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METRIZATION OF PRO-MORPHISM SETS

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Abstract. Every pair of inverse systems $\boldsymbol{X}$, $\boldsymbol{Y}$ in a category $\mathcal{A}$, where $\boldsymbol{Y}$ is cofinite, admits a complete (ultra)metric structure on the set $pro\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$. The corresponding hom-bifunctor is not, generally, an internal Hom. However, there exists a subcategory of $pro\text{-}\mathcal{A}$, containing $tow\text{-}\mathcal{A}$, for which the hom-bifunctor is an invariant Hom into the category of complete metric spaces. Application to the sets $tow\text{-}HcANR(\boldsymbol{X},\boldsymbol{Y})$ yields several new interesting results concerning Borsuk's quasi-equivalence.

1. Introduction

In the last decade several papers were published seeking a "natural" structure of the shape morphism sets ([3,4,15–19]). It has become clear that, in general, there is no unique topological structure on those sets. The original idea was to consider the shape morphisms as certain classes of Cauchy sequences, i.e., to obtain the shape as a Cantor completion process analogous to the construction of the real numbers (irrationals) from the rationals. It should be mentioned that their starting point was *not* a metric (not even a pseudometric). Although not unique, the obtained (ultra)metric and topological structures on the shape morphism sets yield some interesting and useful results. In the first place, they permit relations between rather distant theories and the shape theory. Further, they admit constructions of some new shape invariants, in addition to simpler expressions of the old ones by means of the new technique.

Our main goal is to obtain, by using a metric, a better view into some classifications of compacta which are strictly coarser than the shape type classification. Therefore, in this paper the starting point is a pseudometric

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on a set $inv\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$, where $\boldsymbol{Y}$ is a cofinite inverse system. It induces a complete (ultra)metric structure on the corresponding pro-set. Then, we are studying the relevant properties of the complete metric space $(\boldsymbol{Y}^{\boldsymbol{X}},d)$, where $\boldsymbol{Y}^{\boldsymbol{X}}$ denotes the set $pro\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$. This approach, of course, immediately requires to involve the hom-bifunctor

$$\mathrm{hom} : (pro\text{-}\mathcal{A})^{op} \times (pro\text{-}\mathcal{A}) \to Set.$$

We have found necessary and sufficient conditions for hom to be an internal Hom, i.e., to be continuous with respect to the category Met_c of complete metric spaces (Lemma 3.5). Especially, hom is (uniformly) continuous for inverse sequences (Corollary 3.9), i.e., there exists

$$Hom : (tow\text{-}\mathcal{A})^{op} \times (tow\text{-}\mathcal{A}) \to Met_c.$$

Moreover, we have found necessary and sufficient conditions for Hom to be invariant (Theorem 4.1). Especially, Hom is invariant for all inverse sequences (Theorem 4.2).

Finally, we apply the new technique to compact metric spaces, i.e., to sequential $HcANR$- and $HcPol$-expansions, and obtain results which provide a deeper insight into Borsuk's quasi-equivalence ([2]). First, we have proven that the quasi-equivalence differs from shape if and only if it realizes without a pair of Cauchy sequences (Corollary 5.2). Further, among our new results, if an FANR is quasi-dominated by a compactum, then it is shape dominated by the same compactum (Corollary 5.5). It was known (J. M. R. Sanjurjo, [20]) that, on the class of all FANR's, the quasi-domination is equivalent to shape domination. Hereby we have proven that, on the class of all FANR's, the quasi-equivalence reduces to shape type (Corollary 5.8). A slight strengthening of the quasi-equivalence, so-called the $\overline{q}$-equivalence, which admits an appropriate $\overline{q}$-shape theory ([21]), is also considered and several new results are obtained. For instance, the $\overline{q}$-equivalence differs from shape if, and only if, it realizes without any Cauchy sequence (Theorem 5.9). Further, the semi-stability, movability and strong movability (i.e., being an FANR) are hereditary $\overline{q}$-shape properties (Lemma 5.11), and thus they are invariants of the $\overline{q}$-shape (Corollary 5.13).

2. A complete metric for $pro\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$

Let $\mathcal{A}$ be a category, and let $inv\text{-}\mathcal{A}$ be the corresponding inv-category of $\mathcal{A}$, ([13]), i.e., the objects of $inv\text{-}\mathcal{A}$ are all the inverse systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{A}$, and $inv\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$ is the set of all morphisms $(f,f_\mu) : \boldsymbol{X} \to \boldsymbol{Y} = (\boldsymbol{Y}_\mu, q_{\mu\mu'}, M)$, defined by the following condition

$$(\forall \mu \leq \mu')(\exists \lambda \geq f(\mu), f(\mu') f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}.$$

The composition is defined by $(g,g_\nu)(f,f_\mu) = (fg, g_\nu f_{g(\nu)})$, and the identity on an $\boldsymbol{X}$ is $(1_\Lambda, 1_{X_\lambda})$.

Two morphisms $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ of *inv*-$\mathcal{A}$ are said to be equivalent (homotopic), denoted by $(f, f_\mu) \simeq (f', f'_\mu)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$, such that

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

This relation is an equivalence relation that is compatible with composition in *inv*-$\mathcal{A}$. Therefore, there exists the corresponding quotient category (pro-category) *inv*-$\mathcal{A}/(\simeq) \equiv$ *pro*-$\mathcal{A}$. A morphism $[(f, f_\mu)]$ of *pro*-$\mathcal{A}$ is denoted by $\boldsymbol{f}$.

By following Definition 1 of [22], let us introduce the relation "to be n-homotopyc" on the morphism sets of *inv*-$\mathcal{A}$, $n \in \mathbb{N}$. First, an auxiliary definition.

DEFINITION 2.1. *Let $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be morphisms of inv-$\mathcal{A}$, and let $\mu \in M$. Then (f, f_μ) is said to be μ-**homotopic to** (f', f'_μ), denoted by $(f, f_\mu) \simeq_\mu (f', f'_\mu)$, provided there exists a $\lambda \in \Lambda$, $\lambda \geq f(\mu), f'(\mu)$, such that*

$$f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

The next lemma is obviously true by the above definition.

LEMMA 2.2. (i) *The relation $\simeq_\mu$ is an equivalence relation on each set inv-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$.*

(ii) *If $(f, f_\mu) \simeq_\mu (f', f'_\mu)$ and $\mu' \leq \mu$, then $(f, f_\mu) \simeq_{\mu'} (f', f'_\mu)$.*

(iii) *If $(f, f_\mu) \simeq_\mu (f', f'_\mu)$, then $(f, f_\mu)(h, h_\lambda) \simeq_\mu (f', f'_\mu)(h, h_\lambda)$.*

(iv) *If $(f, f_\mu) \simeq_\mu (f', f'_\mu)$, then $(g, g_\nu)(f, f_\mu) \simeq_\nu (g, g_\nu)(f', f'_\mu)$, whenever $g(\nu) \leq \mu$.*

(v) *$(f, f_\mu) \simeq (f', f'_\mu)$ if and only if $(f, f_\mu) \simeq_\mu (f', f'_\mu)$ for every $\mu \in M$.*

Recall that, for any $\lambda \in \Lambda$, $|\lambda|$ denotes the cardinal of the set of all the predecessors λ' of λ in Λ, $\lambda' < \lambda$ (i.e., $\lambda' \leq \lambda$ and $\lambda' \neq \lambda$). In the case of a cofinite inverse system (indexing set), for every $\lambda \in \Lambda$, $|\lambda|$ is finite, i.e., $|\lambda| = n - 1$ for some $n \in \mathbb{N}$.

DEFINITION 2.3. *Let $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be morphisms of inv-$\mathcal{A}$, and let κ be a cardinal. Then (f, f_μ) is said to be κ-**homotopic to** (f', f'_μ), denoted by $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$, provided that for every $\mu \in M$, such that $|\mu| < \kappa$, $(f, f_\mu) \simeq_\mu (f', f'_\mu)$ holds.*

Notice that in the case of a cofinite $\boldsymbol{Y}$, those cardinals (representatives–numbers) κ range over the set of positive integers $n \in \mathbb{N}$. Furthermore, in the case of an inverse sequence $\boldsymbol{Y}$, the relations $\simeq_n$ and $\simeq_\mu$ coincide ($\mu = |\mu| + 1 = n$). By Definitions 2.1 and 2.3 and by Lemma 2.2, the next lemma is obviously true.

LEMMA 2.4. (i) *The relation $\simeq_\kappa$ is an equivalence relation on each set inv-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$.*

(ii) *If* $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$ *and* $\kappa' \leq \kappa$, *then* $(f, f_\mu) \simeq_{\kappa'} (f', f'_\mu)$.
(iii) *If* $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$ *and* $(f', f'_\mu) \simeq_{\kappa'} (f'', f''_\mu)$, *then* $(f, f_\mu) \simeq_{\kappa''} (f'', f''_\mu)$, *where* $\kappa'' = \min\{\kappa, \kappa'\}$.
(iv) *If* $(f, f_\mu) \simeq_\kappa (g, g_\mu)$, $(f', f'_\mu) \simeq_{\kappa'} (g', g'_\mu)$ *and* $(f, f_\mu) \simeq_\eta (f', f'_\mu)$, *then* $(g, g_\mu) \simeq_{\eta'} (g', g'_\mu)$, *where* $\eta' = \min\{\kappa, \kappa', \eta\}$.
(v) *If* $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$, *then* $(f, f_\mu)(h, h_\lambda) \simeq_\kappa (f', f'_\mu)(h, h_\lambda)$.
(vi) *If* $(f, f_\mu) \simeq_\kappa (f', f'_\mu)$, *then* $(g, g_\nu)(f, f_\mu) \simeq_{\kappa'} (g, g_\nu)(f', f'_\mu)$, *provided, for every* $\nu \in N$, $\mid \nu \mid < \kappa'$ *implies* $\mid g(\nu) \mid < \kappa$.
(vii) *If* $\boldsymbol{Y}$ *is cofinite, then* $(f, f_\mu) \simeq (f', f'_\mu)$ *if and only if* $(f, f_\mu) \simeq_n (f', f'_\mu)$ *for every* $n \in \mathbb{N}$.

Given a pair of inverse systems $\boldsymbol{X}, \boldsymbol{Y}$, where $\boldsymbol{Y}$ is *cofinite*, let us define the function

$$\rho : inv\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \times inv\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \to \mathbb{R}$$

by putting

$$\rho((f, f_\mu), (f', f'_\mu)) = \begin{cases} \inf\{\frac{1}{n+1} \mid (f, f_\mu) \simeq_n (f', f'_\mu),\, n \in \mathbb{N}\} \\ 1, \text{ otherwise} \end{cases}.$$

LEMMA 2.5. *For every* $\boldsymbol{X}$ *and every cofinite* $\boldsymbol{Y}$, *the ordered pair* $(inv\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}), \rho)$ *is a pseudo(ultra)metric space.*

PROOF. Clearly, $\rho((f, f_\mu), (f', f'_\mu)) \geq 0$ and

$$\rho((f, f_\mu), (f', f'_\mu)) = \rho((f', f'_\mu), (f, f_\mu)).$$

Further, if $(f, f_\mu) = (f', f'_\mu)$, then $(f, f_\mu) \simeq (f', f'_\mu)$, and thus, by Lemma 2.4 (vii), $(f, f_\mu) \simeq_n (f', f'_\mu)$ for every $n \in \mathbb{N}$, which is equivalent to $\rho((f, f_\mu), (f', f'_\mu)) = 0$. It remains to prove that

$$\rho((f, f_\mu), (f'', f''_\mu)) \leq \max\{\rho((f, f_\mu), (f', f'_\mu)), \rho((f', f'_\mu), (f'', f''_\mu))\}.$$

This obviously holds true whenever

$$\rho((f, f_\mu), (f', f'_\mu)) = 1 \quad \text{or} \quad \rho((f', f'_\mu), (f'', f''_\mu)) = 1.$$

Further, it also holds whenever $\rho((f, f_\mu), (f', f'_\mu)) = 0$ (i.e., $(f, f_\mu) \simeq (f', f'_\mu)$) or $\rho((f', f'_\mu), (f'', f''_\mu)) = 0$ (i.e., $(f', f'_\mu) \simeq (f'', f''_\mu)$). Namely, by Lemma 2.4 (iii) and (vii), $(f', f'_\mu) \simeq (f'', f''_\mu)$ implies $\rho((f, f_\mu), (f', f'_\mu)) = \rho((f, f_\mu), (f'', f''_\mu))$. Let

$$\rho((f, f_\mu), (f', f'_\mu)) = \frac{1}{n+1} \quad \text{and} \quad \rho((f', f'_\mu), (f'', f''_\mu)) = \frac{1}{n'+1}.$$

Then, by Lemma 2.4 (iii), $(f, f_\mu) \simeq_{n''} (f'', f''_\mu)$, where $n'' = \min\{n, n'\}$, and the conclusion follows. □

Let us briefly denote $pro\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \equiv \boldsymbol{Y}^{\boldsymbol{X}}$. Observe that, by Lemma 2.4, (iii) and (vii), if $(f, f_\mu) \simeq (g, g_\mu)$ and $(f', f'_\mu) \simeq (g', g'_\mu)$ (all of $\boldsymbol{X}$ to $\boldsymbol{Y}$), then $\rho((f, f_\mu), (f', f'_\mu)) = \rho((g, g_\mu), (g', g'_\mu))$. Thus, for every cofinite $\boldsymbol{Y}$, the function

$$d : \boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Y}^{\boldsymbol{X}} \to \mathbb{R}$$

is well defined by putting

$$d(\boldsymbol{f}, \boldsymbol{f}') = \rho((f, f_\mu), (f', f'_\mu)),$$

where $(f, f_\mu) \in \boldsymbol{f}$, $(f', f'_\mu) \in \boldsymbol{f}'$ is any pair of representatives.

Theorem 2.6. *For every* $\boldsymbol{X}$ *and every cofinite* $\boldsymbol{Y}$*, the ordered pair* $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ *is a complete (ultra)metric space.*

Proof. According to Lemma 2.5, it suffices to prove that $d(\boldsymbol{f}, \boldsymbol{f}') = 0$ implies $\boldsymbol{f} = \boldsymbol{f}'$, and the completeness. Let $d(\boldsymbol{f}, \boldsymbol{f}') = 0$. Then, $\rho((f, f_\mu), (f', f'_\mu)) = 0$ for any pair of the representatives. By definition of ρ and Lemma 2.4, it is equivalent to $(f, f_\mu) \simeq (f', f'_\mu)$, i.e., $\boldsymbol{f} = \boldsymbol{f}'$. Let $(\boldsymbol{f}^n)$ be a Cauchy sequence in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$. Then, for every $k \in \mathbb{N}$, there exists an $n_k \in \mathbb{N}$ such that, for every pair $n, m \in \mathbb{N}$, $n, m \geq n_k$,

$$d(\boldsymbol{f}^n, \boldsymbol{f}^m) \leq \frac{1}{k+1}.$$

Without loss of generality, one may assume that $n_{k+1} \geq n_k$. For each $k \in \mathbb{N}$, put $n = n_k$ and consider the sequence (n_k). Let us define, for every $\mu \in M$,

$$f^0_\mu = f^{n_k}_\mu : X_{f^{n_k}(\mu)} \to Y_\mu, \quad k = |\mu| + 1.$$

In this way we have obtained the family $(f^0_\mu)_{\mu \in M}$ of morphisms $f^0_\mu : X_\lambda \to Y_\mu$, $\lambda = f^{n_{|\mu|+1}}(\mu)$, of $\mathcal{A}$. Notice that it defines an index function

$$f^0 : M \to \Lambda, \quad f^0(\mu) = f^{n_{|\mu|+1}}(\mu).$$

Let us show that the ordered pair (f^0, f^0_μ) is a morphism of $\boldsymbol{X}$ to $\boldsymbol{Y}$ in *inv*-$\mathcal{A}$. Given a pair $\mu < \mu'$ in M, we have to prove that there exists a $\lambda \geq f^0(\mu), f^0(\mu')$ such that

$$f^0_\mu p_{f^0(\mu)\lambda} = q_{\mu\mu'} f^0_{\mu'} p_{f^0(\mu')\lambda}.$$

Denote $|\mu| = k - 1$ and $|\mu'| = k' - 1$. Then $k' > k$, and $n_{k'} \geq n_{k+1} \geq n_k$. Since $(f^{n_{k'}}, f^{n_{k'}}_\mu)$ is a morphism of *inv*-$\mathcal{A}$ and since $(f^{n_k}, f^{n_k}_\mu) \simeq_k (f^{n_{k'}}, f^{n_{k'}}_\mu)$, we infer that there exists a $\lambda \in \Lambda$, $\lambda \geq f^{n_k}(\mu), f^{n_{k'}}(\mu), f^{n_{k'}}(\mu')$, such that the diagram below commutes.

