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APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS IN WEIGHTED REARRANGEMENT INVARIANT SPACES

Ali Guven and Daniyal M. Israfilov
Balikesir University, Turkey

Abstract. We investigate the approximation properties of trigonometric polynomials and prove some direct and inverse theorems for polynomial approximation in weighted rearrangement invariant spaces.

1. Introduction and the main results

Let $(\mathcal{R}, \mu)$ be a nonatomic $\sigma-$finite measure space, i.e., a measure space with nonatomic $\sigma-$finite measure μ given on a $\sigma-$algebra of subsets of $\mathcal{R}$. Denote by $\mathcal{M}$ the set of all $\mu-$measurable complex valued functions on $\mathcal{R}$, and let $\mathcal{M}^+$ be the subset of functions from $\mathcal{M}$ whose values lie in $[0, \infty]$. The characteristic function of a $\mu-$measurable set $E \subset \mathcal{R}$ will be denoted by χ_E.

Let a function $\rho : \mathcal{M}^+ \to [0, \infty]$ be given. The function ρ is called a *function norm* if it satisfies the following properties for all functions $f, g, f_n \in \mathcal{M}^+ \, (n \in \mathbb{N})$, for all constants $a \geq 0$ and for all $\mu-$measurable subsets $E \subset \mathcal{R}$:

(1) $\rho(f) = 0 \;\Leftrightarrow\; f = 0 \;\mu - a.e, \rho(af) = a\rho(f), \rho(f + g) \leq \rho(f) + \rho(g),$
(2) $0 \leq g \leq f \;\mu - a.e \;\Rightarrow\; \rho(g) \leq \rho(f),$
(3) $0 \leq f_n \uparrow f \;\mu - a.e \;\Rightarrow\; \rho(f_n) \uparrow \rho(f),$
(4) $\mu(E) < \infty \;\Rightarrow\; \rho(\chi_E) < \infty,$
(5) $\mu(E) < \infty \;\Rightarrow\; \displaystyle\int_E f \, d\mu \leq C_E \rho(f),$

where C_E is a constant depending on E and ρ but independent of f.

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If ρ is a function norm, its *associate* function norm ρ' is defined by

$$\rho'(g) := \sup\left\{ \int_{\mathcal{R}} fg d\mu : f \in \mathcal{M}^+, \rho(f) \leq 1 \right\} \tag{1.1}$$

for $g \in \mathcal{M}^+$. If ρ is a function norm, then ρ' is also a function norm [3, pp. 8-9].

Let ρ be a function norm. We denote by $X = X(\rho)$ the linear space of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$. The space X is called a *Banach function space*. If we define the norm of $f \in X$ by

$$\|f\|_X := \rho(|f|)$$

X will be a Banach space [3, pp. 6-7]. By the property (5), it follows that if the measure space $(\mathcal{R}, \mu)$ is finite, i.e., if $\mu(\mathcal{R}) < \infty$, then $X \subset L_1(\mathcal{R}, \mu)$.

Let ρ be a function norm and ρ' be its associate function norm. The Banach function space determined by ρ' is called the *associate space* of X and denoted by X'. Every Banach function space coincides with its second associate space $X'' = (X')'$ and $\|f\|_X = \|f\|_{X'}$ for all $f \in X$ [3, pp. 10-12]. So we have by (1.1)

$$\|f\|_X = \sup\left\{ \int_{\mathcal{R}} |fg|\, d\mu : g \in X', \|g\|_{X'} \leq 1 \right\} \tag{1.2}$$

and

$$\|g\|_{X'} = \sup\left\{ \int_{\mathcal{R}} |fg|\, d\mu : f \in X, \|f\|_X \leq 1 \right\}. \tag{1.3}$$

For every $f \in X$ and $g \in X'$ the Hölder inequality

$$\int_{\mathcal{R}} |fg|\, d\mu \leq \|f\|_X \|g\|_{X'} \tag{1.4}$$

holds [3, p. 9].

