, *Equidistant sets and their connectivity properties*, Proc. Amer. Math. Soc. **47(2)** (1975), 449–452.

Ema Jurkin
Faculty of Mining, Geology and Petroleum Engineering
University of Zagreb
10 000 Zagreb, Croatia
E-mail: `ema.jurkin@rgn.unizg.hr`