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ON $D(w)$-QUADRUPLES IN THE RINGS OF INTEGERS OF CERTAIN PURE NUMBER FIELDS

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Abstract. The purpose of this paper is to show the non-existence of $D(w)$-quadruples in number fields of odd degree whose rings of integers are of the special form. We derive some elements which can not be represented as difference of squares in such rings and comment the non-existence of corresponding Diophantine quadruples. This relies on the non-solvability of system of congruences which we prove in some low-degree cases.

1. Introduction

Existence of the Diophantine quadruples with the property $D(w)$ (or, $D(w)$-quadruples) consists of finding a set $\{w_1, w_2, w_3, w_4\}$ of four non-zero integers with the property that $w_i \cdot w_j + w$ is a perfect square, for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. This problem represents one of the most interesting and oldest problems in number theory, with important applications and far-reaching generalizations.

It has turned out that this problem is closely related to the description of elements which can be represented as a difference of squares. It has been first shown by Dujella in [3] that if $w \not\equiv 2 \pmod 4$ and $w \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$ then there exists a $D(w)$–quadruple. Further, it is obtained as a consequence of work of some other authors (see [1, 13, 15]) that there is no $D(4k + 2)$-quadruples, for $k \in \mathbb{Z}$.

Similar problems can be studied in rings other than the ring of rational integers and it seems that the most interesting rings are the rings of integers of number fields and rings of polynomials over a commutative ring with identity (for a good exposition of the polynomial variant of this problem we refer the reader to [6, 7] and references therein).

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After a rather complete description of differences of two squares in quadratic number fields has been given in [5], in the series of papers by Franušić [9–11] it has been shown that for an integer w in non-imaginary quadratic number field of the form $\mathbb{Q}(\sqrt{d})$ there exists a Diophantine quadruple with the property $D(w)$ if and only if w can be represented as a difference of squares of two integers, up to finitely many exceptions.

The problem is also solved for Gaussian integers, up to finitely many cases by Dujella in [4], and further studied in some imaginary quadratic fields by Dujella and Soldo ([8]), Soldo ([17]), Muriefah and Al-Rashed ([16]).

However, the main advantage of the considered case of degree two is the fact that description of differences of two squares completely relies on the solvability of certain Pellian equations. Unfortunately, in higher degree number fields one obtains more complicated equations and also has to face with mainly uncomplete description of the rings of integers. Further, every quadratic field K has a power basis, i.e. there is an element $\alpha \in K$ such that the ring of integers in K is of the form $\mathbb{Z} + \mathbb{Z}\alpha$. Such a base, which is also called the cyclic one, represents one of the most important bases in the algebraic number field theory. We note that fields having a power bases are called monogenic and an example of cubic field which is not monogenic was given already by Dedekind.

Unfortunately, there is much less known about the rings of integers in the algebraic number fields of degree greater than four, but it has recently been shown in [14] that there exist infinitely many monogenic quintic fields.

In our previous paper [12], we have proved that a $D(w)$-quadruple does not exist for certain integers in the pure cubic fields of the form $\mathbb{Q}(\sqrt[3]{d})$ with d even that can not be represented as a difference of two squares of integers. Our description of such elements relies on the fact that the rings of integers in pure cubic fields are completely known and given already in [2]. Also, pure cubic fields happen to be rather similar to the monogenic ones. One of the most important consequences of our results is that there are no $D(4k + 2)$-quadruples in such cubic fields.