$$\begin{array}{ccc}
 & X_\lambda & \\
\swarrow & \downarrow & \searrow \\
X_{f^{n_k}(\mu)} & X_{f^{n_{k'}}(\mu)} & X_{f^{n_{k'}}(\mu')} \\
f^0_\mu = \downarrow f^{n_k}_\mu & \swarrow f^{n_{k'}}_\mu & f^{n_{k'}}_{\mu'} \downarrow = f^0_{\mu'} \\
Y_\mu & \leftarrow & Y_{\mu'}
\end{array}.$$

This implies that

$$f^{n_k}_\mu p_{f^{n_k}(\mu)\lambda} = q_{\mu\mu'} f^{n_{k'}}_{\mu'} p_{f^{n_{k'}}(\mu')\lambda},$$

which means that

$$f^0_\mu p_{f^0(\mu)\lambda} = q_{\mu\mu'} f^0_{\mu'} p_{f^0(\mu')\lambda}.$$

This proves that $(f^0, f^0_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism of *inv*-$\mathcal{A}$. Observe that we have proven even more. Namely, for every $k \in \mathbb{N}$ and every $n \geq n_k$,

$$(f^n, f^n_\mu) \simeq_k (f^0, f^0_\mu),$$

i.e.,

$$\rho((f^n, f^n_\mu), (f^0, f^0_\mu)) \leq \frac{1}{k+1}.$$

Therefore, for every $k \in \mathbb{N}$ and every $n \geq n_k$,

$$d(\boldsymbol{f}^n, \boldsymbol{f}^0) \leq \frac{1}{k+1},$$

which means that $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$, i.e., that the sequence $(\boldsymbol{f}^n)$ converges to $\boldsymbol{f}^0$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$. □

Remark 2.7. If $\boldsymbol{Y} = (Y_\mu = Y, q_{\mu\mu'} = 1_Y, M) \in Ob(pro\text{-}\mathcal{A})$ is cofinite, then, for every $\boldsymbol{X} \in Ob(pro\text{-}\mathcal{A})$, the space $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is discrete. However, according to [2] (see also Section 5 below), in the case $\mathcal{A} = HcANR$ there exist mutually quasi-equivalent metric compacta which are not shape equivalent. Consequently, by applying the characterization of Borsuk's quasi-equivalence in terms of associated compact ANR (or polyhedral) inverse sequences, given in [22], one readily sees that a space $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, in general, is *not discrete*. Especially, there exist inverse sequences $\boldsymbol{X}$ such that the spaces $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ are not discrete. An example is given below.

Example 2.8. Let $\boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \mathbb{N})$ in *HcPol* be defined by $X_\lambda = \{x_1, \ldots, x_\lambda\}$ and $[p_{\lambda\lambda+1}]$ $(= \{p_{\lambda\lambda+1}\})$ such that the fibre $p^{-1}_{\lambda\lambda+1}(x_1) = \{x_1, x_{\lambda+1}\}$, while each other fibre $p^{-1}_{\lambda\lambda+1}(x_\lambda)$, $\lambda \neq 1$, is the singleton $\{x_\lambda\}$ (the inverse limit of $\boldsymbol{X}$ is an infinite compact countable space having the only one nonopen point). Then the space $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ is not discrete. Indeed (see the proof below), there exists a sequence $(\boldsymbol{f}^n)$ in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ such that, for every $n \in \mathbb{N}$,

$$d(\boldsymbol{f}^n, \boldsymbol{1_X}) = \frac{1}{n+1}.$$

Observe that every homotopy commutative diagram relating $\boldsymbol{X}$ to itself is strictly commutative. Let, for each $n \in \mathbb{N}$, $f^n : \mathbb{N} \to \mathbb{N}$ be the identity function $1_\mathbb{N}$, and let, for every $\lambda \in \mathbb{N}$, the mapping $f^n_\lambda : X_\lambda \to X_\lambda$ be defined as follows: If $\lambda \leq n$, then $f^n_\lambda = 1_{X_\lambda}$; if $\lambda > n$, then,

$$f^n_\lambda(x_j) = \begin{cases} x_j,\ j \in \{1, \ldots, n\} \\ x_1,\ j \in \{n+1, \ldots, \lambda\} \end{cases}.$$

It is readily seen that, for each n and every related pair $\lambda \leq \lambda'$,

$$f^n_\lambda p_{\lambda\lambda'} = p_{\lambda\lambda'} f^n_{\lambda'}.$$

Thus, $(f^n,[f^n_\lambda]) : \boldsymbol{X} \to \boldsymbol{X}$ is a morphism of $(HcPol)^{\mathbb{N}}$. By construction, for every $n \in \mathbb{N}$,

$$(f^n,[f^n_\lambda]) \simeq_n (1_{\mathbb{N}},[1_{X_\lambda}]),$$

while

$$(f^n,[f^n_\lambda]) \simeq_{n+1} (1_{\mathbb{N}},[1_{X_\lambda}])$$

does *not* hold. Therefore, by definition of pseudometric ρ, for every $n \in \mathbb{N}$,

$$\rho((f^n,[f^n_\lambda]),(1_{\mathbb{N}},[1_{X_\lambda}])) = \frac{1}{n+1}.$$

Finally, by this and definition of metric d,

$$d(\boldsymbol{f}^n, \boldsymbol{1_X}) = \frac{1}{n+1},$$

where $\boldsymbol{f}^n = [(f^n,[f^n_\lambda])] \in \boldsymbol{X}^{\boldsymbol{X}}$, $n \in \mathbb{N}$.

In some considerations, the next technical lemma could help.

Lemma 2.9. *Let $(\boldsymbol{f}^n)$ and $(\boldsymbol{f}'^n)$ be sequences in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, and let (m_n) be an increasing unbounded sequence in $\mathbb{N}$. Suppose that $(\boldsymbol{f}^n)$ is a Cauchy sequence.*

(i) *If, for every n, there exists a pair of representatives $(f^n, f^n_\mu) \in \boldsymbol{f}^n$, $(f'^n, f'^n_\mu) \in \boldsymbol{f}'^n$ such that $(f^n, f^n_\mu) \simeq_{m_n} ((f'^n, f'^n_\mu)$ for almost all n, then $(\boldsymbol{f}'^n)$ is a Cauchy sequence too and $\lim(\boldsymbol{f}^n) = \lim(\boldsymbol{f}'^n)$.*

(ii) *If $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$, then for every representing sequence $((f^n, f^n_\mu))$ of $(\boldsymbol{f}^n)$ and every representative (f^0, f^0_μ) of $\boldsymbol{f}^0$, the following condition holds:*

$$(\forall \mu \in M)(\exists n_\mu \in \mathbb{N})(\forall n \geq n_\mu)(\exists \lambda \geq f^n(\mu), f^0(\mu))\ f^n_\mu p_{f^n(\mu)\lambda} = f^0_\mu p_{f^0(\mu)\lambda}.$$

(iii) *There exists a representing sequence $((f^n, f^n_\mu))$ of $(\boldsymbol{f}^n)$ such that, for every $\mu \in M$, the sequence $(f^n(\mu))$ in Λ is stationary. Moreover, for every $(f^0, f^0_\mu) \in \boldsymbol{f}^0 = \lim(\boldsymbol{f}^n)$, there exists a representing sequence $((f^n, f^n_\mu))$ such that*

$$(\forall \mu \in M)(\exists n_\mu \in \mathbb{N})(\forall n \geq n_\mu) \quad f^n_\mu = f^0_\mu.$$

Proof. By Theorem 2.6, the proof of statement (i) is straightforward. To prove (ii), let $\mu_0 \in M$ be chosen arbitrarily. Then, $|\mu_0| = k_0 - 1$ for some $k_0 \in \mathbb{N}$. Since $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$, for every $k \in \mathbb{N}$, there exists an $n_k \in \mathbb{N}$ such that, for every $n \geq n_k$,

$$d(\boldsymbol{f}^n, \boldsymbol{f}^0) \leq \frac{1}{k+1}.$$

Let $((f^n, f^n_\mu))$ be any representing sequence of $(\boldsymbol{f}^n)$, and let $(f^0, f^0_\mu) \in \boldsymbol{f}^0$ be chosen arbitrarily. Then, for every $n \geq n_k$,

$$\rho((f^n, f^n_\mu),(f^0, f^0_\mu)) \leq \frac{1}{k+1},$$

which implies $(f^n, f^n_\mu) \simeq_k (f^0, f^0_\mu)$. Especially, $(f^n, f^n_\mu) \simeq_\mu (f^0, f^0_\mu)$ for every $\mu \in M$, $|\mu| \leq k-1$. Therefore, if we put $n_{\mu_0} = n_{k_0}$, the relation $(f^n, f^n_\mu) \simeq_{\mu_0} (f^0, f^0_\mu)$ holds for every $n \geq n_{\mu_0}$. This means that, for every $n \geq n_{\mu_0}$, there exists a $\lambda \geq f^n(\mu_0), f^0(\mu_0)$ (depending on μ_0 and n) such that

$$f^n_{\mu_0} p_{f^n(\mu_0)\lambda} = f^0_{\mu_0} p_{f^0(\mu_0)\lambda}.$$

This completes the proof of assertion (ii). The proof of (iii) is by induction on $|\mu| \in \{0\} \cup \mathbb{N}$, $\mu \in M$. For each $k \in \mathbb{N}$, denote

$$M_{k-1} \equiv \{\mu \in M \mid |\mu| = k-1\} \subseteq M.$$

Notice that M is the disjoint union of all M_{k-1}, $k \in \mathbb{N}$ (some of them may be empty). Let $(f^n, f^n_\mu) \in \boldsymbol{f}^n$, $n \in \mathbb{N}$, let $(f^0, f^0_\mu) \in \boldsymbol{f}^0$, and let, $\mu \in M_0$, i.e., $|\mu| = 0$. By (ii), there is an $n_\mu \in \mathbb{N}$ such that, for every $n \geq n_\mu$, there is a $\lambda_n \geq f^n(\mu), f^0(\mu)$ satisfying

$$f^n_\mu p_{f^n(\mu)\lambda_n} = f^0_\mu p_{f^0(\mu)\lambda_n}.$$

Clearly, for every $n \geq n_\mu$ and every $\mu' \geq \mu$, there exists a $\lambda \geq \lambda_n, f^n(\mu')$, such that

$$f^n_\mu p_{f^n(\mu)\lambda} = q_{\mu\mu'} f^n_{\mu'} p_{f^n(\mu')\lambda}.$$

Therefore,

$$f^0_\mu p_{f^0(\mu)\lambda} = q_{\mu\mu'} f^n_{\mu'} p_{f^n(\mu')\lambda}.$$

This shows that, for each $\mu \in M_0$ and all $n \geq n_\mu$, the values $f^n(\mu)$ may be replaced by $f^0(\mu)$ as well as the morphisms f^n_μ by f^0_μ. It yields the new representing sequence $((f'^n, f'^n_\mu))$ of $(\boldsymbol{f}^n)$ that satisfies the stationary condition for $(f'^n(\mu))$, $\mu \in M_0$. Let $k \in \mathbb{N}$, and let us assume that assertion (iii) is proved for every $\mu \in M_l$, $l \leq k-1$, i.e., for all $\mu \in M$, $|\mu| < k$. Let $\mu \in M_k$. By (ii), there exists an $n_\mu \in \mathbb{N}$ such that, for every $n \geq n_\mu$, there is a $\lambda_n \geq f^n(\mu), f^0(\mu)$ satisfying

$$f^n_\mu p_{f^n(\mu)\lambda_n} = f^0_\mu p_{f^0(\mu)\lambda_n}.$$

Again, for every $n \geq n_\mu$ and every $\mu' \geq \mu$, there exists a $\lambda \geq \lambda_n, f^n(\mu')$, such that

$$f^n_\mu p_{f^n(\mu)\lambda} = q_{\mu\mu'} f^n_{\mu'} p_{f^n(\mu')\lambda}.$$

Thus,

$$f^0_\mu p_{f^0(\mu)\lambda} = q_{\mu\mu'} f^n_{\mu'} p_{f^n(\mu')\lambda}.$$

This shows that, for each $\mu \in M_k$ and all $n \geq n_\mu$, the values $f^n(\mu)$ may be replaced by $f^0(\mu)$ as well as the morphisms f^n_μ by f^0_μ. Observe that, by this replacement, all the relevant terms not related to M_k remain unchanged. It implies that the inductive step $k-1 \mapsto k$ is correct. Clearly, it yields the new representing sequence $((f'^n, f'^n_\mu))$ of $(\boldsymbol{f}^n)$ that satisfies the stationary condition for $(f'^n(\mu))$, $\mu \in M_0 \cup \cdots \cup M_k$. The conclusion follows. ∎

At the end of this section we want to prove the following useful theorem.

THEOREM 2.10. *For every $\boldsymbol{X}$ and every cofinite $\boldsymbol{Y}$, every Cauchy sequence in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ admits a representing sequence having a unique increasing index function.*

PROOF. First, every sequence $(\boldsymbol{f}^n)$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ admits a representing sequence $((f^n, f^n_\mu))$ such that all the index functions are increasing and $f^1 \leq \cdots \leq f^n \leq \cdots$ (this can be achieved by a straightforward inductive construction). Let $(\boldsymbol{f}^n)$ be a Cauchy sequence. Recall the proof of Theorem 2.6, i.e., the construction of the limit $\boldsymbol{f}^0 = \lim(\boldsymbol{f}^n)$: the constructed representative $(f^0, f^0_\mu) \in \boldsymbol{f}^0$ has been defined by means of a subsequence $((f^{n_k}, f^{n_k}_\mu))$, where $n_1 \leq \cdots \leq n_k \leq \cdots$, such that

$$f^0(\mu) = f^{n_{|\mu|+1}}(\mu) \quad \text{and} \quad f^0_\mu = f^{n_{|\mu|+1}}_\mu.$$

Notice that, in this case, $f^0 : M \to \Lambda$ is an increasing function. Let $\mu \in M$, $|\mu| = 0$. Since $f^1 \leq \cdots \leq f^{n_1}$, one can, for every $i = 1, \ldots, n_1$, replace $f^i(\mu)$ with $f'^i(\mu) = f^0(\mu) = f^{n_1}(\mu)$ and f^i_μ with $f'^i_\mu = f^i_\mu p_{f^i(\mu) f^0(\mu)}$. In the next step, since

$$f^1 \leq \cdots \leq f^{n_1} \leq f^{n_1+1} \leq \cdots \leq f^{n_2},$$

given a $\mu \in M$, $|\mu| = 1$, one can, for every $i = 1, \ldots, n_2$, replace $f^i(\mu)$ with $f'^i(\mu) = f^0(\mu) = f^{n_2}(\mu)$ and f^i_μ with $f'^i_\mu = f^i_\mu p_{f^i(\mu) f^0(\mu)}$. Moreover, for every $\mu' \in M$, $|\mu'| = 0$, and every $i = n_1 + 1, \ldots, n_2$, one can replace $f^i(\mu')$ with $f'^i(\mu') = f^0(\mu') = f^{n_2}(\mu')$ and $f^i_{\mu'}$ with $f'^i_{\mu'} = f^i_{\mu'} p_{f^i(\mu') f^0(\mu')}$.

The construction proceeds in an obvious way by induction on $|\mu| + 1 = k \in \mathbb{N}$ through the sequence (n_k). Thus, in the inductive step $k \mapsto k + 1$, one also must correctly move every f^i, $i = n_k + 1, \ldots, n_{k+1}$, for all $\mu \in M$, $|\mu| \leq k$. Observe that, for every $n \in \mathbb{N}$, $(f'^n, f'^n_\mu) \simeq (f^n, f^n_\mu)$. Clearly, the new representing sequence $((f'^n, f'^n_\mu))$ has the unique increasing index function $f^0 = f'^n$ for all n. ∎

3. CONTINUITY OF THE HOM-BIFUNCTOR

Recall (see [8]) that, for every category $\mathcal{K}$, there exists the hom-bifunctor

$$\hom : \mathcal{K}^{op} \times \mathcal{K} \to Set$$

defined by $\hom(X, Y) = \mathcal{K}(X, Y)$ and $\hom(u, v)(f) = vfu$. More precisely, for each pair of (pairs of) objects

$$(X, Y), (X', Y') \in Ob(\mathcal{K}^{op} \times \mathcal{K}) = Ob\mathcal{K}^{op} \times Ob\mathcal{K} = Ob\mathcal{K} \times Ob\mathcal{K},$$

$$\begin{aligned} &\hom^{X,Y}_{X',Y'} : (\mathcal{K}^{op} \times \mathcal{K})((X, Y), (X', Y')) \\ &\qquad (= \mathcal{K}^{op}(X, X') \times \mathcal{K}(Y, Y') = \mathcal{K}(X', X) \times \mathcal{K}(Y, Y')) \to \\ &\qquad \to Set(\mathcal{K}(X, Y), \mathcal{K}(X', Y')), \\ &(u, v) \mapsto (\hom^{X,Y}_{X',Y'}(u, v) : \mathcal{K}(X, Y) \to \mathcal{K}(X', Y')), \end{aligned}$$

is defined by the composition, i.e., $\hom_{X',Y'}^{X,Y}(u,v)(f) = vfu$.