Let $\mathcal{M}_0$ and $\mathcal{M}_0^+$ be the classes of $\mu - a.e.$ finite functions from $\mathcal{M}$ and $\mathcal{M}^+$ respectively. The distribution function μ_f of $f \in \mathcal{M}_0$ is defined by

$$\mu_f(\lambda) := \mu\{x \in \mathcal{R} : |f(x)| > \lambda\}$$

for $\lambda \geq 0$. Two functions $f, g \in \mathcal{M}_0$ are said to be *equimeasurable* if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$.

Definition 1.1 ([3, p. 59]). *If $\rho(f) = \rho(g)$ for every pair of equimeasurable functions $f, g \in M_0^+$, the function norm ρ is called a rearrangement invariant function norm. In this case, the Banach function space generated by ρ is called a rearrangement invariant space.*

Let $f \in \mathcal{M}_0$. The function f^* defined by

$$f^*(t) := \inf\{\lambda : \mu_f(\lambda) \le t\}, \quad t \ge 0$$

is called the *decreasing rearrangement* of the function f.

Let X be a rearrangement-invariant space over a nonatomic finite measure space $(\mathcal{R}, \mu)$. By the Luxemburg representation theorem [3, pp. 62-64], there is a (not necessarily unique) rearrangement invariant function norm $\overline{\rho}$ over $\mathbb{R}_+ = [0, \infty)$ with the Lebesgue measure m such that

$$\rho(f) = \overline{\rho}(f^*)$$

for every $f \in \mathcal{M}_0^+$.

The rearrangement invariant space over $(\mathbb{R}_+, m)$ generated by $\overline{\rho}$ is denoted by $\overline{X}$.

Let's consider the operator E_x, $x > 0$ defined on $\mathcal{M}_0(\mathbb{R}_+, m)$ by

$$(E_x f)(t) := \begin{cases} f(xt), & xt \in [0, \mu(\mathcal{R})] \\ 0, & xt \notin [0, \mu(\mathcal{R})] \end{cases}, \quad t > 0.$$

It is known that [3, pp. 165] $E_{1/x} \in \mathcal{B}(\overline{X})$ for each $x > 0$, where $\mathcal{B}(\overline{X})$ is the Banach algebra of bounded linear operators on $\overline{X}$. Let $h_X(x)$ be the operator norm of $E_{1/x}$, i.e., $h_X(x) := \|E_{1/x}\|_{\mathcal{B}(\overline{X})}$.

The numbers α_X and β_X defined by

$$\alpha_X := \sup_{0<x<1} \frac{\log h_X(x)}{\log x}, \quad \beta_X := \inf_{1<x<\infty} \frac{\log h_X(x)}{\log x}$$

are called the *lower* and *upper Boyd indices* of X, respectively. It is known that [3, p. 149] the Boyd indices satisfy

$$0 \le \alpha_X \le \beta_X \le 1.$$

The Boyd indices are said to be *nontrivial* if $0 < \alpha_X \le \beta_X < 1$.

Let $\mathbb{T}$ be the unit circle $\{e^{i\theta} : \theta \in [-\pi, \pi]\}$, or the interval $[-\pi, \pi]$, $\mathbb{C}$ be the complex plane and $L_p(\mathbb{T}), 1 \le p \le \infty$, be the Lebesgue space of measurable functions on $\mathbb{T}$. Further, any rearrangement invariant space over $\mathbb{T}$ will be denoted by $X(\mathbb{T})$.

A measurable function $\omega : \mathbb{T} \to [0, \infty]$ is called a *weight function* if the set $\omega^{-1}(\{0, \infty\})$ has Lebesgue measure zero.

Let $X(\mathbb{T})$ be a rearrangement invariant space over $\mathbb{T}$ and ω be a weight function. We denote by $X(\mathbb{T}, \omega)$ the class of all measurable functions f such that $f\omega \in X(\mathbb{T})$, which is equipped with the norm

$$\|f\|_{X(\mathbb{T},\omega)} := \|f\omega\|_{X(\mathbb{T})}. \tag{1.5}$$

The space $X(\mathbb{T}, \omega)$ is called a *weighted rearrangement invariant* space.