In the present paper we generalize our approach to larger family of rings, i.e. to the rings of the form $\mathbb{Z}[1, \sqrt[n]{d}, \dots, \sqrt[n]{d^{n-1}}]$, where n is odd and d is an even integer. Such rings represent important example of the rings of integers in monogenic pure number fields of odd degree, especially for square-free d, and happen to be appropriate for obtaining some necessary conditions which an element of the ring of integers that can be written as a difference of squares in the same ring has to satisfy. In certain cases we show the non-existence of $D(w)$-quadruples for element w which does not satisfy the obtained conditions. In particular, we show the non-existence of $D(4k+2)$-quadruples in such cases, that is different from the case of some quadratic fields. The non-existence of such Diophantine quadruples is a direct consequence of the non-solvability of particular systems of congruences in several variables, which we obtain. Although in those interesting systems appear only congruences modulo 2 and modulo 4, for pure number fields of larger degree we obtain highly complicated systems of congruences in large number of variables. There does not seem to be a general method for testing the solvability of the obtained systems of congruences, since there appear many subcases that have to be treated in an essentially different way. For this reason, we choose to restrict our attention to number fields of small degree and study the solvability of mentioned systems of congruences using a case-by-case consideration.

We take a moment to describe the content of the paper. In the following section we determine some congruence conditions which differences of squares in number fields whose rings of integers have a cyclic base have to satisfy. Using these congruence conditions, in the third section we deduce certain necessary conditions for the existence of a $D(w)$-quadruple, which enables us to obtain a family of elements in the ring of integers without a corresponding Diophantine quadruple. Section 4 is devoted to the proof of the non-existence of $D(4k+2)$-quadruples in cyclic rings of integers in pure number fields of degree 5 and 7.

2. Differences of squares in pure number fields

We will consider number fields of the form $\mathbb{Q}(\sqrt[n]{d})$, where n is odd and d is even and restrict our attention to such number fields whose rings of integers have a cyclic base. In other words, in considered number field $\mathbb{Q}(\sqrt[n]{d})$ the ring of integers is given by $\mathbb{Z}[1, \sqrt[n]{d}, \sqrt[n]{d^2}, \ldots, \sqrt[n]{d^{n-1}}]$, where d is an even integer. We denote this ring of integers briefly by $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$.

First, let us observe how the product of two elements in mentioned ring of integers looks like. We denote by a and b elements in $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$ given by

$$a = a_0 + a_1 \sqrt[n]{d} + \ldots + a_{n-1} \sqrt[n]{d^{n-1}}$$

and

$$b = b_0 + b_1 \sqrt[n]{d} + \ldots + b_{n-1} \sqrt[n]{d^{n-1}}.$$

The product $a \cdot b$ is of the form

$$a_0 b_0 + \sum_{i=1}^{n-1} a_i b_{n-i} d + \sum_{j=1}^{n-1} \big(\sum_{i=0}^{j} a_i b_{j-i} + \sum_{i=j+1}^{n-1} a_i b_{n+j-i} d \big) \sqrt[n]{d^j}. \tag{2.1}$$

It is now easily seen that the following important relation holds for a square of an element $a = a_0 + a_1 \sqrt[n]{d} + \ldots + a_{n-1} \sqrt[n]{d^{n-1}}$ in the ring of integers of

$\mathbb{Q}(\sqrt[n]{d})$ (we emphasize that n is odd):

$$\begin{aligned} a^2 &= a_0^2 + 2d\sum_{i=1}^{\frac{n-1}{2}} a_i a_{n-i} + \sum_{\substack{j=1 \\ j\ odd}}^{n-2} \left(2\sum_{i=0}^{\frac{j-1}{2}} a_i a_{j-i} + 2d \sum_{i=j+1}^{n-1} a_i a_{n+j-i} \right) \sqrt[n]{d^j} \\ &+ \sum_{\substack{j=2 \\ j\ even}}^{n-1} \left(2\sum_{i=0}^{\frac{j}{2}-1} a_i a_{j-i} + a_{\frac{j}{2}}^2 + 2d \sum_{i=j+1}^{n-1} a_i a_{n+j-i} \right) \sqrt[n]{d^j}. \end{aligned} \tag{2.2}$$

Since $2d \equiv 0 \pmod 4$, using equality (2.2) one directly obtains the following statement:

Proposition 2.1. *Suppose that* $a = a_0 + a_1\sqrt[n]{d} + a_2\sqrt[n]{d^2} + \cdots + a_{n-1}\sqrt[n]{d^{n-1}}$, *where* $a_0, a_1, a_2, \ldots, a_{n-1} \in \mathbb{Z}$ *such that* $a_0 \equiv 2 \pmod 4$ *or* $a_i \equiv 1 \pmod 2$ *for some odd* $i \in \{1, 3, \ldots, n-2\}$. *Then* a *cannot be written as a difference of squares in the ring of integers* $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$ *in pure number field* $\mathbb{Q}(\sqrt[n]{d})$ *where* n *is odd and* d *is even.*

3. Non-existence of certain Diophantine quadruples in the ring $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$

In this section we will try to relate the non-existence of $D(w)$-quadruples with the impossibility of representing w as a difference of squares in the ring of integers. In the following lemma we give answer in several cases.

Lemma 3.1. *Suppose that* $w = w_0 + w_1\sqrt[n]{d} + w_2\sqrt[n]{d^2} + \cdots + w_{n-1}\sqrt[n]{d^{n-1}}$ *denote an element of the ring of integers* $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$ *in the number field* $\mathbb{Q}(\sqrt[n]{d})$, *where* d *is an even integer, such that a* $D(w)$*-quadruple exists in* $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$ $(n > 5)$. *Then* w_1, w_3 *and* w_5 *are even.*

Proof. First, we prove that w_1 is even.

Suppose, contrary to our claim, that w_1 is odd and that the set $\{m_1, m_2, m_3, m_4\}$ is a $D(w)$-quadruple in $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$. For simplicity of notation, let us write

$$m_i = x_i + y_i\sqrt[n]{d} + m_i',$$

where $m_i' \in \mathbb{Z}[\sqrt[n]{d^2}, \ldots, \sqrt[n]{d^{n-1}}]$.

Since d is even, using the definition of $D(w)$-quadruple, together with relations (2.1) and (2.2), we obtain that for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, there exist integers a_{ij}, b_{ij} such that $x_i y_j + x_j y_i + 2a_{ij} + w_1 = 2b_{ij}$. It follows that $x_i y_j + x_j y_i$ is an odd integer for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Now one obtains a contradiction in the same way as in the proof of [16, Proposition 1].

Let us now prove that w_3 is even. We will again assume, on the contrary, that w_3 is odd and the set $\{m_1, m_2, m_3, m_4\}$ is a $D(w)$-quadruple in $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$. Let us write

$$m_i = x_i + y_i \sqrt[n]{d} + z_i \sqrt[n]{d^2} + u_i \sqrt[n]{d^3} + m_i',$$

where $m_i' \in \mathbb{Z}[\sqrt[n]{d^4}, \ldots, \sqrt[n]{d^{n-1}}]$.

Using relations (2.1) and (2.2), together with the assumption that w_3 is odd, we obtain the following system of congruences

$$x_i u_j + x_j u_i + y_i z_j + y_j z_i \equiv 1 \pmod 2, \tag{3.1}$$

for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

Since we have already proved that w_1 is even, we also have

$$x_i y_j + x_j y_i \equiv 0 \pmod 2 \tag{3.2}$$

for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

We will show that the obtained system of congruences (3.1) and (3.2) has no solution using case-by-case consideration. Note that the assumption that x_i is even for $i = 1, 2, 3, 4$ gives the system of congruences $y_i z_j + y_j z_i \equiv 1 \pmod 2$, for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, which is equivalent to already considered one and has no solution. Thus, we may assume that there is some $i \in \{1, 2, 3, 4\}$ such that x_i is odd.

- Suppose that there is at most one $i \in \{1, 2, 3, 4\}$ such that x_i is even.