If the sets $\mathcal{K}(X,Y)$ are enriched in a natural way with a structure, and if the hom-bifunctor preserves the structure, then notation hom is usually changed into Hom (the "internal" Hom-bifunctor), having an appropriate codomain category (instead of Set).

Let us now consider the case $\mathcal{K} = pro\text{-}\mathcal{A}$ for an arbitrary category $\mathcal{A}$, i.e.,

$$\hom : (pro\text{-}\mathcal{A})^{op} \times (pro\text{-}\mathcal{A}) \to Set,$$
$$\hom(\boldsymbol{X},\boldsymbol{Y}) = pro\text{-}\mathcal{A}(\boldsymbol{X},\boldsymbol{Y}) \equiv \boldsymbol{Y}^{\boldsymbol{X}}$$

and

$$\hom_{\boldsymbol{X}',\boldsymbol{Y}'}^{\boldsymbol{X},\boldsymbol{Y}} : \boldsymbol{X}^{\boldsymbol{X}'} \times \boldsymbol{Y}'^{\boldsymbol{Y}} \to Set(\boldsymbol{Y}^{\boldsymbol{X}}, \boldsymbol{Y}'^{\boldsymbol{X}'}),$$

where

$$\hom_{\boldsymbol{X}',\boldsymbol{Y}'}^{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{u},\boldsymbol{v}) : \boldsymbol{Y}^{\boldsymbol{X}} \to \boldsymbol{Y}'^{\boldsymbol{X}'}$$

is defined by $\hom_{\boldsymbol{X}',\boldsymbol{Y}'}^{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{u},\boldsymbol{v})(\boldsymbol{f}) = \boldsymbol{v}\boldsymbol{f}\boldsymbol{u}$, i.e.,

$$\begin{array}{ccc} \boldsymbol{X} & \overset{\boldsymbol{u}}{\leftarrow} & \boldsymbol{X}' \\ \boldsymbol{f}\downarrow & \overset{\hom(\boldsymbol{u},\boldsymbol{v})}{\mapsto} & \downarrow \boldsymbol{v}\boldsymbol{f}\boldsymbol{u} \\ \boldsymbol{Y} & \underset{\boldsymbol{v}}{\rightarrow} & \boldsymbol{Y}' \end{array}.$$

We assume in the sequel that all inverse systems are *cofinite*. The natural question arises: Does the hom-bifunctor preserve the complete (ultra)metric structure of $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$? In other words: is the function

$$\hom(\boldsymbol{u},\boldsymbol{v}) : (\boldsymbol{Y}^{\boldsymbol{X}}, d) \to (\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$$

continuous for all (some) $\boldsymbol{u} : \boldsymbol{X}' \to \boldsymbol{X}$ and $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$? In general, the answer is negative (see Theorem 3.4 below). First, recall the notion of *semi-stability* (the complementary part of the strong movability ([22, Definition 3 and Lemma 4])) of an inverse sequence $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$:

$$(\exists i_0 \in \mathbb{N})(\forall i \geq i_0)(\forall i' \geq i)(\exists r : X_i \to X_{i'})(\exists i_1 \geq i')(\forall i'' \geq i_1) \quad rp_{ii''} = p_{i'i''}.$$

It is readily seen that an $\boldsymbol{X}$ of $tow\text{-}\mathcal{A} \subseteq pro\text{-}\mathcal{A}$ is semi-stable if and only if every morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of $pro\text{-}\mathcal{A}$ admit an $i_0 \in \mathbb{N}$ such that $\boldsymbol{f} = [(c_{i_0}, f_\mu)]$. Clearly, every stable $\boldsymbol{X}$ is semi-stable. Also, every strongly movable $\boldsymbol{X}$ is semi-stable. For instance, every object $\boldsymbol{X}$ of $tow\text{-}HcANR$ ($\subseteq pro\text{-}HcANR \subseteq pro\text{-}HTop$) associated with an FANR X is semi-stable.

Remark 3.1. Since the quasi-equivalence of compacta is not transitive (in general, see [10]), but it is transitive on the class of all quasi-stable compacta (including all semi-stable compacta ([22, Definition 5, Theorem 2 and Corollary 1])), it follows that nonsemi-stable inverse sequences of $tow\text{-}HcANR$ exist. For instance, any compact ANR inverse sequence associated with the continuum $Y \subseteq \mathbb{R}^3$ constructed in [10] (see Lemma 9 of [22]) is not semi-stable. Much simpler, any compact ANR inverse sequence associated with the Hawaiian earring is not semi-stable.

Consider now the following general example.

EXAMPLE 3.2. Let $\boldsymbol{Y} = (Y_j, q_{jj'}, \mathbb{N})$ be an inverse sequence in a category $\mathcal{A}$, and let $\boldsymbol{Y'} = (Y'_\mu, q'_{\mu\mu'}, M)$ be the (countable and cofinite) inverse system associated with $\boldsymbol{Y}$ by the well known "Mardešić trick", i.e.,

$$\begin{aligned} &M = F(\mathbb{N}) = \{\mu \mid \mu \subseteq \mathbb{N} \text{ finite}\} \subseteq 2^{\mathbb{N}}, \\ &\mu \leq \mu' \Leftrightarrow \mu \subseteq \mu', \\ &Y'_\mu = Y_j, \quad j = \max \mu, \\ &q'_{\mu\mu'} = q_{jj'} : Y'_{\mu'} = Y_{j'} \to Y_j = Y'_\mu. \end{aligned}$$

Then, clearly, $\boldsymbol{Y} \cong \boldsymbol{Y'}$ in *pro*-$\mathcal{A}$. However, the following fact occurs:

LEMMA 3.3. *If $\boldsymbol{Y}$ is* not *semi-stable, then for every section $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y'}$ (especially, for every isomorphism), every its representative (v, v_μ) has the following property:*

$$(\forall j \in \mathbb{N})(\exists \mu \in M, |\mu| = 0) \quad v(\mu) > j.$$

PROOF. Let us assume to the contrary. Then there exist a section $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y'}$, a representative $(v, v_\mu) \in \boldsymbol{v}$ and a $j_0 \in \mathbb{N}$ such that $v(\mu) \leq j_0$, whenever $|\mu| = 0$. For each $k \in \{0\} \cup \mathbb{N}$, denote

$$M_k = \{\mu \mid \mu \in M, |\mu| = k\} \subseteq M.$$

Then M is the (disjoint) union of all M_k, $k \geq 0$. More precisely, $M_0 = \{\{j\} \mid j \in \mathbb{N}\}$, $M_1 = \varnothing$, $M_2 = \{\{j, j'\} \mid j, j' \in \mathbb{N}\}$, $M_3 = M_4 = M_5 = \varnothing$, $M_6 = \{\{j, j', j''\} \mid j, j', j'' \in \mathbb{N}\}, \dots$ (for every $k \geq 0$, M_k is not empty if and only if $k = 2^r - 2$, $r \in \mathbb{N}$, i.e., $\mu \in M_k$ if and only if $card(\mu) = \log_2(k + 2)$). Notice that, for each k and every pair $\mu, \mu' \in M_k$, the elements μ and μ' are not related.

At first, our intention is to construct a representative (v', v'_μ) of $\boldsymbol{v}$, hence $(v', v'_\mu) \simeq (v, v_\mu)$, such that $v' : M \to \mathbb{N}$ is the constant function into j_0. For every $\mu = \{j\} \in M_0$, let the morphisms $v'_\mu : Y_{j_0} \to Y'_\mu$ be defined by $v_\mu q_{v(\mu)j_0}$ (since $v(\mu) \leq j_0$). We now proceed by induction through all the sets M_k, $k > 0$. Since $M_1 = \varnothing$, let $\mu = \{j, j'\} \in M_2$, $j < j'$. Then $Y'_\mu = Y_{j'}$, and $q'_{\{j\}\mu} : Y'_\mu \to Y'_{\{j\}}$, $q'_{\{j'\}\mu} : Y'_\mu \to Y'_{\{j'\}}$ are $q_{jj'} : Y_{j'} \to Y_j$, $1 : Y_{j'} \to Y_{j'}$ respectively. If $v(\mu) \leq j_0$, then put (as before) $v'_\mu = v_\mu q_{v(\mu)j_0} : Y_{j_0} \to Y'_\mu$. If $v(\mu) > j_0$, then observe that there exists a $j_1 \geq v(\mu)$ such that, for every $j'' \geq j_1$,

$$v_{\{j\}} q_{v(\{j\})j''} = q_{jj'} v_\mu q_{v(\mu)j''} \quad \text{and} \quad v_{\{j'\}} q_{v(\{j'\})j''} = 1_{Y_{j'}} v_\mu q_{v(\mu)j''}.$$

Thus, we can put $v'_\mu = v_{\{j'\}}$ $(= v_{\{\max \mu\}})$ to assure desired commutativity. Assume that the construction is well done for all M_l, $l < k \geq 2$. Given any $\mu = \{j_1, \dots, j_r\} \in M_k$, if $v(\mu) \leq j_0$ put, as before, $v'_\mu = v_\mu q_{v(\mu)j_0} : Y_{j_0} \to Y'_\mu$, while in the case $v(\mu) > j_0$ put $v'_\mu = v_{\{\max \mu\}}$. The needed commutativity relations go straightforwardly (it suffices to verify them for all $\mu' < \mu$, where μ' belongs to the closest nonempty M_l, i.e., for all $\mu' = \{j'_1, \dots, j'_{r-1}\} \subseteq \mu$).

Since $\boldsymbol{v}$ is a section, there exists a $\boldsymbol{u} : \boldsymbol{Y}' \to \boldsymbol{Y}$ such that $\boldsymbol{uv} = \boldsymbol{1_Y}$. Let (u, u_j) be any representative of $\boldsymbol{u}$. Then $(u, u_j)(v', v'_\mu) \simeq (1_\mathbb{N}, 1_{Y_j})$, i.e., for every $j \in \mathbb{N}$, there exists a $j_1 \geq j_0, j$ $(v'u(j) = j_0)$ such that, for every $j'' \geq j_1$,

$$u_j v'_{u(j)} q_{j_0 j''} = q_{jj''}.$$

Finally, given a pair $j, j' \in \mathbb{N}$ such that $j' \geq j \geq j_0$, put

$$s = u_{j'} v'_{u(j')} q_{j_0 j} : Y_j \to Y_{j'},$$

and check that

$$s q_{jj''} = u_{j'} v'_{u(j')} q_{j_0 j} q_{jj'} = u_{j'} v'_{u(j')} q_{j_0 j''} = q_{j' j''},$$

for every large enough j''. This shows that the inverse sequence $\boldsymbol{Y}$ is semi-stable, contradicting the assumption. □

THEOREM 3.4. *The hom-bifunctor on pro-$\mathcal{A}$, in general, does not preserve the metric structure on pro-morphism sets. More precisely, there exist an inverse sequence $\boldsymbol{Y}$ and an inverse system $\boldsymbol{Y}'$ isomorphic to $\boldsymbol{Y}$, $\boldsymbol{Y}' \cong \boldsymbol{Y}$ in pro-$\mathcal{A}$, such that, for every inverse system $\boldsymbol{X}$ yielding the nondiscrete space $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, every (nontrivial) morphism $\boldsymbol{u} : \boldsymbol{X}' \to \boldsymbol{X}$ and every section $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$, the function*

$$\hom(\boldsymbol{u}, \boldsymbol{v}) : (\boldsymbol{Y}^{\boldsymbol{X}}, d) \to (\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$$

is not *continuous. Especially, for $\boldsymbol{X}' = \boldsymbol{X} = \boldsymbol{Y}$, $\boldsymbol{u} = \boldsymbol{1_Y} : \boldsymbol{Y} \to \boldsymbol{Y}$ and every isomorphism $\boldsymbol{v}$, the bijection*

$$\hom(\boldsymbol{1_Y}, \boldsymbol{v}) : (\boldsymbol{Y}^{\boldsymbol{Y}}, d) \to (\boldsymbol{Y}'^{\boldsymbol{Y}}, d)$$

is not continuous.

PROOF. Let $\boldsymbol{Y}$ be a nonsemi-stable inverse sequence, let $\boldsymbol{Y}'$ be associated with $\boldsymbol{Y}$ by the "Mardešić trick" and let $\boldsymbol{X}$ be any inverse system such that the space $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is not discrete. Let $\boldsymbol{v} \in pro\text{-}\mathcal{A}(\boldsymbol{Y}, \boldsymbol{Y}')$ be a section and let a nontrivial $\boldsymbol{u} \in pro\text{-}\mathcal{A}(\boldsymbol{X}', \boldsymbol{X})$ be chosen arbitrarily. Since $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is not discrete, there exists a nonstationary convergent sequence $(\boldsymbol{f}^n)$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$. Then, for every $k \in \mathbb{N}$, there exists an $n_k \in \mathbb{N}$ such that, for every $n \geq n_k$,

$$d(\boldsymbol{f}^n, \boldsymbol{f}^0) \leq \frac{1}{k+1}.$$

By Lemma 2.4 (v), $d(\boldsymbol{f}^n \boldsymbol{u}, \boldsymbol{f}^0 \boldsymbol{u}) \leq \frac{1}{k+1}$ also holds for $n \geq n_k$. By Lemma 3.3, for every representative (v, v_μ) of $\boldsymbol{v}$ and every $k \in \mathbb{N}$, there exists a $\mu \in M$,

$|\mu| = 0$, such that $v(\mu) > k$. This implies, by definition of the metric, that $d(\boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u}, \boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u}) \not\leq \frac{1}{k+1}$ for any k and $n \geq n_k$, and thus, $d(\boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u}, \boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u}) = 1$ for every $n \geq n_1$. Consequently, the sequence $(\boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u})$ in $(\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$ does not converge to $\boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u}$. Since $\hom(\boldsymbol{u}, \boldsymbol{v})(\boldsymbol{f}^n) = \boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u}$ and $\hom(\boldsymbol{u}, \boldsymbol{v})(\boldsymbol{f}^0) = \boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u}$, the function $\hom(\boldsymbol{u}, \boldsymbol{v}) : (\boldsymbol{Y}^{\boldsymbol{X}}, d) \to (\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$ is not continuous. Finally, in the special case $\boldsymbol{X}' = \boldsymbol{X} = \boldsymbol{Y}$, we have to choose a nonsemi-stable $\boldsymbol{Y}$ such that the space $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$ is not discrete (see Remark 2.7). ∎

The next lemma shows that the continuity of $\hom(\boldsymbol{u}, \boldsymbol{v})$ depends only on the existence of a certain representative of $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$, having a specific index function.