From the Hölder inequality it follows that if $\omega \in X(\mathbb{T})$ and $1/\omega \in X'(\mathbb{T})$ then $L_\infty(\mathbb{T}) \subset X(\mathbb{T}, \omega) \subset L_1(\mathbb{T})$.

Let $1 < p < \infty$ and $1/p + 1/q = 1$. A weight function ω belongs to the *Muckenhoupt class* $A_p(\mathbb{T})$ if

$$\left(\frac{1}{|J|} \int_J \omega^p(x)\, dx \right)^{1/p} \left(\frac{1}{|J|} \int_J \omega^{-q}(x)\, dx \right)^{1/q} \leq C$$

with a finite constant C independent of J, where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes the length of J.

Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space with nontrivial Boyd indices α_X and β_X, and ω be a weight function such that $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$. For a given function $f \in X(\mathbb{T}, \omega)$ we define the shift operator σ_h

$$(\sigma_h f)(x) := \frac{1}{2h} \int_{-h}^{h} f(x+t)\, dt, \quad 0 < h < \pi,\ x \in \mathbb{T},$$

and later the $k-$*modulus of smoothness* $\Omega^k_{X,\omega}(\cdot, f)$ $(k = 1, 2, \ldots)$

$$\Omega^k_{X,\omega}(\delta, f) := \sup_{\substack{0<h_i\leq\delta \\ 1\leq i\leq k}} \left\| \prod_{i=1}^{k} (I - \sigma_{h_i}) f \right\|_{X(\mathbb{T},\omega)}, \quad \delta > 0,$$

where I is the identity operator. This modulus of smoothness is well defined, because we will prove (Lemma 2.2) that the operator σ_h is a bounded linear operator in $X(\mathbb{T}, \omega)$.

We define the shift operator σ_h and the modulus of smoothness $\Omega^k_{X,\omega}$ in such way since the space $X(\mathbb{T}, \omega)$ is noninvariant, in general, under the usual shift $f(x) \to f(x+h)$.

In the case of $k = 0$ we assume $\Omega^0_{X,\omega}(\delta, f) := \|f\|_{X(\mathbb{T},\omega)}$ and if $k = 1$ we write $\Omega_{X,\omega}(\delta, f) := \Omega^1_{X,\omega}(\delta, f)$. The modulus of smoothness $\Omega^k_{X,\omega}(\cdot, f)$ is nondecreasing, nonnegative, continuous function and

$$\Omega^k_{X,\omega}(\delta, f + g) \leq \Omega^k_{X,\omega}(\delta, f) + \Omega^k_{X,\omega}(\delta, g) \tag{1.6}$$

for $f, g \in X(\mathbb{T}, \omega)$.

We denote by $E_n(f)_{X,\,\omega}$ $(n = 0, 1, 2, \ldots)$ the best approximation of $f \in X(\mathbb{T}, \omega)$ by trigonometric polynomials of degree not exceeding n, *i. e.*,

$$E_n(f)_{X,\omega} = \inf \left\{ \|f - T_n\|_{X(\mathbb{T},\omega)} : T_n \in \Pi_n \right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n. Note that the existence of the trigonometric polynomial $T^*_n \in \Pi_n$ such that

$$E_n(f)_{X,\omega} = \|f - T^*_n\|_{X(\mathbb{T},\omega)},$$

follows, for example, from Theorem 1.1 in [8, p. 59].

In the literature there are sufficiently many results, where investigated the approximation problems and obtained, in particular, the direct and inverse theorems of approximation theory by trigonometric polynomials in weighted and nonweighted Lebesgue spaces. The elegant representation of the corresponding result in the nonweighted Lebesgue spaces $L^p(\mathbb{T})$, $1 \leq p \leq \infty$, can be found in [8, 34, 35]. The best approximation problem by trigonometric polynomials in weighted spaces with weights satisfying the so-called $A_p(\mathbb{T})$ −condition was investigated in [15, 26, 27]. In particular, using the $L^p(\mathbb{T},\omega)$ version of the $k-$modulus of smoothness $\Omega^k_{X,\omega}(\cdot, f)$, $k = 1, 2, \ldots$, some direct and inverse theorems in the weighted Lebesgue spaces were obtained in [15, 27]. The generalizations of the last results for the weighted Lebesgue spaces, defined on the curves of the complex plane were proved in [18–20]. The similar results in the nonweighted Lebesgue spaces were obtained in [1, 7, 25].