There is no loss of generality in assuming that x_2, x_3, x_4 are odd. System of congruences (3.2) yields $y_i + y_j \equiv 0 \pmod 2$ for $i, j \in \{2, 3, 4\}$, $i \neq j$. If $y_i \equiv 0 \pmod 2$ for $i = 2, 3, 4$, we obtain that there is no solution of system (3.1) in the same way as in the first case considered. Thus, we may suppose $y_i \equiv 1 \pmod 2$ for $i = 2, 3, 4$. This gives a new system of congruences

$$u_i + u_j + z_i + z_j \equiv 1 \pmod 2 \tag{3.3}$$

for $i, j \in \{2, 3, 4\}$, $i \neq j$. By adding all three congruences in (3.3) together, we obtain a contradiction.

- Suppose that there are $i, j, k \in \{1, 2, 3, 4\}$, $i \neq j$, such that $x_i \equiv x_j \equiv 0 \pmod 2$ and $x_k \equiv 1 \pmod 2$.

We may assume that $x_1 \equiv x_2 \equiv 0 \pmod 2$ and $x_3 \equiv 1 \pmod 2$. Using (3.2) we get $x_3 y_i \equiv 0 \pmod 2$ for $i = 1, 2$ and, consequently, $y_1 \equiv y_2 \equiv 0 \pmod 2$. This contradicts (3.1) for $i = 1$ and $j = 2$ and completes the proof.

It remains to prove that w_5 is even.

We will again suppose, contrary to our assumption, that w_5 is odd and there is a $D(w)$-quadruple $\{m_1, m_2, m_3, m_4\}$ in $\mathbb{Z}[1, \sqrt[n]{d}, \ldots, \sqrt[n]{d^{n-1}}]$. To simplify notation, we write

$$m_i = x_i + y_i \sqrt[n]{d} + z_i \sqrt[n]{d^2} + u_i \sqrt[n]{d^3} + v_i \sqrt[n]{d^4} + t_i \sqrt[n]{d^5} + m_i'$$

where $m_i' \in \mathbb{Z}[\sqrt[n]{d^6}, \ldots, \sqrt[n]{d^{n-1}}]$.

Repeating the same arguments as in the proof of previously considered cases, we obtain the following system of congruences:

$$x_i y_j + x_j y_i \equiv 0 \pmod 2,$$

$$x_i u_j + x_j u_i + y_i z_j + y_j z_i \equiv 0 \pmod 2, \tag{3.4}$$

$$x_i t_j + x_j t_i + y_i v_j + y_j v_i + z_i u_j + z_j u_i \equiv 1 \pmod 2, \tag{3.5}$$

for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.

We note that the first part of previous system of congruences coincides with (3.2).

In the rest of the proof we will show that system of congruences (3.2), (3.4) and (3.5) does not have a solution. This will again be done using case-by-case consideration. Note that the assumption that x_i is even for $i = 1, 2, 3, 4$ directly reduces this case to the previously considered one, so once again, we can assume that there is some $i \in \{1, 2, 3, 4\}$ such that x_i is odd.

- Suppose that there is $i \in \{1, 2, 3, 4\}$ such that $x_i \equiv 1 \pmod 2$ and $j, k \in \{1, 2, 3, 4\}$, $j \neq k$, such that $x_j \equiv x_k \equiv 0 \pmod 2$.

We may assume $x_1 \equiv x_2 \equiv 0 \pmod 2$ and $x_3 \equiv 1 \pmod 2$. Using (3.2), (3.4) and (3.5) we deduce $y_1 \equiv y_2 \equiv 0 \pmod 2$ and $y_3 \equiv 1 \pmod 2$. Now (3.4) leads to $u_1 + z_1 \equiv 0 \pmod 2$ and $u_2 + z_2 \equiv 0 \pmod 2$, while (3.5) shows $z_1 u_2 + z_2 u_1 \equiv 1 \pmod 2$. This is impossible since derived congruences $u_1 \equiv z_1 \pmod 2$ and $u_2 \equiv z_2 \pmod 2$ imply that $z_1 u_2 + z_2 u_1$ is even.

- Suppose that there is at most one $i \in \{1, 2, 3, 4\}$ such that $x_i \equiv 0 \pmod 2$.