LEMMA 3.5. *Let $\boldsymbol{u} : \boldsymbol{X}' \to \boldsymbol{X}$ and $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$ be morphisms of pro-$\mathcal{A}$, and suppose that the space $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is not discrete. Then the function*

$$\hom(\boldsymbol{u}, \boldsymbol{v}) : (\boldsymbol{Y}^{\boldsymbol{X}}, d) \to (\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$$

is (uniformly) continuous if and only if $\boldsymbol{v}$ admits a representative $(v, v_{\mu'})$ satisfying the following "uniformity" condition:

$$(U) \qquad (\forall k \in \mathbb{N})(\exists s_k \in \mathbb{N})(\forall \mu' \in M')\, |\mu'| \leq k - 1 \Rightarrow |v(\mu')| \leq s_k - 1.$$

PROOF. First, the sufficiency part. To prove continuity, it is enough to show that the function $\hom(\boldsymbol{u}, \boldsymbol{v})$ preserves convergent sequences. Let $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$. We are to prove that the sequence

$$(\hom(\boldsymbol{u}, \boldsymbol{v})(\boldsymbol{f}^n)) = (\boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u})$$

in $(\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$ converges to $\hom(\boldsymbol{u}, \boldsymbol{v})(\boldsymbol{f}^0) = \boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u}$. Let $(f^n, f^n_\mu) \in \boldsymbol{f}^n$, $n \in \mathbb{N}$, $(f^0, f^0_\mu) \in \boldsymbol{f}^0$ and $(u, u_\lambda) \in \boldsymbol{u}$ be chosen arbitrarily, and let $(v, v_{\mu'}) \in \boldsymbol{v}$ be a representative satisfying condition (U). By Lemma 2.4, (v) and (vi), if $(f^n, f^n_\mu) \simeq_k (f^0, f^0_\mu)$, then $(f^n, f^n_\mu)(u, u_\lambda) \simeq_k (f^0, f^0_\mu)(u, u_\lambda)$, and $(v, v_{\mu'})(f^n, f^n_\mu) \simeq_{k'} (v, v_{\mu'})(f^0, f^0_\mu)$ provided $|\mu'| \leq k' - 1$ implies $|v(\mu')| \leq k - 1$. Since $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$,

$$d(\boldsymbol{f}^n, \boldsymbol{f}^0) = \rho((f^n, f^n_\mu), (f^0, f^0_\mu))$$

becomes arbitrarily small when n increases, i.e., for every $k \in \mathbb{N}$, there exists an $n_k \in \mathbb{N}$ such that, for every $n \geq n_k$,

$$\rho((f^n, f^n_\mu), (f^0, f^0_\mu)) \leq \frac{1}{k+1}.$$

Hence, $(f^n, f^n_\mu) \simeq_k (f^0, f^0_\mu)$, $n \geq n_k$, and thus,

$$(v, v_{\mu'})(f^n, f^n_\mu)(u, u_\lambda) \simeq_{k'} (v, v_{\mu'})(f^0, f^0_\mu)(u, u_\lambda), \quad n \geq n_k,$$

provided $|\mu'| \leq k' - 1$ implies $|v(\mu')| \leq k - 1$. Since, by assumption, for every k, there exists an s_k such that, for every $\mu' \in M'$, $|\mu'| \leq k - 1$ implies

$|v(\mu')| \leq s_k - 1$, we infer that, for every k and every $n \geq n_{s_k}$,

$$(v, v_{\mu'})(f^n, f^n_\mu)(u, u_\lambda) \simeq_k (v, v_{\mu'})(f^0, f^0_\mu)(u, u_\lambda)$$

holds. Thus,

$$d(\boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u}, \boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u}) = \rho((v, v_{\mu'})(f^n, f^n_\mu)(u, u_\lambda), (v, v_{\mu'})(f^0, f^0_\mu)(u, u_\lambda)) \leq \frac{1}{k+1},$$

for every $n \geq n_{s_k}$. This means that $\lim(\boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u}) = \boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u}$, which proves the continuity of $\hom(\boldsymbol{u}, \boldsymbol{v})$. Finally, notice that a $\delta > 0$ (for continuity of $\hom(\boldsymbol{u}, \boldsymbol{v})$) does not depend on any particular point $\boldsymbol{f} \in \boldsymbol{Y}^{\boldsymbol{X}}$. Namely, given any $\varepsilon = \frac{1}{k+1} > 0$, one may put $\delta = \frac{1}{s_k+1} > 0$. Therefore, $\hom(\boldsymbol{u}, \boldsymbol{v})$ is uniformly continuous.

Conversely, suppose to the contrary, i.e., that $\hom(\boldsymbol{u}, \boldsymbol{v})$ is continuous and that, for every representative $(v, v_{\mu'})$ of $\boldsymbol{v}$, the following condition is fulfilled:

$$(\exists k_0 \in \mathbb{N})(\forall s \in \mathbb{N})(\exists \mu' \in M')(|\mu'| \leq k_0 - 1 \wedge |v(\mu')| > s - 1).$$

Since $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is not discrete, there exists a nonstationary sequence $(\boldsymbol{f}^n)$ in $\boldsymbol{Y}^{\boldsymbol{X}}$ converging to a point $\boldsymbol{f}^0 \in \boldsymbol{Y}^{\boldsymbol{X}}$. Then the continuity of $\hom(\boldsymbol{u}, \boldsymbol{v})$ implies that $d(\boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u}, \boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u})$ becomes arbitrarily small when n increases. However, by the above condition and definition of the metric, for every large enough $n \in \mathbb{N}$,

$$d(\boldsymbol{v}\boldsymbol{f}^n\boldsymbol{u}, \boldsymbol{v}\boldsymbol{f}^0\boldsymbol{u}) > \frac{1}{k_0+1}$$

must hold - a contradiction. □

Observe that property (U) of some morphisms of *inv*-$\mathcal{A}$ is preserved by composition. Since each identity morphism $(1_\Lambda, 1_{X_\lambda})$ obviously satisfies condition (U), there exists a subcategory inv_{U}-$\mathcal{A} \subseteq inv$-$\mathcal{A}$ such that

$$Ob(inv_{\mathrm{U}}\text{-}\mathcal{A}) = Ob(inv\text{-}\mathcal{A})$$

and

$$inv_{\mathrm{U}}\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) = \{(f, f_\mu) \mid (f, f_\mu) \text{ satisfies (U)}\} \subseteq inv\text{-}\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}).$$

Let pro_{U}-$\mathcal{A} \subseteq pro$-$\mathcal{A}$ be the subcategory on the same object class such that every morphism $\boldsymbol{f}$ of pro_{U}-$\mathcal{A}$ admits a representative in inv_{U}-$\mathcal{A}$. Let us briefly denote pro_{U}-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y}) \equiv \boldsymbol{Y}^{\boldsymbol{X}}_{\mathrm{U}}$. By assuming the restriction to all *cofinite* inverse systems, the following theorem holds.

THEOREM 3.6. *The hom-bifunctor for the subcategory* pro_U-$\mathcal{A}$ *is a structure preserving (continuous) one, i.e., it is*

$$Hom : (pro_U\text{-}\mathcal{A})^{op} \times (pro_U\text{-}\mathcal{A}) \to Met_c,$$

where Met_c *is the category of complete metric spaces.*

PROOF. According to Theorem 2.6 and Lemma 3.5, it suffices to prove that $(\boldsymbol{Y}^{\boldsymbol{X}}_{\mathrm{U}}, d) \subseteq (\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is a closed subspace. Therefore, the proof follows by the next lemma. □

LEMMA 3.7. *$(\boldsymbol{Y}^{\boldsymbol{X}}_U, d) \subseteq (\boldsymbol{Y}^{\boldsymbol{X}}, d)$ is a closed subspace.*

PROOF. Suppose that a sequence $(\boldsymbol{f}^n)$ in $\boldsymbol{Y}^{\boldsymbol{X}}_{\mathrm{U}}$ converges to an $\boldsymbol{f}^0$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$. We have to prove that $\boldsymbol{f}^0 \in \boldsymbol{Y}^{\boldsymbol{X}}_{\mathrm{U}}$. Recall the construction of the limit morphism $\boldsymbol{f}^0$ in the proof of Theorem 2.6. Given any representing sequence $((f^n, f^n_\mu))$ of $(\boldsymbol{f}^n)$, the representing mappings f^0_μ, $\mu \in M$, have been defined to be $f^{n_k}_\mu : X_{f^{n_k}(\mu)} \to Y_\mu$, for all $k \in \mathbb{N}$ and all $\mu \in M$ with $|\mu| = k-1$. In this case one should, in addition, choose a representing sequence in inv_{U}-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. Then the obtained (f^0, f^0_μ) satisfies condition (U), i.e., it belongs to inv_{U}-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. Indeed, $f^0(\mu) = f^{n_k}(\mu)$, for every $k \in \mathbb{N}$ and every $\mu \in M$ with $|\mu| = k-1$. Therefore,

$$(\forall n \in \mathbb{N})(\forall k \in \mathbb{N})(\exists s^{n_k}_k \in \mathbb{N})(\forall \mu \in M)$$
$$|\mu| \le k-1 \Rightarrow \left|f^0(\mu)\right| = |f^{n_k}(\mu)| \le s^{n_k}_k - 1.$$

(without loss of generality, we may assume that all f^n are increasing, and thus, it suffices to verify condition (U) only for $\mu \in M$, $|\mu| = k-1$). □

An inverse system $\boldsymbol{X}$ is said to have property (F) provided, for every $k \in \mathbb{N}$, the subset

$$\Lambda_{k-1} \equiv \{\lambda \in \Lambda \mid |\lambda| = k-1\} \subseteq \Lambda$$

is finite. Clearly, every inverse sequence $\boldsymbol{X} = (X_i, p_{ii'}, \mathbb{N})$ has property (F). Let inv_{F}-$\mathcal{A} \subseteq inv$-$\mathcal{A}$ be the full subcategory containing all the cofinite objects which have property (F). Let pro_{F}-$\mathcal{A} \subseteq pro$-$\mathcal{A}$ be the corresponding pro-category. Then (for inverse sequences), tow-$\mathcal{A} \subseteq pro_{\mathrm{F}}$-$\mathcal{A}$ is a full subcategory.

COROLLARY 3.8. *The hom-bifunctor for the subcategory pro_F-$\mathcal{A} \subseteq pro$-$\mathcal{A}$ is structure preserving (continuous), i.e., it is*

$$Hom : (pro_F\text{-}\mathcal{A})^{op} \times (pro_F\text{-}\mathcal{A}) \to Met_c.$$

PROOF. Observe that pro_{F}-$\mathcal{A} \subseteq pro_{\mathrm{U}}$-$\mathcal{A}$ is a full subcategory, because every morphism of inv_{F}-$\mathcal{A}$ satisfies condition (U). Hence, the conclusion follows by Theorem 3.6. □

COROLLARY 3.9. *The hom-bifunctor for the tower category tow-$\mathcal{A}$ is structure preserving (continuous), i.e., it is*

$$Hom : (tow\text{-}\mathcal{A})^{op} \times (tow\text{-}\mathcal{A}) \to Met_c.$$

PROOF. Every inverse sequence has property (F). Thus, the conclusion follows by Corollary 3.8. □

Let

$$(\boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Z}^{\boldsymbol{Y}}, d') = (\boldsymbol{Y}^{\boldsymbol{X}}, d) \times (\boldsymbol{Z}^{\boldsymbol{Y}}, d)$$

be the product space endowed with an appropriate metric d' (for instance, d_2, d_1 or d_∞ with respect to the metrics on the factors). Then the function

$$\omega : (\boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Z}^{\boldsymbol{Y}}, d') \to (\boldsymbol{Z}^{\boldsymbol{X}}, d),$$

defined by the composition, $(\boldsymbol{f}, \boldsymbol{g}) \mapsto \boldsymbol{g}\boldsymbol{f}$, naturally arises. According to preceding results, ω cannot be continuous in general. However, the following fact holds as a consequence of Lemma 3.5 and Theorem 3.6.

Corollary 3.10. *The function (restriction)*

$$\omega : (\boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Z}^{\boldsymbol{Y}}_U, d') \to (\boldsymbol{Z}^{\boldsymbol{X}}, d), \omega(\boldsymbol{f}, \boldsymbol{g}) = \boldsymbol{g}\boldsymbol{f},$$

is (uniformly) continuous. Especially, for all inverse sequences $\boldsymbol{X}$*,* $\boldsymbol{Y}$ *and* $\boldsymbol{Z}$ *in* $\mathcal{A}$*, the function*

$$\omega : (\boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Z}^{\boldsymbol{Y}}, d') \to (\boldsymbol{Z}^{\boldsymbol{X}}, d)$$

is (uniformly) continuous. Moreover, for every section $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$*, the hom-bifunctor commutes with* ω*, i.e., the diagram*

$$\begin{array}{ccc} \boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Z}^{\boldsymbol{Y}} & \overset{\hom(\boldsymbol{u},\boldsymbol{v}) \times \hom(\boldsymbol{v}',\boldsymbol{w})}{\rightarrow} & \boldsymbol{Y}'^{\boldsymbol{X}'} \times \boldsymbol{Z}'^{\boldsymbol{Y}'} \\ \omega \downarrow & & \downarrow \omega \\ \boldsymbol{Z}^{\boldsymbol{X}} & \overset{\hom(\boldsymbol{u},\boldsymbol{w})}{\rightarrow} & \boldsymbol{Z}'^{\boldsymbol{X}'} \end{array}$$

is commutative. More precisely,

$$\omega \circ (\hom(\boldsymbol{u}, \boldsymbol{v}) \times \hom(\boldsymbol{v}', \boldsymbol{w})) = \hom(\boldsymbol{u}, \boldsymbol{w}) \circ \omega,$$

where $\boldsymbol{v}' : \boldsymbol{Y}' \to \boldsymbol{Y}$ *is a left inverse of* $\boldsymbol{v}$*,* $\boldsymbol{v}'\boldsymbol{v} = \boldsymbol{1}_{\boldsymbol{Y}}$*.*

Proof. It suffices to prove that $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ and $\lim(\boldsymbol{g}^n) = \boldsymbol{g}^0$ in $(\boldsymbol{Z}^{\boldsymbol{Y}}_{\mathrm{U}}, d)$ imply $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{g}^0\boldsymbol{f}^0 = \lim(\boldsymbol{g}^n)\lim(\boldsymbol{f}^n)$ in $(\boldsymbol{Z}^{\boldsymbol{X}}, d)$. By Lemma 2.4 (v), for each $m \in \mathbb{N}$, $\lim(\boldsymbol{g}^n\boldsymbol{f}^m) = \boldsymbol{g}^0\boldsymbol{f}^m$. Since $\boldsymbol{g}^0 \in \boldsymbol{Z}^{\boldsymbol{Y}}_{\mathrm{U}}$, the function $\hom(\boldsymbol{1}_{\boldsymbol{X}}, \boldsymbol{g}^0)$ is (uniformly) continuous. The conclusion follows (observe that we have only needed $\boldsymbol{g}^0 \in \boldsymbol{Z}^{\boldsymbol{Y}}_{\mathrm{U}}$. Thus, the sequence $(\boldsymbol{g}^n)$ may be in $(\boldsymbol{Z}^{\boldsymbol{Y}}, d)$ as well). The commutativity of the diagram goes straightforwardly. □

4. The invariance of the hom-bifunctor

Our intention now is to answer the question concerning invariance of the hom-bifunctor, i.e., under what conditions, $\boldsymbol{X} \cong \boldsymbol{X}'$ and $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in *pro*-$\mathcal{A}$ imply that the spaces $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ and $(\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$ are homeomorphic. Clearly, every pair of isomorphisms $\boldsymbol{u} : \boldsymbol{X}' \to \boldsymbol{X}$, $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$ yields a set bijection $\boldsymbol{Y}^{\boldsymbol{X}} \to \boldsymbol{Y}'^{\boldsymbol{X}'}$, $\boldsymbol{f} \mapsto \boldsymbol{v}\boldsymbol{f}\boldsymbol{u}$, having the inverse $\boldsymbol{Y}'^{\boldsymbol{X}'} \to \boldsymbol{Y}^{\boldsymbol{X}}$, $\boldsymbol{f}' \mapsto \boldsymbol{v}^{-1}\boldsymbol{f}'\boldsymbol{u}^{-1}$. Therefore, for every such a pair of isomorphisms, the function

$$\hom(\boldsymbol{u}, \boldsymbol{v}) : \boldsymbol{Y}^{\boldsymbol{X}} \to \boldsymbol{Y}'^{\boldsymbol{X}'}$$

is a bijection with the inverse $\hom(\boldsymbol{u}, \boldsymbol{v})^{-1} = \hom(\boldsymbol{u}^{-1}, \boldsymbol{v}^{-1})$. According to Lemma 3.5, the following theorem holds.