For the more general doubling weights, approximation by trigonometric polynomials in the periodic case and other related problems were studied in [4, 29–31]. The direct and converse results in case of the exponential weights given on the real line were obtained in [13, 14]. Some interesting results concerning to the best polynomial approximation in weighted Lebesgue spaces were also proved in [9, 10]. The detailed information on the weighted polynomial approximation can be found in the books: [11, 32]. In the non-weighted rearrangement invariant spaces the direct theorems can be found in [8]. Some other aspects of the approximation theory in the more general spaces were investigated by many authors (see, for example: [28]).

To the best of the authors' knowledge there are no results, where studied the approximation problems by trigonometric polynomials in the weighted rearrangement invariant spaces. These spaces are sufficiently wide; the Lebesgue, Orlicz, Lorentz spaces are examples of rearrangement invariant spaces. In this work we prove some direct and inverse theorems of approximation theory in the weighted rearrangement invariant spaces $X(\mathbb{T}, \omega)$. In particular, we obtain a result on the constructive characteristic of the generalized Lipschitz classes defined in these spaces.

Let $r = 1, 2, \ldots$. If we denote the space of functions $f \in X(\mathbb{T}, \omega)$ for which $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in X(\mathbb{T}, \omega)$ by $W^r_X(\mathbb{T}, \omega)$, it become a normed space with respect to the norm

$$\|f\|_{W^r_X(\mathbb{T},\omega)} := \|f\|_{X(\mathbb{T},\omega)} + \left\|f^{(r)}\right\|_{X(\mathbb{T},\omega)}. \tag{1.7}$$

Our main results are the following.

Theorem 1.2. *Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space with nontrivial Boyd indices α_X and β_X, and ω be a weight function such that $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$. Then for every $f \in W^r_X(\mathbb{T}, \omega)$ $(r = 1, 2, \ldots)$, the inequality*

$$E_n(f)_{X,\omega} \leq \frac{c}{(n+1)^r} E_n\left(f^{(r)}\right)_{X,\omega}, \quad n = 1, 2, \ldots \tag{1.8}$$

holds with a constant $c > 0$ independent of n.

THEOREM 1.3. *Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space with nontrivial Boyd indices α_X and β_X, and ω be a weight function such that $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$. Then for every $f \in X(\mathbb{T},\omega)$ and $k = 1, 2, \ldots,$ the estimate*

$$E_n(f)_{X,\omega} \le c\Omega^k_{X,\omega}\left(\frac{1}{n+1}, f\right) \tag{1.9}$$

holds with a positive constant $c = c(k)$ independent of n.

In weighted Lebesgue spaces $L_p(\mathbb{T},\omega)$ similar results were proved in [15] and [27].

Let $\mathbb{D}$ be the unit disk in the complex plane and $H_1(\mathbb{D})$ be the Hardy space of analytic functions in $\mathbb{D}$. It is known that every function $f \in H_1(\mathbb{D})$ admits nontangential boundary limits a. e. on $\mathbb{T}$ and the limit function belongs to $L_1(\mathbb{T})$ [12, p. 23].

Let $X(\mathbb{T}, \omega)$ be a weighted rearrangement invariant space on $\mathbb{T}$ and let $H_X(\mathbb{D},\omega)$ be the class of analytic functions in $\mathbb{D}$ defined as:

$$H_X(\mathbb{D},\omega) := \{f \in H_1(\mathbb{D}) : f \in X(\mathbb{T},\omega)\}.$$

Then from Theorem 1.3 we obtain the following result.