We may assume $x_j \equiv 1 \pmod 2$ for $j \in \{1, 2, 3\}$. Using (3.2) and (3.4) we get $y_j \equiv 1 \pmod 2$ for $j \in \{1, 2, 3\}$. From (3.4) and (3.5) one immediately obtains

$$u_i + u_j + z_i + z_j \equiv 0 \pmod 2, \tag{3.6}$$

$$t_i + t_j + v_i + v_j + z_i u_j + z_j u_i \equiv 1 \pmod 2, \tag{3.7}$$

for $i, j \in \{1, 2, 3\}$, $i \neq j$. Two possible subcases will be considered separately.

1. Suppose $z_i \equiv z_j \pmod 2$ for $i, j \in \{1, 2, 3\}$. From (3.6) we obtain $z_i u_j + z_j u_i \equiv 0 \pmod 2$, which implies that $t_i + t_j + v_i + v_j \equiv 1 \pmod 2$ holds for $i, j \in \{1, 2, 3\}$, $i \neq j$. Adding all three congruences together immediately leads to a contradiction.
2. Suppose that there are $i_1, i_2 \in \{1, 2, 3\}$, $i_1 \neq i_2$, such that $z_{i_1} \equiv z_{i_2} \pmod 2$ and $j \in \{1, 2, 3\}$ such that $z_j \not\equiv z_{i_1} \pmod 2$. Obviously, we may assume $z_1 \equiv z_2 \pmod 2$ and $z_3 \not\equiv z_1 \pmod 2$. Using (3.6) we see

at once that $u_1 \equiv u_2 \pmod 2$ and $u_1 \not\equiv u_3 \pmod 2$ hold. From (3.7) we have

$$\begin{aligned} t_1 + t_2 + v_1 + v_2 &\equiv 1 \pmod 2, \\ t_1 + t_3 + v_1 + v_3 + z_1 u_3 + z_3 u_1 &\equiv 1 \pmod 2, \\ t_2 + t_3 + v_2 + v_3 + z_2 u_3 + z_3 u_2 &\equiv 1 \pmod 2. \end{aligned}$$

Summing the last two congruences we deduce $t_1 + t_2 + v_1 + v_2 \equiv 0 \pmod 2$, contrary to the first one. This ends the proof.

□

We emphasize that it is natural to expect that if there is a $D(w)$-quadruple in $\mathbb{Z}[1, \sqrt[n]{d}, \dots, \sqrt[n]{d^{n-1}}]$, for $w = w_0 + w_1 \sqrt[n]{d} + w_2 \sqrt[n]{d^2} + \cdots + w_{n-1} \sqrt[n]{d^{n-1}}$, then w_i is even for odd i.

We also note the following result for certain even d which are not square-free.

Proposition 3.2. *If $d \equiv 0 \pmod 4$ and there is a $D(w)$-quadruple in $\mathbb{Z}[1, \sqrt[n]{d}, \dots, \sqrt[n]{d^{n-1}}]$ for $w = w_0 + w_1 \sqrt[n]{d} + w_2 \sqrt[n]{d^2} + \cdots + w_{n-1} \sqrt[n]{d^{n-1}}$, then $w_0 \not\equiv 2 \pmod 4$.*

Proof. To obtain a contradiction, suppose that there is a $D(w)$-quadruple $\{m_1, m_2, m_3, m_4\}$ in $\mathbb{Z}[1, \sqrt[n]{d}, \dots, \sqrt[n]{d^{n-1}}]$ for $w = w_0 + w_1 \sqrt[n]{d} + w_2 \sqrt[n]{d^2} + \cdots + w_{n-1} \sqrt[n]{d^{n-1}}$, $k \in \mathbb{Z}$. Let us write $m_i = x_i + y_i \sqrt[n]{d}$, where $x_i \in \mathbb{Z}$ and $y_i \in \mathbb{Z}[1, \sqrt[n]{d}, \dots, \sqrt[n]{d^{n-2}}]$.