Theorem 4.1. *Let* $\boldsymbol{u} : \boldsymbol{X}' \to \boldsymbol{X}$ *and* $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$ *be isomorphisms of pro-*$\mathcal{A}$*, and suppose that the spaces* $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ *and* $(\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$ *are not discrete. Then,*

$$\hom(\boldsymbol{u}, \boldsymbol{v}) : (\boldsymbol{Y}^{\boldsymbol{X}}, d) \to (\boldsymbol{Y}'^{\boldsymbol{X}'}, d)$$

is a (uniform) homeomorphism (of complete (ultra)metric spaces), if and only if $\boldsymbol{v}$ *and* $\boldsymbol{v}^{-1}$ *belong to pro*$_U$*-*$\mathcal{A}$*.*

Proof. By Lemma 3.5, $\hom(\boldsymbol{u}, \boldsymbol{v})$ and $\hom(\boldsymbol{u}, \boldsymbol{v})^{-1} = \hom(\boldsymbol{u}^{-1}, \boldsymbol{v}^{-1})$ are (uniformly) continuous if and only if $\boldsymbol{v}$ and $\boldsymbol{v}^{-1}$ admit representatives $(v, v_{\mu'})$ and (v', v'_{μ}), respectively, both of them satisfying condition (U). The conclusion follows. ◻

Theorem 4.2. *For every category* $\mathcal{A}$*, the hom-bifunctor for pro-*$\mathcal{A}$ *is invariant (and continuous into* Met_c*) with respect to the object isomorphisms in the following (sub)pro-categories: tow-*$\mathcal{A}$*, pro*$_F$*-*$\mathcal{A}$ *and pro*$_U$*-*$\mathcal{A}$*.*

Proof. Apply Theorem 4.1 together with Corollary 3.9, Corollary 3.8 and Theorem 3.6 respectively. ◻

Remark 4.3. (a) By Theorem 4.1, for every (cofinite) $\boldsymbol{Y}$ and every pair $\boldsymbol{X} \cong \boldsymbol{X}'$ in *pro*-$\mathcal{A}$, $(\boldsymbol{Y}^{\boldsymbol{X}}, d) \approx (\boldsymbol{Y}^{\boldsymbol{X}'}, d)$ in Met_c holds via the hom-bifunctor. Moreover, it is readily seen that, for every isomorphism $\boldsymbol{u} : \boldsymbol{X}' \to \boldsymbol{X}$, the homeomorphism $\hom(\boldsymbol{u}, \boldsymbol{1_Y})$ is an *isometry.* On the other hand, by Example 3.2 and Theorem 3.4, there exist an inverse sequence $\boldsymbol{Y}$ and a (countable and cofinite) inverse system $\boldsymbol{Y}'$ isomorphic to $\boldsymbol{Y}$, $\boldsymbol{Y} \cong \boldsymbol{Y}'$ in *pro*-$\mathcal{A}$, such that, for every isomorphism $\boldsymbol{v} : \boldsymbol{Y} \to \boldsymbol{Y}'$, the bijection $\hom(\boldsymbol{1_Y}, \boldsymbol{v}) : (\boldsymbol{Y}^{\boldsymbol{Y}}, d) \to (\boldsymbol{Y}'^{\boldsymbol{Y}}, d)$ is not continuous. Moreover, there is such a pair of metric spaces which are *not* homeomorphic (see Example 4.4 below). An important implication of this fact is that, in general, there is *no unique* canonical metrization of the shape morphism sets. Nevertheless, in some special cases (for instance, compact metrizable spaces, by using only sequential *HcANR*- or *HcPol*-expansions) a unique canonical complete (ultra)metrization of the shape morphism sets is possible.

(b) In the last decade several papers dealing with (ultra)metric and topology structures on the (standard) shape morphism sets were written: [3,4,15–19], ... The obtained results are interesting and useful because, in the first place, they have closely related many rather distant theories to the shape theory. Also, they admit to construct some new shape invariants. Looking for the basic idea which they exploit (as well as we do), one readily sees that it is the notion of *being* μ-homotopic (Definition 2.1). However, we ought to say that the germ of this idea goes back to 1976 when K. Borsuk [2] introduced the notion of quasi-equivalence of metric compacta. This is, indeed, quite clear after seeing the characterization (reinterpretation) of the quasi-equivalence in terms of sequences of morphisms of inverse sequences ([22]).

Example 4.4. Let $\boldsymbol{Y} = (Y_j, q_{jj'}, \mathbb{N})$ be an inverse sequence in a category $\mathcal{A}$, and let $\boldsymbol{Y}' = (Y'_\mu, q'_{\mu\mu'}, M)$ be associated with $\boldsymbol{Y}$ by the "Mardešić trick" (see Example 3.2). Then the space $(\boldsymbol{Y}'^{\boldsymbol{Y}}, d)$ is discrete (see the proof below). Therefore, by choosing a $\boldsymbol{Y}$ in *tow-HcANR* such that $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$ is not discrete (see Remark 2.7 and Example 2.8), one provides an example with $\boldsymbol{Y} \cong \boldsymbol{Y}'$ such that the spaces $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$ and $(\boldsymbol{Y}'^{\boldsymbol{Y}}, d)$ are *not* homeomorphic.

Let us prove that $(\boldsymbol{Y}'^{\boldsymbol{Y}}, d)$ of Example 4.4 is a discrete space. Since $diam(\boldsymbol{Y}'^{\boldsymbol{Y}}, d) \leq 1$, let us consider a pair $\boldsymbol{f}, \boldsymbol{f}' \in \boldsymbol{Y}'^{\boldsymbol{Y}}$ such that $d(\boldsymbol{f}, \boldsymbol{f}') < 1$, or equivalently, $d(\boldsymbol{f}, \boldsymbol{f}') \leq \frac{1}{2}$ (because d takes its values in $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$). We are to prove that $d(\boldsymbol{f}, \boldsymbol{f}') = 0$, i.e., that $\boldsymbol{f} = \boldsymbol{f}'$. Let $(f, f_\mu) \in \boldsymbol{f}$ and $(f', f'_\mu) \in \boldsymbol{f}'$ be any representatives. Then

$$\rho((f, f_\mu), (f', f'_\mu)) \leq \frac{1}{2},$$

which implies $(f, f_\mu) \simeq_1 (f', f'_\mu)$, i.e., $(f, f_\mu) \simeq_\mu (f', f'_\mu)$ for every $\mu \in M$, $|\mu| = 0$. By construction, $|\mu| = 0$ means $\mu = \{j\} \in M_0 \subseteq M$ $(= \bigsqcup_{k \in \mathbb{N}} M_{k-1}$, see the proof of Lemma 3.3) and $Y'_\mu = Y_j$, $j \in \mathbb{N}$. Thus,

$$(\forall j \in \mathbb{N})(\exists i_j \geq f(\{j\}), f'(\{j\}))(\forall i \geq i_j) \quad f_{\{j\}} q_{f(\{j\})i} = f'_{\{j\}} q_{f'(\{j\})i}.$$

Since $M_1 = \varnothing$, consider any $\mu = \{j, j'\} \in M_2 \subseteq M$, $j < j'$. Then $\{j\}, \{j'\} < \mu$, $Y'_\mu = Y_{j'}$, $q'_{\{j\}\mu} = q_{jj'}$ and $q'_{\{j'\}\mu} = 1_{Y_{j'}}$. Since $q'_{\{j'\}\mu} = 1_{Y_{j'}}$, the above relation and properties of morphisms of *inv*-$\mathcal{A}$ imply that there exists an $i_\mu \geq i_{j'}, f(\mu), f'(\mu)$ such that, for every $i \geq i_\mu$, $f_\mu q_{f(\mu)i} = f'_\mu q_{f'(\mu)i}$ holds. This shows that $(f, f_\mu) \simeq_\mu (f', f'_\mu)$ for every $\mu \in M$, $|\mu| \leq 2$, i.e., $(f, f_\mu) \simeq_3 (f', f'_\mu)$, and thus,

$$\rho((f, f_\mu), (f', f'_\mu)) \leq \frac{1}{4}.$$

Now, by induction on $k \in \mathbb{N}$, assuming that

$$\rho((f, f_\mu), (f', f'_\mu)) \leq \frac{1}{k+1},$$

one can prove, in the same way as above, that

$$\rho((f, f_\mu), (f', f'_\mu)) \leq \frac{1}{k+1+l_k},$$

holds, for some $l_k \in \mathbb{N}$. Therefore, $d(\boldsymbol{f}, \boldsymbol{f}') = \rho((f, f_\mu), (f', f'_\mu)) = 0$, i.e., $\boldsymbol{f} = \boldsymbol{f}'$. This shows that, for each $\boldsymbol{f} \in \boldsymbol{Y}'^{\boldsymbol{Y}}$ and every $0 < \varepsilon \leq 1$, the open ball $B(\boldsymbol{f}, \varepsilon) = \{\boldsymbol{f}\} \subseteq (\boldsymbol{Y}'^{\boldsymbol{Y}}, d)$, which completes the proof.

5. Applications

Our aim is to show that the introduced complete metric structure on the sets $\boldsymbol{Y}^{\boldsymbol{X}}$ admits a much better view into quasi-equivalence ([2]) as well as into its strengthening, so called $\overline{q}$-equivalence ([22]).

5.1. *Borsuk's quasi-equivalence.* Let us briefly recall the quasi equivalence of metric compacta. It was originally defined and studied in [2] by means of fundamental sequences ([1]) and neighborhoods in a pair of AR ambient spaces. Afterwards, it was characterized by sequences of morphisms of compact ANR inverse sequences ([22, Section 4]). We are now able to reinterpret it in the metric terms introduced in this paper:

Two metric compacta X and Y are quasi-equivalent, $X \overset{q}{\simeq} Y$, if and only if there is a (equivalently, for every) pair of associated $\boldsymbol{X}$ and $\boldsymbol{Y}$ of *tow-HcANR* and there is a pair of sequences $(\boldsymbol{f}^n)$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, $(\boldsymbol{g}^n)$ in $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$ such that $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{1}_{\boldsymbol{X}}$ in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ and $\lim(\boldsymbol{f}^n\boldsymbol{g}^n) = \boldsymbol{1}_{\boldsymbol{Y}}$ in $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$, i.e., for every $n \in \mathbb{N}$,

$$d(\boldsymbol{g}^n\boldsymbol{f}^n, \boldsymbol{1}_{\boldsymbol{X}}) \leq \frac{1}{n+1} \quad \text{and} \quad d(\boldsymbol{f}^n\boldsymbol{g}^n, \boldsymbol{1}_{\boldsymbol{Y}}) \leq \frac{1}{n+1}.$$

Notice that our Corollary 3.10 holds true because of the following fact: If $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ and $\lim(\boldsymbol{g}^n) = \boldsymbol{g}^0$ in $(\boldsymbol{Z}^{\boldsymbol{Y}}, d)$, $\boldsymbol{g}^0 \in \boldsymbol{Z}^{\boldsymbol{Y}}_{\mathrm{U}}$, then $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{g}^0\boldsymbol{f}^0$ in $(\boldsymbol{Z}^{\boldsymbol{X}}, d)$. Clearly, the converse does not hold, i.e., if $(\boldsymbol{g}^n\boldsymbol{f}^n)$ converges, then the sequence $(\boldsymbol{f}^n)$ $((\boldsymbol{g}^n))$ might not converge even if $(\boldsymbol{g}^n)$ $((\boldsymbol{f}^n))$ converges. It is enough to take for $\boldsymbol{Z}$ $(\boldsymbol{X})$ the trivial inverse sequence, and for $\boldsymbol{X}$, $\boldsymbol{Y}$ $(\boldsymbol{Y}, \boldsymbol{Z})$ an appropriate pair. We pay a special attention to the case $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{1}_{\boldsymbol{X}}$ and $\lim(\boldsymbol{f}^n\boldsymbol{g}^n) = \boldsymbol{1}_{\boldsymbol{Y}}$. Then again, in general, the sequences $(\boldsymbol{f}^n)$ and $(\boldsymbol{g}^n)$ do not converge. This immediately confirms the well known fact that the quasi-equivalence is strictly coarser than the shape type classification. It also indicates the reason why the quasi-equivalence, in general, is not transitive ([10]). Further, if $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{w}$, then $\boldsymbol{w}$, in general, does not admit a factorization through $\boldsymbol{Y}$. Indeed, if this would hold, then the quasi-equivalence would imply the shape domination, which is not the case. Namely, if $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{1}_{\boldsymbol{X}}$ and $\lim(\boldsymbol{f}^n\boldsymbol{g}^n) = \boldsymbol{1}_{\boldsymbol{Y}}$ would imply $\boldsymbol{1}_{\boldsymbol{X}} = \boldsymbol{g}^0\boldsymbol{f}^0$ or $\boldsymbol{1}_{\boldsymbol{Y}} = \boldsymbol{f}'^0\boldsymbol{g}'^0$, then $\boldsymbol{X} \leq \boldsymbol{Y}$ or $\boldsymbol{Y} \leq \boldsymbol{X}$ in *tow-HcANR*, which contradicts the known examples ([2,22]).

By the above characterization, the notion of quasi-equivalence can be defined generally in any category *pro-*$\mathcal{A}$, especially, in the category *tow-*$\mathcal{A}$, for any $\mathcal{A}$. Then we can characterize a pair of isomorphic objects of *tow-*$\mathcal{A}$ as follows.

THEOREM 5.1. *(i) Two inverse sequences $\boldsymbol{X}$, $\boldsymbol{Y}$ in a category $\mathcal{A}$ are isomorphic, $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow-$\mathcal{A}$, if and only if $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ and there exists a pair of Cauchy sequences realizing this quasi-equivalence.*

(ii) If $\boldsymbol{X} \overset{q}{\leq} \boldsymbol{Y}$ ($\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$), then $\boldsymbol{X} \leq \boldsymbol{Y}$ ($\boldsymbol{X} \cong \boldsymbol{Y}$) in tow-$\mathcal{A}$ provided the space $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ (as well as $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$) is discrete.

PROOF. (i). It is enough to prove the sufficiency part. Let $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$, i.e., $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{1}_{\boldsymbol{X}}$ and $\lim(\boldsymbol{f}^n\boldsymbol{g}^n) = \boldsymbol{1}_{\boldsymbol{Y}}$, where $(\boldsymbol{f}^n)$, $(\boldsymbol{g}^n)$ are Cauchy sequences in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ and $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$ respectively. By Theorem 2.6, the sequences

$(\boldsymbol{f}^n)$ and $(\boldsymbol{g}^n)$ converge, $\lim(\boldsymbol{f}^n) = \boldsymbol{f}^0$ and $\lim(\boldsymbol{g}^n) = \boldsymbol{g}^0$. By Corollary 3.10, $\boldsymbol{g}^0\boldsymbol{f}^0 = \mathbf{1}_{\boldsymbol{X}}$ and $\boldsymbol{f}^0\boldsymbol{g}^0 = \mathbf{1}_{\boldsymbol{Y}}$. Thus, $\boldsymbol{X} \cong \boldsymbol{Y}$ in *tow*-$\mathcal{A}$.

(ii). Let $\boldsymbol{X} \overset{q}{\leq} \boldsymbol{Y}$. Then there exists a pair of sequences $(\boldsymbol{f}^n)$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, $(\boldsymbol{g}^n)$ in $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$ such that $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \mathbf{1}_{\boldsymbol{X}}$ in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$. Since $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ is discrete, there exists an $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\boldsymbol{g}^n\boldsymbol{f}^n = \mathbf{1}_{\boldsymbol{X}}$. Hence, $\boldsymbol{X} \leq \boldsymbol{Y}$. In the case $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$, the proof is quite similar. □

Corollary 5.2. (i) *Two metrizable compacta X and Y have the same shape, $Sh(X) = Sh(Y)$, if and only if they are quasi-equivalent, $X \overset{q}{\simeq} Y$, and there exists a pair of Cauchy sequences realizing this quasi-equivalence.*

(ii) *If $X \overset{q}{\leq} Y$ ($X \overset{q}{\simeq} Y$), then $Sh(X) \leq Sh(Y)$ ($Sh(X) = Sh(Y)$) provided every/some associated space $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ (as well as every/some $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$) is discrete.*

Proof. Choose a pair $\boldsymbol{X}$, $\boldsymbol{Y}$ of compact ANR inverse sequences associated with X, Y respectively ($\lim \boldsymbol{X} = X$ and $\lim \boldsymbol{Y} = Y$), put the new bonding mappings to be the homotopy classes, and apply Theorem 5.1. □

Remark 5.3. Observe that the quasi-equivalence, generally, realizes *without* any Cauchy sequence. Indeed, if in every case one of $(\boldsymbol{f}^n)$, $(\boldsymbol{g}^n)$ would be a Cauchy sequence, then it could be replaced by its limit morphism. However, then the quasi-equivalence would be transitive ([22, proof of Lemma 9 and Remark 4]), which contradicts the main result of [10]. Thus, one may say that Corollary 5.2 shows (measures) how far the quasi-equivalence is from the shape type.

If we want to study objects of a category by means of inverse systems, we ought to consider a category pair $(\mathcal{C}, \mathcal{D})$, $\mathcal{D} \subseteq \mathcal{C}$, such that every $\mathcal{C}$-object X admits a $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{X} \in Ob(pro\text{-}\mathcal{D})$ ([13]). In some special cases, the sequential subpro-category $pro^{\mathbb{N}}\text{-}\mathcal{D} \equiv tow\text{-}\mathcal{D}$ suffices. Then one usually says that $\mathcal{D}$ is *sequentially dense* in $\mathcal{C}$. In that case, there exists the corresponding (abstract) shape category $Sh_{(\mathcal{C},\mathcal{D})}$ realized via *tow*-$\mathcal{D}$, i.e.,

$$Sh(X, Y) \approx tow\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y}).$$

Especially, if $Y \in Ob\mathcal{D}$, then every shape morphism $\phi : X \to Y$, i.e., every $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of *tow*-$\mathcal{D}$, admits a unique representative $f : X \to Y$ of $\mathcal{C}$. The most interesting example is $\mathcal{C} = HcM$ (the homotopy category of metrizable compacta) and $\mathcal{D} = HcANR$ (the homotopy category of compact ANR's) or $\mathcal{D} = HcPol$ (the homotopy category of compact polyhedra). We hereby also want to involve in our considerations the S-equivalence ([11,12]) and S^*-equivalence ([14]) (as well as the S_n- and S_n^+-equivalence of [23] and [5]). These equivalences and corresponding dominations are well defined in every category *tow*-$\mathcal{A}$.