THEOREM 1.4. *Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space with nontrivial Boyd indices α_X and β_X, ω be a weight function such that $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, and $f \in H_X(\mathbb{D},\omega)$. If $\sum_{j=0}^{\infty} a_j(f) z^j$ is the Taylor series of f at the origin, then*

$$\left\| f(z) - \sum_{j=0}^{n} a_j(f) z^j \right\|_{X(\mathbb{T},\omega)} \le c\Omega^k_{X,\omega}\left(\frac{1}{n+1}, f\right), \quad k = 1, 2, \ldots \tag{1.10}$$

with a constant $c = c(k) > 0$, which is independent of n.

THEOREM 1.5. *Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space with nontrivial Boyd indices α_X, β_X, and let the ω be a weight function such that $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$. Then for $f \in X(\mathbb{T},\omega)$ and $k = 1, 2, \ldots,$ the estimate*

$$\Omega^k_{X,\omega}\left(\frac{1}{n}, f\right) \le \frac{c}{n^{2k}}\left\{ E_0(f)_{X,\omega} + \sum_{j=1}^{n} j^{2k-1} E_j(f)_{X,\omega} \right\} \tag{1.11}$$

holds with some positive constant $c = c(k)$ independent of n.

From Theorem 1.5 we obtain the following result.

Corollary 1.6. *If*

$$E_n(f)_{X,\omega} = \mathcal{O}\left(n^{-\alpha}\right), \quad \alpha > 0,\ n = 1, 2, \ldots,$$

for $f \in X(\mathbb{T}, \omega)$*, then for any natural number* k *and* $\delta > 0$

$$\Omega^k_{X,\omega}(\delta, f) = \begin{cases} \mathcal{O}(\delta^\alpha), & k > \alpha/2 \\ \mathcal{O}(\delta^\alpha \log(1/\delta)), & k = \alpha/2 \\ \mathcal{O}\left(\delta^{2k}\right), & k < \alpha/2. \end{cases}$$

Hence if we define the generalized Lipschitz class $Lip^*\alpha(X, \omega)$ for $\alpha > 0$ and $k := [\alpha/2] + 1$ as

$$Lip^*\alpha(X, \omega) := \left\{ f \in X(\mathbb{T}, \omega) : \Omega^k_{X,\omega}(\delta, f) \le c\delta^\alpha, \delta > 0 \right\},$$

then by virtue of Corollary 1.6 we obtain the following

Corollary 1.7. *If*

$$E_n(f)_{X,\omega} = \mathcal{O}\left(n^{-\alpha}\right), \quad \alpha > 0,\ n = 1, 2, \ldots,$$

for $f \in X(\mathbb{T}, \omega)$*, then* $f \in Lip^*\alpha(X, \omega)$.

Combining this with Direct Theorem we get the following constructive description of classes $Lip^*\alpha(X, \omega)$.

Theorem 1.8. *For* $\alpha > 0$ *the following assertions are equivalent:*

(i) $f \in Lip^*\alpha(X, \omega)$;
(ii) $E_n(f)_{X,\omega} = \mathcal{O}(n^{-\alpha})$ *for all* $n = 1, 2, \ldots$.

We use c, c_1, $c_2, \ldots$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest.

2. Auxiliary results

The following interpolation theorem was proved in [5].

Theorem 2.1. *Let* $1 < q < p < \infty$*. If a linear operator is bounded in the Lebesgue spaces* $L_p(\mathbb{T})$ *and* $L_q(\mathbb{T})$*, then it is bounded in every rearrangement invariant space* $X(\mathbb{T})$ *whose Boyd indices satisfy* $1/p < \alpha_X \le \beta_X < 1/q$.

In the proof of the following lemma, we will use the method used by A. Yu. Karlovich in [23].

Lemma 2.2. *Let* $X(\mathbb{T})$ *be a rearrangement invariant space with nontrivial Boyd indices* α_X *and* β_X*, and* ω *be a weight function. If* $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$*, then the operator* σ_h *is bounded in the space* $X(\mathbb{T}, \omega)$.