Since the set $\{m_1, m_2, m_3, m_4\}$ is a $D(w)$-quadruple, using equalities (2.1) and (2.2) we obtain a system of congruences $x_i x_j \equiv 2$ or $3 \pmod 4$, for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, which does not have a solution by the pigeonhole principle. This completes the proof. □

4. Non-existence of $D(4k+2)$-quadruples

The purpose of this section is to show that there are no $D(4k+2)$-quadruples in cyclic rings of integers in some pure number fields of low dimension, using more general results obtained in the previous section. Since the case $\mathbb{Q}(\sqrt[3]{d})$ has been completely solved in [12], we will restrict our attention on the pure number fields $\mathbb{Q}(\sqrt[5]{d})$ and $\mathbb{Q}(\sqrt[7]{d})$ and rings $\mathbb{Z}[1, \sqrt[p]{d}, \dots, \sqrt[p]{d^{p-1}}]$ for $p \in \{5, 7\}$, where d is an even integer. Proposition 3.2 enables us to assume $d \equiv 2 \pmod 4$.

Theorem 4.1. *If a $D(w)$-quadruple exists in $\mathbb{Z}[1, \sqrt[p]{d}, \dots, \sqrt[p]{d^{p-1}}]$, for $w = w_0 + w_1 \sqrt[p]{d} + w_2 \sqrt[p]{d^2} + \ldots + w_{p-1} \sqrt[p]{d^{p-1}}$, where $p \in \{5, 7\}$ and d is even, then $w_0 \not\equiv 2 \pmod 4$.*

PROOF. Let us suppose, on the contrary, that there is a $D(w)$-quadruple in $\mathbb{Z}[1, \sqrt[p]{d}, \dots, \sqrt[p]{d^{p-1}}]$ and w_0 is of the form $4k+2$ for some $k \in \mathbb{Z}$.

The first case considered in Lemma 3.1 shows that w_1 is even while the second case considered in the same lemma shows that w_3 is also even.

We shall first consider the case $p = 5$. Similarly as in the proof of Proposition 3.2, we suppose that the set $\{m_1, m_2, m_3, m_4\}$ is a $D(w)$-quadruple in $\mathbb{Z}[1, \sqrt[5]{d}, \dots, \sqrt[5]{d^4}]$. For abbreviation, we write

$$m_i = x_i + y_i \sqrt[5]{d} + z_i \sqrt[5]{d^2} + u_i \sqrt[5]{d^3} + v_i \sqrt[5]{d^4}.$$

Using the definition of $D(w)$-quadruple, together with equalities (2.1), (2.2) and the fact that d is even, we obtain that for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, the following congruences hold:

$$x_i y_j + x_j y_i \equiv 0 \pmod 2, \tag{4.1}$$

$$x_i u_j + x_j u_i + y_i z_j + y_j z_i \equiv 0 \pmod 2. \tag{4.2}$$

Also, for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, one of the following holds:

$$\begin{aligned} &x_i x_j + 2(y_i v_j + y_j v_i + z_i u_j + z_j u_i) \equiv 2 \pmod 4, \\ &x_i x_j + 2(y_i v_j + y_j v_i + z_i u_j + z_j u_i) \equiv 3 \pmod 4. \end{aligned} \tag{4.3}$$

We will show that the system of congruences (4.1), (4.2), (4.3) does not have a solution. It is easy to see that the assumption that x_i is even for $i \in \{1, 2, 3, 4\}$ yields a system of congruences which does not have a solution by Lemma 3.1. Let us briefly comment on other possible cases.

First, let us suppose that there is exactly one $i \in \{1, 2, 3, 4\}$ such that $x_i \equiv 1 \pmod 2$. Clearly, we may assume $x_1 \equiv 1 \pmod 2$. It follows directly from (4.1), (4.2) and (4.3) that $u_j \equiv z_j \pmod 2$ and $z_i u_j + z_j u_i \equiv 1 \pmod 2$ hold for $i, j \in \{2, 3, 4\}$, $i \neq j$, which is impossible.