Theorem 5.4. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in a category $\mathcal{A}$.*

(i) *Let $\boldsymbol{X}$ be semi-stable. Then, $\boldsymbol{X}$ is dominated by $\boldsymbol{Y}$ in tow-$\mathcal{A}$ if and only if $\boldsymbol{X}$ is quasi-dominated by $\boldsymbol{Y}$, i.e.,*

$$\boldsymbol{X} \leq \boldsymbol{Y} \Leftrightarrow \boldsymbol{X} \overset{q}{\leq} \boldsymbol{Y}.$$

Consequently, $\boldsymbol{X} \overset{q}{\leq} \boldsymbol{Y}$ implies $S_0^+(\boldsymbol{X}) \leq S_0^+(\boldsymbol{Y})$, whenever $\boldsymbol{X}$ is semi-stable.

(ii) *If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are regularly movable or they both are stable, then $S_0^+(\boldsymbol{X}) \leq S_0^+(\boldsymbol{Y})$ implies $\boldsymbol{X} \overset{q}{\leq} \boldsymbol{Y}$.*

Proof. First, we will show that, for every semi-stable $\boldsymbol{X}$ and every $\boldsymbol{Z} = (Z_\nu, s_{\nu\nu'}, N)$, the space $(\boldsymbol{X}^{\boldsymbol{Z}}, d)$ is discrete. It suffices to prove that there exists a $k_{\boldsymbol{X}} \in \mathbb{N}$ such that, for every pair $\boldsymbol{h}, \boldsymbol{h}' : \boldsymbol{Z} \to \boldsymbol{X}$,

$$d(\boldsymbol{h}, \boldsymbol{h}') \leq \frac{1}{k_{\boldsymbol{X}} + 1} \Rightarrow \boldsymbol{h} = \boldsymbol{h}'.$$

Since $\boldsymbol{X}$ is semi-stable, there exists an $i_0 \in \mathbb{N}$ such that

$$(\forall i \geq i_0)(\forall i' \geq i)(\exists r : X_i \to X_{i'})(\exists i_1 \geq i')(\forall i'' \geq i_1) \quad r p_{ii''} = p_{i'i''}.$$

Put $k_{\boldsymbol{X}} = i_0$, and let $\boldsymbol{h}, \boldsymbol{h}' : \boldsymbol{Z} \to \boldsymbol{X}$ be given such that $d(\boldsymbol{h}, \boldsymbol{h}') \leq \frac{1}{k_{\boldsymbol{X}}+1}$. This means that, for every pair of representatives $(h, h_i) \in \boldsymbol{h}$, $(h', h'_i) \in \boldsymbol{h}'$,

$$(h, h_i) \simeq_k (h', h'_i),$$

whenever $k \leq k_{\boldsymbol{X}} = i_0$. Without loss of generality, we may suppose that $h \leq h'$. Let $k \equiv i' > i_0$. Then, for $i = i_0$ and $i' > i$, there exist an $r : X_{i_0} \to X_{i'}$ and an $i_1 \geq i'$ such that, for every $i'' \geq i_1$, $r p_{i_0 i''} = p_{i'i''}$ holds. Choose a $\nu'' \in N$ to be a commutativity index for (h, h_i) and (h', h'_i) with respect to $i \leq i' \leq i'' = i_1$. It is readily seen, by chasing the diagram

$$\begin{array}{ccccccc}
\nu_0 \leftarrow \nu'_0 & \longleftarrow & \nu \leftarrow \nu' & \longleftarrow & \nu_1 \leftarrow \nu'_1 & \leftarrow \nu'' & \\
h_\cdot \downarrow\swarrow h'_\cdot & & h_\cdot \downarrow\swarrow h'_\cdot & & h_\cdot \downarrow\swarrow h'_\cdot & & , \\
i = i_0 & \underset{\overrightarrow{r}}{\longleftarrow} & i' = k & \longleftarrow & i'' = i_1 & &
\end{array}$$

that $h_{i'} s_{\nu\nu''} = h'_{i'} s_{\nu'\nu''}$, which implies that $(h, h_i) \simeq_k (h', h'_i)$. This proves that $(h, h_i) \simeq_k (h', h'_i)$ for all $k \in \mathbb{N}$. Thus, by Lemma 2.2 (v), $(h, h_i) \simeq (h', h'_i)$, i.e., $\boldsymbol{h} = \boldsymbol{h}'$.

To prove the first assertion, we need to prove the sufficiency part only. Consider a pair of sequences $(\boldsymbol{f}^n)$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, $(\boldsymbol{g}^n)$ in $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$ such that $\lim(\boldsymbol{g}^n \boldsymbol{f}^n) = \boldsymbol{1}_{\boldsymbol{X}}$ in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$. Since the space $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ is discrete, the sequence $(\boldsymbol{g}^n \boldsymbol{f}^n)$ must be a stationary one. It implies that there exists an $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\boldsymbol{g}^n \boldsymbol{f}^n = \boldsymbol{1}_{\boldsymbol{X}}$. Therefore, $\boldsymbol{X} \leq \boldsymbol{Y}$ in *tow-$\mathcal{A}$*. The second assertion of (i) follows now by [23, Theorem 2.15] (a general case; the notation according to [5, Definition 1]).

Let $S_0^+(\boldsymbol{X}) \leq S_0^+(\boldsymbol{Y})$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are regularly movable, i.e., all p_{ii+1} and q_{jj+1} are retractions of $\mathcal{A}$. Then condition $S_0^+(\boldsymbol{Y},\boldsymbol{X})$ holds, which means

$$(\forall i_1 \in \mathbb{N})(\exists j_1 \in \mathbb{N})(\forall j_1' \geq j_1)(\exists i_1' \geq i_1)$$

and there exist $\mathcal{A}$-morphisms $g_1 : Y_{j_1} \to X_{i_1}$ and $f_1 : X_{i_1'} \to Y_{j_1'}$ making the following diagram commutative

$$\begin{array}{ccc} Y_{j_1} & \leftarrow & Y_{j_1'} \\ g_1 \downarrow & & \uparrow f_1 \\ X_{i_1} & \leftarrow & X_{i_1'} \end{array}.$$

Since $\boldsymbol{X}$ and $\boldsymbol{Y}$ are regularly movable, the morphisms g_1 and f_1 generate in an obvious way the morphisms $\boldsymbol{g}^{i_1} : \boldsymbol{Y} \to \boldsymbol{X}$ and $\boldsymbol{f}^{j_1'} : \boldsymbol{X} \to \boldsymbol{Y}$ of *tow*-$\mathcal{A}$, respectively, such that

$$d(\boldsymbol{g}^{i_1}\boldsymbol{f}^{j_1'}, \boldsymbol{1_X}) \leq \frac{1}{i_1+1}.$$

Namely, one has to put $g_i = p_{ii_1}g_1$, $i \leq i_1$, and $g_i = u_i g_1$, $i > i_1$, where $u_i : X_{i_1} \to X_i$ is the corresponding section; similarly for f_j via f_1, $j \in \mathbb{N}$. Observe that it holds for each $i \in \mathbb{N}$ and each corresponding j'. Therefore, by appropriate inductive construction, there exist sequences $(\boldsymbol{g}^n)$ in $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$ and $(\boldsymbol{f}^n)$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ such that $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{1_X}$ in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$. Hence, $\boldsymbol{X} \overset{q}{\leq} \boldsymbol{Y}$. In the case of $\boldsymbol{X}$ and $\boldsymbol{Y}$ stable, the proof is much simpler. Namely, then there exists a pair $P, Q \in Ob\mathcal{A}$ such that $\boldsymbol{X} \cong \boldsymbol{P}$ and $\boldsymbol{Y} \cong \boldsymbol{Q}$ in *tow*-$\mathcal{A}$, where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are inverse sequences generated by the identities 1_P and 1_Q respectively. Then, clearly, $S_0^+(\boldsymbol{X}) \leq S_0^+(\boldsymbol{Y})$ implies $S_0^+(\boldsymbol{P}) \leq S_0^+(\boldsymbol{Q})$, which reduces to a pair of $\mathcal{A}$-morphisms $g : Q \to P$, $f : P \to Q$ such that the diagram

$$\begin{array}{ccc} Q & \overset{1_Q}{\leftarrow} & Q \\ g \downarrow & & \uparrow f \\ P & \overset{1_P}{\leftarrow} & P \end{array}$$

commutes. It follows that $P \leq Q$ in $\mathcal{A}$, and consequently, $\boldsymbol{P} \leq \boldsymbol{Q}$ in *tow*-$\mathcal{A}$. Then $\boldsymbol{P} \overset{\boldsymbol{q}}{\leq} \boldsymbol{Q}$ follows trivially. Since the quasi-domination is invariant with respect to isomorphisms of *tow*-$\mathcal{A}$ ([22, Lemma 2], a general case), it follows that $\boldsymbol{X} \overset{q}{\leq} \boldsymbol{Y}$. □

Recall that the stability (strictly) implies strong movability ([24,7]). Further, an FANR is characterized by the strong movability of any associated inverse sequence in $HcANR$ or $HcPol$ ([13]). Finally, the strong movability does *not* imply regular movability ([6,7,9]). However, the next corollary holds.

COROLLARY 5.5. *Let X and Y be compact metrizable spaces.*

(i) *Let X be semi-stable (especially, an FANR). Then X is shape dominated by Y if and only if X is quasi-dominated by Y, i.e.,*

$$Sh(X) \leq Sh(Y) \Leftrightarrow X \overset{q}{\leq} Y.$$

Consequently, $X \overset{q}{\leq} Y$ implies $S_0^+(X) \leq S_0^+(Y)$, whenever X is semi-stable.

(ii) *If X and Y are regularly movable or they both are FANR's, then $S_0^+(X) \leq S_0^+(Y)$ implies $X \overset{q}{\leq} Y$.*

PROOF. We only have to prove that $S_0^+(X) \leq S_0^+(Y)$, where X and Y are FANR's, implies $X \overset{q}{\leq} Y$. Namely, in general, a strongly movable inverse sequence of *tow-HcANR* (associated with an FANR) is *not* stable ([7,24]). Hence, we may *not* apply the appropriate statement of Theorem 5.4. However, every such a sequence (FANR) is stable with respect to *HANR* ([23, Lemma 2.13 and Remark 2.14 (b)]). Therefore, one only has to verify that the corresponding part of the proof of Theorem 5.4 works for the noncompact ANR inverse sequences $\boldsymbol{P}$ and $\boldsymbol{Q}$ as well. ∎

REMARK 5.6. J. M. R. Sanjurjo proved in his paper [20] that the quasi-domination on the class of all FANR's is equivalent to the *shape* domination. The result from above (Corollary 5.5 (i)) strengthens the former because it assumes that only the dominated compactum is an FANR.

In the case of quasi-equivalence on the semi-stable inverse sequences (compacta), one can get even more.

THEOREM 5.7. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in a category $\mathcal{A}$.*

(i) *If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are semi-stable, then, $\boldsymbol{X}$ is isomorphic to $\boldsymbol{Y}$ in tow-$\mathcal{A}$ if and only if $\boldsymbol{X}$ is quasi-equivalent to $\boldsymbol{Y}$, i.e.,*

$$\boldsymbol{X} \cong \boldsymbol{Y} \Leftrightarrow \boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}.$$

Consequently, $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ implies $S^*(\boldsymbol{X}) = S^*(\boldsymbol{Y})$, whenever $\boldsymbol{X}$ and $\boldsymbol{Y}$ are semi-stable.

(ii) *If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are regularly movable or they both are stable, then $S_1(\boldsymbol{X}) \leq S_1(\boldsymbol{Y})$ (or $S_1(\boldsymbol{Y}) \leq S_1(\boldsymbol{X})$) implies $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$.*

PROOF. The necessity part of assertion (i) is trivial. Conversely, as in the proof of Theorem 5.4, if $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$, where $\boldsymbol{X}$ ($\boldsymbol{Y}$) is semi-stable, then $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{1_X}$ ($\lim(\boldsymbol{f}^n\boldsymbol{g}^n) = \boldsymbol{1_Y}$) implies that the sequence $(\boldsymbol{g}^n\boldsymbol{f}^n)$ $((\boldsymbol{f}^n\boldsymbol{g}^n))$ is a stationary one. Thus, there exists an $n_1 \in \mathbb{N}$ ($n_2 \in \mathbb{N}$) such that, for every $n \geq n_1$ ($n \geq n_2$), $\boldsymbol{g}^n\boldsymbol{f}^n = \boldsymbol{1_X}$ ($\boldsymbol{f}^n\boldsymbol{g}^n = \boldsymbol{1_Y}$). Hence, if $n \geq n_0 = \max\{n_1, n_2\}$, then $\boldsymbol{g}^n\boldsymbol{f}^n = \boldsymbol{1_X}$ and $\boldsymbol{f}^n\boldsymbol{g}^n = \boldsymbol{1_Y}$, which means $\boldsymbol{X} \cong \boldsymbol{Y}$ in *tow*-$\mathcal{A}$.

Let $S_1(\boldsymbol{X}) \leq S_1(\boldsymbol{Y})$, i.e., let condition $S_1(\boldsymbol{X},\boldsymbol{Y})$ is fulfilled ([5, Definition 1]). Then,

$$(\forall j_1 \in \mathbb{N})(\exists i_1 \in \mathbb{N})(\forall i_1' \geq i_1)(\exists j_1' \geq j_1)(\forall j_2 \geq j_1')(\exists i_2 \geq i_1')$$

and there exist $\mathcal{A}$-morphisms $f_1 : X_{i_1} \to Y_{j_1}$, $g_1 : Y_{j_1'} \to X_{i_1'}$ and $f_2 : X_{i_2} \to Y_{j_2}$ such that the following diagram in $\mathcal{D}$ commutes:

$$\begin{array}{ccccc} X_{i_1} & \leftarrow & X_{i_1'} & \leftarrow & X_{i_2} \\ f_1 \downarrow & & g_1 \uparrow & & \downarrow f_2 \\ Y_{j_1} & \leftarrow & Y_{j_1'} & \leftarrow & Y_{j_2} \end{array} .$$

Now, if $\boldsymbol{Y}$ and $\boldsymbol{X}$ are regularly movable, the morphisms f_1, f_2 and g_1 generate (see the proof of Theorem 5.4) morphisms $\boldsymbol{f}^{j_1} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g}^{j_1'} : \boldsymbol{X} \to \boldsymbol{Y}$ of *tow*-$\mathcal{A}$, respectively, such that

$$d(\boldsymbol{g}^{i_1'}\boldsymbol{f}^{j_1}, \mathbf{1}_{\boldsymbol{X}}) \leq \frac{1}{j_1+1} \quad \text{and} \quad d(\boldsymbol{f}^{j_1}\boldsymbol{g}^{i_1'}, \mathbf{1}_{\boldsymbol{Y}}) \leq \frac{1}{i_1'+1}.$$

Observe that the above relations hold for each $j \in \mathbb{N}$ and each corresponding i'. Therefore, by an inductive construction, there exist sequences $(\boldsymbol{f}^n)$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ and $(\boldsymbol{g}^n)$ in $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$ such that $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \mathbf{1}_{\boldsymbol{X}}$ in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ and $\lim(\boldsymbol{f}^n\boldsymbol{g}^n) = \mathbf{1}_{\boldsymbol{Y}}$ in $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$. Hence, $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$. In the case of $\boldsymbol{X}$ and $\boldsymbol{Y}$ stable, the proof may be as follows. There exists a pair $P, Q \in Ob\mathcal{A}$ such that $\boldsymbol{X} \cong \boldsymbol{P}$ and $\boldsymbol{Y} \cong \boldsymbol{Q}$ in *tow*-$\mathcal{A}$, where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are inverse sequences generated by the identities 1_P and 1_Q respectively. Then, clearly, $S_1(\boldsymbol{X}) \leq S_1(\boldsymbol{Y})$ implies $S_1(\boldsymbol{P}) \leq S_1(\boldsymbol{Q})$, which reduces to three $\mathcal{A}$-morphisms $u_1 : P \to Q$, $v_1 : Q \to P$, $u_2 : P \to Q$ making the diagram

$$\begin{array}{ccccc} P & \overset{1}{\leftarrow} & P & \overset{1}{\leftarrow} & P \\ u_1 \downarrow & & v_1 \uparrow & & \downarrow u_2 \\ Q & \overset{1}{\leftarrow} & Q & \overset{1}{\leftarrow} & Q \end{array}$$

commutative. Thus, $u_1v_1 = 1_Q$, $v_1u_2 = 1_P$ and $u_1 = u_2$. It follows that $P \cong Q$ in $\mathcal{A}$, and consequently, $\boldsymbol{P} \cong \boldsymbol{Q}$ in *tow*-$\mathcal{A}$. Then $\boldsymbol{P} \overset{q}{\simeq} \boldsymbol{Q}$ follows trivially. Since the quasi-equivalence is invariant with respect to isomorphisms of *tow*-$\mathcal{A}$ ([22, Lemma 2], a general case), it follows that $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$. ∎

COROLLARY 5.8. *Let X and Y be compact metrizable spaces.*

(i) *If X and Y are semi-stable (especially, FANR's), then, X is shape equivalent to Y if and only if X is quasi-equivalent to Y, i.e.,*

$$Sh(X) = Sh(Y) \Leftrightarrow X \overset{q}{\simeq} Y.$$

Consequently, $X \overset{q}{\simeq} Y$ implies $S^*(X) = S^*(Y)$, whenever X and Y are semi-stable.