Proof. Since $0 < \alpha_X \leq \beta_X < 1$, we can find the numbers q and p such that

$$1 < q < 1/\beta_X \leq 1/\alpha_X < p < \infty$$

and $\omega \in A_p(\mathbb{T}) \cap A_q(\mathbb{T})$ [6, p. 58]. As follows from the continuity of the maximal operator in weighted Lebesgue spaces (see [33]), the operator σ_h is bounded in the spaces $L_p(\mathbb{T}, \omega)$ and $L_q(\mathbb{T}, \omega)$. In that case the operator $A_h := \omega\sigma_h\omega^{-1}I$ is bounded in the Lebesgue spaces $L_p(\mathbb{T})$ and $L_q(\mathbb{T})$. Hence by Theorem 2.1, the operator A_h is bounded in the rearrangement invariant space $X(\mathbb{T})$. This implies the boundedness of the operator σ_h in the space $X(\mathbb{T}, \omega)$. ∎

From this Lemma and the density of the continuous functions in $X(\mathbb{T}, \omega)$ (see [22]) we obtain the following result.

Corollary 2.3. *For $f \in X(\mathbb{T}, \omega)$ we have*

$$\lim_{h \to 0} \|f - \sigma_h f\|_{X(\mathbb{T},\omega)} = 0$$

and hence

$$\lim_{\delta \to 0} \Omega^k_{X,\omega}(\delta, f) = 0, k = 1, 2, \ldots$$

Moreover

$$\Omega^k_{X,\omega}(\delta, f) \leq c \|f\|_{X(T,\omega)}$$

holds with some constant c independent of f.

Let $S_n(\cdot, f)$ $(n = 1, 2, \ldots)$ be the nth partial sums of the Fourier series of the function $f \in L_1(T)$, i. e.

$$S_n(x, f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx,$$

where

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx.$$

Then [2, Vol. 1, pp. 95-96]

$$S_n(x, f) = \frac{1}{\pi} \int_{\mathbb{T}} f(t) D_n(x - t)\, dt$$

with the Dirichlet kernel

$$D_n(t) := \frac{1}{2} + \sum_{k=1}^{n} \cos kt$$

of order n. Consider the sequence $\{K_n(\cdot, f)\}$ of the Fejer means defined by

$$K_n(x, f) := \frac{S_0(x, f) + S_1(x, f) + \cdots + S_n(x, f)}{n + 1}, \quad n = 0, 1, 2, \ldots$$

with $K_0(x,f) = S_0(x,f) := a_0/2$.

It is known [2, Vol. 1, p. 133] that

$$K_n(x,f) = \frac{1}{\pi}\int_{\mathbb{T}} f(t)\, F_n(x-t)\, dt,$$

where the expression

$$F_n(t) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(t)$$

is the Fejer kernel of order n (for more information see: [2, vol. 1, pp. 133-137]).

LEMMA 2.4. *Let $X(\mathbb{T})$ be a rearrangement invariant space with nontrivial Boyd indices α_X, β_X, and ω be a weight function such that $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$. Then the sequence $\{K_n\}$ of the Fejer means is uniformly bounded in the space $X(\mathbb{T},\omega)$, i. e.*

$$\|K_n(\cdot,f)\|_{X(\mathbb{T},\omega)} \le c\,\|f\|_{X(\mathbb{T},\omega)}, \quad f \in X(\mathbb{T},\omega) \tag{2.1}$$

with a constant c, independent of n.

The proof of Lemma 2.4 is similar to proof of Lemma 2.2.

Now we can state and prove Bernstein's inequality for weighted rearrangement invariant spaces.

LEMMA 2.5. *Let $X(\mathbb{T})$ be a rearrangement invariant space with nontrivial Boyd indices α_X, β_X. If $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, then for every trigonometric polynomial T_n of degree n, the inequality*

$$\|T_n'\|_{X(\mathbb{T},\omega)} \le cn\,\|T_n\|_{X(\mathbb{T},\omega)} \tag{2.2}$$

holds with a constant c, independent of n.