The case when there are $i_1, i_2 \in \{1, 2, 3, 4\}$, $i_1 \neq i_2$, such that $x_{i_1} \equiv x_{i_2} \equiv 0 \pmod 2$ and $j_1, j_2 \in \{1, 2, 3, 4\}$, $j_1 \neq j_2$, such that $x_{j_1} \equiv x_{j_2} \equiv 1 \pmod 2$ can be handled in completely same way as the previous one.

The remaining case, when there is at most one $i \in \{1, 2, 3, 4\}$ such that x_i is even, happens to be the most complicated one. Obviously, we may assume that x_2, x_3, x_4 are odd. Using (4.1) we obtain $y_i \equiv y_j \pmod 2$ for $i, j \in \{2, 3, 4\}$.

The assumption $y_i \equiv 0 \pmod 2$ for $i \in \{2, 3, 4\}$ leads either to a system $x_i x_j \equiv 3 \pmod 4$ for $i, j \in \{2, 3, 4\}$, $i \neq j$, or to a system $x_i x_j + 2(z_i + z_j) \equiv 3 \pmod 4$ for $i, j \in \{2, 3, 4\}$, $i \neq j$, and it can be directly verified that neither of these two systems has a solution.

It remains to consider the case $y_i \equiv 1 \pmod 2$ for $i \in \{2, 3, 4\}$. From (4.2) and (4.3) we obtain

$$\begin{aligned} u_i + u_j + z_i + z_j &\equiv 0 \pmod 2, \\ x_i x_j + 2(v_i + v_j + z_i u_j + z_j u_i) &\equiv 3 \pmod 4 \end{aligned}$$

for $i, j \in \{2,3,4\}$, $i \neq j$. Examining several possible subcases, it is not hard to see that the obtained system of congruences does not have a solution. This ends the proof for the number fields of degree 5.

In the following part of the proof, we will show the equivalent result for pure number fields of degree 7.

In the same fashion as in the previous part of the proof, starting from the assumption that there is a $D(w)$-quadruple $\{m_1, m_2, m_3, m_4\}$ in $\mathbb{Z}[1, \sqrt[7]{d}, \ldots, \sqrt[7]{d^6}]$ with

$$m_i = x_i + y_i\sqrt[7]{d} + z_i\sqrt[7]{d^2} + u_i\sqrt[7]{d^3} + v_i\sqrt[7]{d^4} + s_i\sqrt[7]{d^5} + t_i\sqrt[7]{d^6},$$

we deduce that the following congruences

$$\begin{aligned} x_iy_j + x_jy_i &\equiv 0 \pmod 2, \\ x_iu_j + x_ju_i + y_iz_j + y_jz_i &\equiv 0 \pmod 2, \\ x_is_j + x_js_i + y_iv_j + y_jv_i + z_iu_j + z_ju_i &\equiv 0 \pmod 2 \end{aligned}$$

hold for $i, j \in \{1,2,3,4\}$, $i \neq j$. Further, for $i, j \in \{1,2,3,4\}$, $i \neq j$, one of the following holds:

$$\begin{aligned} x_ix_j + 2(y_it_j + y_jt_i + z_is_j + z_js_i + u_iv_j + u_jv_i) &\equiv 2 \pmod 4, \\ x_ix_j + 2(y_it_j + y_jt_i + z_is_j + z_js_i + u_iv_j + u_jv_i) &\equiv 3 \pmod 4. \end{aligned}$$

Again, it can be shown that the obtained system of congruences has no solution examining several possible cases. We note that if x_i is even for all $i \in \{1,2,3,4\}$ we have the system of congruences which is analogous to one observed in the third part of the proof of Lemma 3.1, which has no solution. Thus, it can be assumed that at least one x_i is odd. Other possible cases happen to be very similar to those obtained in the previous part of the proof, so the details are left to the reader. This ends the proof. $\square$

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