(ii) *If X and Y are regularly movable or they both are FANR's, then $S_1(X) \leq S_1(Y)$ (or $S_1(Y) \leq S_1(X)$) implies $X \overset{q}{\simeq} Y$.*

Proof. Only assertion (ii) in the case of FANR's needs an extra proof. However, the corresponding part of the proof of Theorem 5.7 works for non-compact ANR inverse sequences $\boldsymbol{P}$ and $\boldsymbol{Q}$ ("associated" with X and Y respectively) as well. The conclusion follows. ∎

5.2. *The $\overline{q}$-equivalence.* Recall now the $\overline{q}$-equivalence of inverse sequences in *tow-HcANR* (associated with metrizable compacta) introduced in [22], Section 5. By the definition and full category characterization ([22, Theorem 6]), an $\boldsymbol{X}$ is *$\overline{q}$-equivalent to* a $\boldsymbol{Y}$, $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$, if and only if $\boldsymbol{X}$ is quasi-equivalent to $\boldsymbol{Y}$, $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$, and there exists a pair of realizing sequences $(\boldsymbol{f}^n), (\boldsymbol{g}^n)$ having *unique* increasing index functions. In terms of this paper, it means that

$$\lim(\boldsymbol{g}^n \boldsymbol{f}^n) = \mathbf{1}_{\boldsymbol{X}} \quad \text{and} \quad \lim(\boldsymbol{f}^n \boldsymbol{g}^n) = \mathbf{1}_{\boldsymbol{Y}},$$

in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ and $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$ respectively, and for every $n \in \mathbb{N}$, $\boldsymbol{f}^n = [(f, [f_j^n])]$ and $\boldsymbol{g}^n = [(g, [g_i^n])]$. Clearly, the $\overline{q}$-equivalence is a kind of "uniformization" of quasi-equivalence with respect to the index functions. It is strictly finer than the quasi-equivalence, because, for instance, it is an equivalence relation ([22]), while the quasi-equivalence is not (transitive, [10]). Notice that there is an obvious generalization to *tow-$\mathcal{A}$*, for any category $\mathcal{A}$. Two metrizable compacta X and Y are $\overline{q}$-equivalent, $X \overset{\overline{q}}{\simeq} Y$, if and only if there is a (equivalently, for every) pair of associated $\boldsymbol{X}$ and $\boldsymbol{Y}$ in *tow-HcANR* such that $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$. According to Theorem 5.4 (Corollary 5.5), the relations $\overset{q}{\leq}$, $\overset{\overline{q}}{\leq}$ and the category domination coincide on the class of all semi-stable inverse sequences of a category *tow-$\mathcal{A}$* (especially, on all FANR's in HcM). Further, by Theorem 5.7 (Corollary 5.8), the relations $\overset{q}{\simeq}$, $\overset{\overline{q}}{\simeq}$ and the category isomorphiness coincide on the class of all semi-stable inverse sequences of a category *tow-$\mathcal{A}$* (especially, on all FANR's in HcM).

By applying the previously developed technique and results, we are able to prove the following analogue of Theorem 5.1.

Theorem 5.9. *Two inverse sequences $\boldsymbol{X}$, $\boldsymbol{Y}$ in a category $\mathcal{A}$ are isomorphic, $\boldsymbol{X} \cong \boldsymbol{Y}$ in tow-$\mathcal{A}$, if and only if $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$ and there is a pair of realizing sequences such that one of them is a Cauchy sequence.*

First, some general auxiliary facts.

Lemma 5.10. *Let $\boldsymbol{u}, \boldsymbol{u}' \in (\boldsymbol{Y}^{\boldsymbol{X}}, d)$, $\boldsymbol{v}, \boldsymbol{v}' \in (\boldsymbol{Z}^{\boldsymbol{Y}}, d)$ and $k \in \mathbb{N}$, such that $d(\boldsymbol{vu}, \boldsymbol{v}'\boldsymbol{u}') \leq \frac{1}{k+1}$ in $(\boldsymbol{Z}^{\boldsymbol{X}}, d)$.*

(i) *If $d(\boldsymbol{u}, \boldsymbol{u}') \leq \frac{1}{l+1}$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$ and $v(k), v'(k) \leq l \in \mathbb{N}$, then the distance between any pair of points of $\{\boldsymbol{vu}, \boldsymbol{v}'\boldsymbol{u}, \boldsymbol{vu}', \boldsymbol{v}'\boldsymbol{u}'\} \subseteq (\boldsymbol{Z}^{\boldsymbol{X}}, d)$ is less or equal $\frac{1}{k+1}$. If, in addition, $\boldsymbol{u}$ (or $\boldsymbol{u}'$) is a retraction of tow-$\mathcal{A}$, then $d(\boldsymbol{v}, \boldsymbol{v}') \leq \frac{1}{k+1}$ in $(\boldsymbol{Z}^{\boldsymbol{Y}}, d)$.*

(ii) *If* $d(\boldsymbol{v},\boldsymbol{v}') \leq \frac{1}{l+1}$ *in* $(\boldsymbol{Z}^{\boldsymbol{Y}}, d)$, *then the distance between any pair of points of* $\{\boldsymbol{vu}, \boldsymbol{v}'\boldsymbol{u}, \boldsymbol{vu}', \boldsymbol{v}'\boldsymbol{u}'\} \subseteq (\boldsymbol{Z}^{\boldsymbol{X}}, d)$ *is less or equal* $\frac{1}{r+1}$, *where* $r = \min\{k,l\}$. *If, in addition,* $\boldsymbol{v}$ *(or* $\boldsymbol{v}'$*) is a section of tow-*$\mathcal{A}$, *then* $d(\boldsymbol{u},\boldsymbol{u}') \leq \frac{1}{s+1}$ *in* $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, *provided there exists a representative* (w, w_j), w *increasing, (or* (w', w'_j), w' *increasing) of a left inverse of* $\boldsymbol{v}$ *(or* $\boldsymbol{v}'$*) such that* $w(s) \leq r$ $(w'(s) \leq r)$.

Proof. Let $d(\boldsymbol{u},\boldsymbol{u}') \leq \frac{1}{l+1}$ and $v(k), v'(k) \leq l$. Then, by Lemma 2.4 (vi), $d(\boldsymbol{vu},\boldsymbol{vu}') \leq \frac{1}{k+1}$ and $d(\boldsymbol{v}'\boldsymbol{u}',\boldsymbol{v}'\boldsymbol{u}) \leq \frac{1}{k+1}$. Since $d(\boldsymbol{vu},\boldsymbol{v}'\boldsymbol{u}') \leq \frac{1}{k+1}$ and d is an ultrametric (Theorem 2.6), the first assertion follows (by assuming that u and v are increasing, and $u \leq u'$, $v \leq v'$ and $u'v \leq uv'$, one can provide a direct proof by chasing the diagram

$$\begin{array}{llll} uv(k) \leftarrow u'v(k) \leftarrow & uv'(k) \leftarrow u'v'(k) & \longleftarrow & i \\ u_{\cdot} \downarrow\swarrow u'_{\cdot} & u_{\cdot} \downarrow\swarrow u'_{\cdot} & & \\ v(k) \longleftarrow & v'(k) \longleftarrow & j = l & \\ v_{\cdot} \downarrow\swarrow v' & & & \\ \quad k & & & \end{array} \quad).$$

If $\boldsymbol{u}$ $(\boldsymbol{u}')$ is a retraction, then by Lemma 2.4 (v) $d(\boldsymbol{vu},\boldsymbol{v}'\boldsymbol{u}) \leq \frac{1}{k+1}$ $(d(\boldsymbol{vu}',\boldsymbol{v}'\boldsymbol{u}') \leq \frac{1}{k+1})$ implies $d(\boldsymbol{v},\boldsymbol{v}') \leq \frac{1}{k+1}$. This proves statement (i).

Let $d(\boldsymbol{v},\boldsymbol{v}') \leq \frac{1}{l+1}$. Then, by Lemma 2.4 (v), $d(\boldsymbol{vu},\boldsymbol{v}'\boldsymbol{u}) \leq \frac{1}{l+1}$ and $d(\boldsymbol{vu}',\boldsymbol{v}'\boldsymbol{u}') \leq \frac{1}{l+1}$. Since $d(\boldsymbol{vu},\boldsymbol{v}'\boldsymbol{u}') \leq \frac{1}{k+1}$ and d is an ultrametric, the first assertion of (ii) follows (by assuming that u and v are increasing, and $u \leq u'$, $v \leq v'$ and $u'v \leq uv'$, one can provide a direct proof by chasing the diagram

$$\begin{array}{llll} uv(r) \leftarrow u'v(r) \leftarrow & uv'(r) \leftarrow u'v'(r) \leftarrow i & \leftarrow uv(j') \leftarrow u'v(j') \leftarrow i' & \\ u_{\cdot} \downarrow\swarrow u'_{\cdot} & u_{\cdot} \downarrow\swarrow u'_{\cdot} & u_{\cdot} \downarrow\swarrow u'_{\cdot} & \\ v(r) \longleftarrow & v'(r) \leftarrow j \leftarrow & j' &). \\ v_{\cdot} \downarrow\swarrow v' & & & \\ r = \min\{k,l\} & \leftarrow & & \end{array}$$

If $\boldsymbol{v}$ $(\boldsymbol{v}')$ is a section such that there exists a desired representative of a left inverse, then by Lemma 2.4 (vi), $d(\boldsymbol{vu},\boldsymbol{vu}') \leq \frac{1}{r+1}$ $(d(\boldsymbol{v}'\boldsymbol{u},\boldsymbol{v}'\boldsymbol{u}') \leq \frac{1}{r+1})$ implies $d(\boldsymbol{u},\boldsymbol{u}') \leq \frac{1}{s+1}$. This proves statement (ii). □

Proof of Theorem 5.9. We need to prove the sufficiency part only. Let $\lim(\boldsymbol{g}^n\boldsymbol{f}^n) = \boldsymbol{1}_{\boldsymbol{X}}$ in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ and $\lim(\boldsymbol{f}^n\boldsymbol{g}^n) = \boldsymbol{1}_{\boldsymbol{Y}}$ in $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$ such that, for every $n \in \mathbb{N}$, $\boldsymbol{f}^n = [(f, f_j^n)]$ and $\boldsymbol{g}^n = [(g, g_i^n)]$, and let $(\boldsymbol{f}^n)$ be a Cauchy sequence in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$. Without loss of generality, we may suppose that $g \geq 1_{\mathbb{N}}$ increases. Let $k \in \mathbb{N}$ be chosen arbitrarily and let $l = g(k)$. By assumptions on $(\boldsymbol{g}^n\boldsymbol{f}^n)$ and $(\boldsymbol{f}^n)$, there exists an $n_k \in \mathbb{N}$ such that, for all $n, n' \geq n_k$,

$$d(\boldsymbol{g}^n\boldsymbol{f}^n, \boldsymbol{g}^{n'}\boldsymbol{f}^{n'}) \leq \frac{1}{k+1} \quad \text{and} \quad d(\boldsymbol{f}^n, \boldsymbol{f}^{n'}) \leq \frac{1}{l+1}.$$

Then, given any $n, n', n'' \geq n_k$, by the first claim of Lemma 5.10 (i),

$$d(\boldsymbol{g}^n \boldsymbol{f}^n, \boldsymbol{g}^{n'} \boldsymbol{f}^n) \leq \frac{1}{k+1} \quad \text{and} \quad d(\boldsymbol{g}^n \boldsymbol{f}^n, \boldsymbol{g}^{n''} \boldsymbol{f}^n) \leq \frac{1}{k+1}.$$

Since d is an ultrametric,

$$d(\boldsymbol{g}^{n'} \boldsymbol{f}^n, \boldsymbol{g}^{n''} \boldsymbol{f}^n) \leq \frac{1}{k+1}$$

also holds. By Lemma 2.4 (v),

$$d(\boldsymbol{g}^{n'} \boldsymbol{f}^n \boldsymbol{g}^n, \boldsymbol{g}^{n''} \boldsymbol{f}^n \boldsymbol{g}^n) \leq \frac{1}{k+1} \tag{1}$$

holds as well. Fix a pair $n', n'' \geq n_k$. Let $(\boldsymbol{u}^n)$ be the constant sequence $\boldsymbol{u}^n = \boldsymbol{u} = \boldsymbol{g}^{n'}$, and let $(\boldsymbol{v}^n)$ be the constant sequence $\boldsymbol{v}^n = \boldsymbol{v} = \boldsymbol{g}^{n''}$. Then, clearly, $\lim(\boldsymbol{u}^n) = \boldsymbol{g}^{n'}$ and $\lim(\boldsymbol{v}^n) = \boldsymbol{g}^{n''}$. Notice that inequality (1) turns then into

$$d(\boldsymbol{u}^n \boldsymbol{f}^n \boldsymbol{g}^n, \boldsymbol{v}^n \boldsymbol{f}^n \boldsymbol{g}^n) \leq \frac{1}{k+1}, \quad n \geq n_k.$$

Since $(\boldsymbol{f}^n \boldsymbol{g}^n)$, $(\boldsymbol{u}^n)$ and $(\boldsymbol{v}^n)$ are convergent sequences, Corollary 3.10 implies that $(\boldsymbol{u}^n \boldsymbol{f}^n \boldsymbol{g}^n)$ and $(\boldsymbol{v}^n \boldsymbol{f}^n \boldsymbol{g}^n)$ are convergent too, and also that

$$\lim(\boldsymbol{u}^n \boldsymbol{f}^n \boldsymbol{g}^n) = \lim(\boldsymbol{u}^n) \lim(\boldsymbol{f}^n \boldsymbol{g}^n) = \boldsymbol{g}^{n'} \boldsymbol{1_Y} = \boldsymbol{g}^{n'}$$

and

$$\lim(\boldsymbol{v}^n \boldsymbol{f}^n \boldsymbol{g}^n) = \lim(\boldsymbol{v}^n) \lim(\boldsymbol{f}^n \boldsymbol{g}^n) = \boldsymbol{g}^{n''} \boldsymbol{1_Y} = \boldsymbol{g}^{n''}.$$

Therefore, there exists an $n'_k \in \mathbb{N}$ such that, for every $n \geq n'_k$,

$$d(\boldsymbol{g}^{n'} \boldsymbol{f}^n \boldsymbol{g}^n, \boldsymbol{g}^{n'}) = d(\boldsymbol{u}^n \boldsymbol{f}^n \boldsymbol{g}^n, \boldsymbol{g}^{n'}) \leq \frac{1}{k+1} \tag{2}$$

and

$$d(\boldsymbol{g}^{n''} \boldsymbol{f}^n \boldsymbol{g}^n, \boldsymbol{g}^{n''}) = d(\boldsymbol{v}^n \boldsymbol{f}^n \boldsymbol{g}^n, \boldsymbol{g}^{n''}) \leq \frac{1}{k+1}. \tag{3}$$

Since d is an ultrametric, (1), (2) and (3 imply that, for every pair $n', n'' \geq \max\{n_k, n'_k\}$,

$$d(\boldsymbol{g}^{n'}, \boldsymbol{g}^{n''}) \leq \frac{1}{k+1}$$

holds, which proves that $(\boldsymbol{g}^n)$ is a Cauchy sequence in $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$. In the same (symmetric) way the assumption that $(\boldsymbol{g}^n)$ is a Cauchy sequence implies that $(\boldsymbol{f}^n)$ is a Cauchy sequence as well. Finally, the conclusion now follows by Theorem 5.1. □

According to [22, Section 5], there exists a $\overline{q}$-shape theory for compacta (yielding the classification coarser than that by the shape type), modeled on the corresponding quotient category $\overline{\mathcal{Q}}$. Observe that there also exists an abstract $\overline{q}$-shape theory for every category pair $(\mathcal{C}, \mathcal{D})$, whenever $\mathcal{D} \subseteq \mathcal{C}$ is a *sequentially* dense subcategory. By means of this new metric technique, we can prove the following facts:

LEMMA 5.11. *The semi-stability, movability and strong movability are the hereditary $\overline{q}$-shape properties.*

PROOF. Let $X, Y \in Ob\mathcal{C}$ such that Y is $\overline{q}$-shape dominated by X, and let X be semi-stable. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be sequential $\mathcal{D}$-expansions of X, Y respectively. Then $\boldsymbol{Y} \overset{\overline{q}}{\leq} \boldsymbol{X}$ and $\boldsymbol{X}$ is a semi-stable inverse sequence in $\mathcal{D}$ (see [22, Lemma 6], which holds in any abstract case). Let $(\boldsymbol{f}^n)$, $(\boldsymbol{g}^n)$ be a pair of realizing sequences for $\boldsymbol{Y} \overset{\overline{q}}{\leq} \boldsymbol{X}$, i.e., $\lim(\boldsymbol{f}^n \boldsymbol{g}^n) = \boldsymbol{!_Y}$ in $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$. Recall that, for every $n \in \mathbb{N}$, $\boldsymbol{f}^n = [(f, f_j^n)]$ and $\boldsymbol{g}^n = [(g, g_i^n)]$. Without loss of generality, we may assume that f and g are increasing and $f, g \geq 1_{\mathbb{N}}$. Let $i_0 \in \mathbb{N}$ be the semi-stability index for $\boldsymbol{X}$. Put $j_0 = g(i_0)$, and let $j' \geq j \geq j_0$ be chosen arbitrarily. Let $i \geq i_0$ be maximal such that $g(i) \leq j$. Put $i' = f(j')$. Since $\boldsymbol{X}$ is semi-stable, there exist an $\mathcal{A}$-morphism $r : X_i \to X_{i'}$ and an $i_1 \geq i'$ such that, for every $i'' \geq i_1$, $rp_{ii''} = p_{i'i''}$ holds. Choose a $j_* > j'$ such that $f(j_*) \geq i_1$. Finally, put $n = j_*$ and

$$s = f_{j'}^n r g_i^n q_{g(i)j} : Y_j \to Y_{j'},$$

and let a $j_1 \geq gf(j_*)$ be chosen according to $d(\boldsymbol{f}^n \boldsymbol{g}^n, \boldsymbol{1_Y}) \leq \frac{1}{n+1}$, i.e.,

$$(f, f_j^n)(g, g_i^n) \simeq_n (1_{\mathbb{N}}, 1_{Y_j}).$$

Now, given any $j'' \geq j_1$, the following diagram occurs:

$$\begin{array}{ccccccc}
i & \overset{r}{\underset{\longleftarrow}{\to}} & i' = f(j') & \leftarrow i_1 \leftarrow & f(j^*) & & \\
g_\cdot^n \uparrow & & \downarrow f_\cdot^n & & \downarrow f^n \nwarrow g_\cdot^n & & \\
g(i) \leftarrow \; j & \overset{\longleftarrow}{\underset{s}{\to}} & j' \leftarrow & & j_* \leftarrow gf(j_*) & \leftarrow j_1 \leftarrow j'' &
\end{array}.$$

By chasing the diagram, one readily verifies that

$$sq_{jj''} = f_{j'}^n r g_i^n q_{g(i)j} q_{jj''} = q_{j'j''}.$$

This proves that $\boldsymbol{Y}$ is a semi-stable inverse sequence in $\mathcal{D}$. Finally, by Lemma 6 of [22], Y is semi-stable because $\boldsymbol{Y}$ is a sequential $\mathcal{D}$-expansion of Y.

If $\boldsymbol{X}$ is movable and $\boldsymbol{Y} \overset{\overline{\boldsymbol{q}}}{\leq} \boldsymbol{X}$, then $\boldsymbol{Y}$ is movable because the analogue for the quasi-domination is already proved in [2]. Finally, if $\boldsymbol{X}$ is strongly movable, one has to obtain a construction quite similar to that of the first part of the proof, taking into account that both properties (movability and semi-stability) are satisfied by a unique morphism $r : X_{i'} \to X_{i''}$. □

That following facts make the $\overline{q}$-equivalence ($\overline{q}$-shape) analogues of Theorem 5.7 and Corollary 5.8 respectively.

THEOREM 5.12. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse sequences in a category $\mathcal{A}$,*

(i) *Let $\boldsymbol{X}$ (or $\boldsymbol{Y}$) be semi-stable. Then $\boldsymbol{X}$ is isomorphic to $\boldsymbol{Y}$ in tow-$\mathcal{A}$ if and only if $\boldsymbol{X}$ is $\overline{q}$-equivalent to $\boldsymbol{Y}$, i.e.,*

$$\boldsymbol{X} \cong \boldsymbol{Y} \Leftrightarrow \boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}.$$

Consequently, $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$ implies $S^*(\boldsymbol{X}) = S^*(\boldsymbol{Y})$, whenever $\boldsymbol{X}$ (or $\boldsymbol{Y}$) is semi-stable.

(ii) *If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are stable, then $S_1(\boldsymbol{X}) \leq S_1(\boldsymbol{Y})$ (or $S_1(\boldsymbol{Y}) \leq S_1(\boldsymbol{X})$) is equivalent to $\boldsymbol{X} \cong \boldsymbol{Y}$.*

PROOF. Statement (i) follows by Theorem 5.7 and Lemma 5.11. To prove (ii), first observe that Theorem 5.7 (ii) assures that $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$. Notice that $\boldsymbol{X} \cong \boldsymbol{P}$, $\boldsymbol{Y} \cong \boldsymbol{Q}$ and (obtained) $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ imply $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{P}$, $\boldsymbol{Y} \overset{\overline{q}}{\simeq} \boldsymbol{Q}$ and $\boldsymbol{P} \overset{\overline{q}}{\simeq} \boldsymbol{Q}$. Then $\boldsymbol{X} \overset{\overline{q}}{\simeq} \boldsymbol{Y}$ follows by transitivity of the $\overline{q}$-equivalence ([21, Theorem 6 (iii)], general case). Finally, by (i), $\boldsymbol{X} \cong \boldsymbol{Y}$. The converse is trivial. □

COROLLARY 5.13. *Let X and Y be compact metrizable spaces.*

(i) *Let X (or Y) be semi-stable (especially, an FANR). Then, X is shape equivalent to Y if and only if X is $\overline{q}$-shape equivalent to Y,*

$$Sh(X) = Sh(Y) \Leftrightarrow X \overset{\overline{q}}{\simeq} Y.$$

Consequently, the stability and strong movability (being an FANR) of metrizable compacta are invariants of the $\overline{q}$-equivalence, i.e., of the $\overline{q}$-shape type. Further,

$$X \overset{\overline{q}}{\simeq} Y \text{ implies } S^*(X) = S^*(Y),$$

whenever X (or Y) is semi-stable.

(ii) *If X and Y are FANR's (especially, stable), then $S_1(X) \leq S_1(Y)$ (or $S_1(Y) \leq S_1(X)$) is equivalent to $Sh(X) = Sh(Y)$.*

PROOF. We only have to verify the necessity in the last assertions concerning FANR's. The conclusion $X \overset{q}{\simeq} Y$ holds as in the proof of Corollary 5.8, while then $X \overset{\overline{q}}{\simeq} Y$ follows by applying the corresponding part of the proof of Theorem 5.12 to *tow-HANR*. Namely, as we mentioned before, FANR's are stable with respect to (noncompact) ANR's. Then apply (i). □

REMARK 5.14. The full analogues of Theorem 5.7 (ii) and Corollary 5.8 (ii) do *not* hold for the $\overline{q}$-equivalence ($\overline{q}$-shape). Namely, there exists a pair of regularly movable inverse sequences $\boldsymbol{X}$, $\boldsymbol{Y}$ in *cPol* (regularly movable compacta X, Y; see Example 5.15 below) such that $S_1(\boldsymbol{X}) = S_1(\boldsymbol{Y})$, while $\boldsymbol{X}$ is not $\overline{q}$-equivalent to $\boldsymbol{Y}$ ($S_1(X) = S_1(Y)$, while X and Y have different $\overline{q}$-shape types).

Example 5.15. Let X be the image of a nonstationary convergent sequence including the limit point in the Euclidean space $\mathbb{R}$. For instance,

$$X = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}.$$

Let $Y = X \sqcup X$ (disjoint union). Let $\boldsymbol{X} = (X_i, [p_{ii'}] = \{p_{ii'}\}, \mathbb{N})$ be associated with X, i.e., $\lim \boldsymbol{X} = X$, where X_i is discrete, $|X_i| = i$, $i \in \mathbb{N}$, and $p_{ii'}$ are surjections such that the fibres of all the points, except the "exploding" one, are singletons (see Example 2.8). Let $Y_j = X_j \sqcup X_j$, $j \in \mathbb{N}$, and let $q_{jj'} = p_{jj'} \sqcup p_{jj'}$. Then $\boldsymbol{Y} = \boldsymbol{X} \sqcup \boldsymbol{X}$ is associated with Y. Observe that all the bonding mappings $p_{ii'}$ and $q_{jj'}$ are retractions. Thus, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are regularly movable. By [6], X and Y are regularly movable as well. By [23], Example 2.9, $S_1(\boldsymbol{X}) = S_1(\boldsymbol{Y})$ $(S_1(X) = S_1(Y))$ holds. Then, by Theorem 5.7 (Corollary 5.8 or, directly, by (6.3) Theorem of [2]), $\boldsymbol{X} \overset{q}{\simeq} \boldsymbol{Y}$ $(X \overset{q}{\simeq} Y)$. However, $\boldsymbol{X}$ is not $\overline{q}$-equivalent to $\boldsymbol{Y}$ (X and Y have different $\overline{q}$-shape types) - see the proof below.

Consider any pair of realizing sequences $(\boldsymbol{f}^n)$ in $(\boldsymbol{Y}^{\boldsymbol{X}}, d)$, $(\boldsymbol{g}^n)$ in $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$, i.e., $\lim(\boldsymbol{g}^n \boldsymbol{f}^n) = \boldsymbol{1}_{\boldsymbol{X}}$ in $(\boldsymbol{X}^{\boldsymbol{X}}, d)$ and $\lim(\boldsymbol{f}^n \boldsymbol{g}^n) = \boldsymbol{1}_{\boldsymbol{Y}}$ in $(\boldsymbol{Y}^{\boldsymbol{Y}}, d)$. Let (f^n, f_j^n) and (g^n, g_i^n) be any representatives of $\boldsymbol{f}^n$ and $\boldsymbol{g}^n$ respectively, $n \in \mathbb{N}$. Notice that every homotopy commutative diagram relating $\boldsymbol{X}$ to $\boldsymbol{Y}$ and vice versa must be (strictly) commutative, and that all the mappings f_j^n and g_i^n must be surjective. Then, a straightforward analysis (compare the proof following [23], Example 2.9) shows that, for every $n \in \mathbb{N}$, the inequality $f^n(1) \geq 2n + 1$ must be satisfied. Consequently, there is no unique index function for any sequence $((f^n, f_j^n))$ representing $(\boldsymbol{f}^n)$. Thus, $\boldsymbol{X}$ (X) cannot be $\overline{q}$-equivalent to $\boldsymbol{Y}$ (Y).

Let us show, in addition, that this example confirms a significance of Theorem 5.9 comparing to its analogue (Theorem 5.1). First, observe that one can provide a Cauchy sequence $(\boldsymbol{g}^n)$ in $(\boldsymbol{X}^{\boldsymbol{Y}}, d)$ satisfying the above relations. Then, by Theorem 2.10, $(\boldsymbol{g}^n)$ admits a representing sequence $((g, g_i^n))$ (moreover, $(\boldsymbol{g}^n)$ may be the constant sequence $\boldsymbol{g}^n = \boldsymbol{r} : \boldsymbol{Y} \to \boldsymbol{X}$, where $\boldsymbol{r}$ is induced by the obvious retractions ("gluing") $r_i : X_i \sqcup X_i \to X_i$, $i \in \mathbb{N}$). Now, if $\boldsymbol{X}$ (X) would be $\overline{q}$-equivalent to $\boldsymbol{Y}$ (Y), then Theorem 5.9 would imply that $\boldsymbol{X} \cong \boldsymbol{Y}$ in *tow-HcPol* (X is shape equivalent to Y). This, finally, would imply that X and Y are homeomorphic ([1], VII. (5.9) Corollary) - a contradiction.

6. Final notes

(a) According to Remark 5.3 and Example 5.15, there is a new equivalence relation on inverse sequences, i.e., on $Ob(tow\text{-}\mathcal{A})$, as well as on compact metrizable spaces. It lies strictly between the quasi-equivalence and $\overline{q}$-equivalence. Namely, by Remark 5.3, only one sequence having a fixed index function is enough for the quasi-equivalence to be transitive. However, by Example 5.15, this does not suffice for it to become the $\overline{q}$-equivalence.

(b) The essential properties of the ultrametric $d : \boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Y}^{\boldsymbol{X}} \to \mathbb{R}$, defined for *cofinite* inverse systems in Section 2, depends almost entirely on $\boldsymbol{Y}$. A slight dependence on $\boldsymbol{X}$ as well can be introduced in the following way. First, for $(f, f_\mu), (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$, one defines $(f, f_\mu) \simeq_\mu^\lambda (f', f'_\mu)$ provided $f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}$. Then, $(f, f_\mu) \simeq_n^{n'} (f', f'_\mu)$, provided $(f, f_\mu) \simeq_\mu^\lambda (f', f'_\mu)$ for every $\mu \in M$ and some $\lambda \in \Lambda$ such that $|\mu| < n$ and $|\lambda| < n'$. Consider the following formula:

$$\rho'((f, f_j), (f', f'_j)) = \frac{1}{n+1} - \frac{1}{m+2},$$

whenever $(f, f_\mu) \simeq_n^{n'} (f', f'_\mu)$, with n maximal and n' minimal, and $m = \max\{n.n'\}$.

Further, let $\rho'((f, f_j), (f', f'_j)) = 1$, provided $(f, f_\mu) \simeq_n (f', f'_\mu)$ does not hold for any n. By passing to the obvious limits, these formulae induce a pseudometric ρ' on the set *inv*-$\mathcal{A}(\boldsymbol{X}, \boldsymbol{Y})$. It is readily seen that it yields a metric

$$d' : \boldsymbol{Y}^{\boldsymbol{X}} \times \boldsymbol{Y}^{\boldsymbol{X}} \to \mathbb{R}, \quad d'(\boldsymbol{f}, \boldsymbol{f}') = \rho'((f, f_j), (f', f'_j)),$$

where f and f' are minimal index functions among all representatives. Clearly, $d' \leq d$. It is not difficult to prove that d' is complete and topologically equivalent to d. However, one can easily verify that d' is *not* an ultrametric.

(c) Finally, a few words about the notions of stability and regular movability. Stability of a system of a pro-category *pro*-$\mathcal{A}$ means being isomorphic (in *pro*-$\mathcal{A}$) to a rudimentary system. However, when the stability of a $\mathcal{C}$-object has to be defined, one should emphasize a certain subcategory of *pro*-$\mathcal{D}$. Namely, a $\mathcal{C}$-object generally admits many expansions belonging to (different subcategories of) *pro*-$\mathcal{D}$. For instance, all FANR's are stable with respect to *tow*-$HANR$ (as well as to *pro*-$HPol$), while there are FANR's that are not stable with respect to *tow*-$HcANR$ (or *pro*-$HcPol$).

The regular movability was defined for metric compacta by A. Trybulec ([21]): A compactum X is said to be regularly movable provided it admits an associated ANR inverse sequence $\boldsymbol{X}$ such that each bonding mapping $p_{i,i+1}$ is a homotopy domination (weak retraction). In [6] was proved that the regular movability of compacta is a shape invariant (the proof essentially depends on specific properties of compact ANR's). Thus, a characterization may be as follows: A compactum X is regularly movable if and only if, *for every* associated *compact* ANR inverse sequence $\boldsymbol{X}$, there exists an $i_0 \in \mathbb{N}$ such that every $i \geq i_0$ admits an $i' > i$ so that $[p_{ii'}]$ is a retraction. In [10] was proved that regular movability is not a hereditary shape property. Clearly, by using the above characterization, one can generally define the regular movability of an inverse sequence (system) as well as of an object. However, the invariance of regular movability of compacta has *no* analogue in the case of a corresponding abstract shape anymore. Namely, this property is not a categorical one. More precisely, a category isomorphism, in general, does not preserve the regular movability. Moreover, the stability does not imply regular movability.

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