PROOF. We use Zygmund's method (see [2, Vol 2, pp. 458-460]). Since

$$T_n(x) = S_n(x,T_n) = \frac{1}{\pi}\int_{\mathbb{T}} T_n(u)\, D_n(u-x)\, du,$$

by differentiation we get

$$\begin{aligned} T_n'(x) &= -\frac{1}{\pi}\int_{\mathbb{T}} T_n(u)\, D_n'(u-x)\, du = -\frac{1}{\pi}\int_{\mathbb{T}} T_n(u+x)\, D_n'(u)\, du \\ &= \frac{1}{\pi}\int_{\mathbb{T}} T_n(u+x)\left(\sum_{k=1}^{n} k\sin ku\right) du \end{aligned}$$

and since T_n is a trigonometric polynomial of degree n,

$$\begin{aligned} T_n'(x) &= \frac{1}{\pi}\int_{\mathbb{T}} T_n(u+x)\left\{\sum_{k=1}^{n} k\sin ku + \sum_{k=1}^{n-1} k\sin(2n-k)u\right\}du \\ &= \frac{1}{\pi}\int_{\mathbb{T}} T_n(u+x)\,2n\sin nu\left\{\frac{1}{2}+\sum_{k=1}^{n-1}\frac{n-k}{n}\cos ku\right\}du \\ &= \frac{2n}{\pi}\int_{\mathbb{T}} T_n(u+x)\sin nu F_{n-1}(u)\,du. \end{aligned}$$

Since F_{n-1} is non-negative, we obtain

$$\begin{aligned} \left|T_n'(x)\right| &\leq \frac{2n}{\pi}\int_{\mathbb{T}} \left|T_n(u+x)\right| F_{n-1}(u)\,du = \frac{2n}{\pi}\int_{\mathbb{T}} \left|T_n(u)\right| F_{n-1}(u-x)\,du \\ &= 2nK_{n-1}(x,|T_n|), \end{aligned}$$

and Lemma 2.4 yields (2.2). ◻

Let $S_n(\cdot,f)$ and $\widetilde{f}$ be the nth partial sums of the Fourier series and the conjugate function of $f \in X(\mathbb{T},\omega)$, respectively. Since the linear operators $f \to S_n(\cdot,f)$ and $f \to \widetilde{f}$ are bounded in the weighted Lebesgue spaces $L_p(\mathbb{T},\omega)$ [16, 17], by using the method of proof of Lemma 2.2, one can show that

$$\left\|S_n(\cdot,f)\right\|_{X(\mathbb{T},\omega)} \leq c\left\|f\right\|_{X(\mathbb{T},\omega)}, \quad \left\|\widetilde{f}\right\|_{X(\mathbb{T},\omega)} \leq c\left\|f\right\|_{X(\mathbb{T},\omega)} \tag{2.3}$$

and as a corollary of these we obtain

$$\left\|f - S_n(\cdot,f)\right\|_{X(\mathbb{T},\omega)} \leq cE_n(f)_{X,\omega}, \quad E_n\left(\widetilde{f}\right)_{X,\omega} \leq cE_n(f)_{X,\omega}. \tag{2.4}$$

Lemma 2.6. *Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space with nontrivial Boyd indices α_X and β_X. If $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, then the class of trigonometric polynomials is dense in $X(\mathbb{T},\omega)$.*

Proof. From the method of proof of Theorem 4.5 in [23] and Lemma 4.2 in [21], can be deduced that the condition $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap \in A_{1/\beta_X}(\mathbb{T})$ implies the conditions $\omega \in X(\mathbb{T})$ and $1/\omega \in X'(\mathbb{T})$. Then the space $X(\mathbb{T},\omega)$ is also reflexive [24, Corollary 2.8] and by Lemmas 1.2 and 1.3 in [22] the class of continuous functions $C(\mathbb{T})$ is dense in $X(\mathbb{T},\omega)$.

Let $f \in X(\mathbb{T},\omega)$ and $\varepsilon > 0$. Since $C(\mathbb{T})$ is dense in $X(\mathbb{T},\omega)$, there is a continuous function f_0 such that

$$\left\|f - f_0\right\|_{X(\mathbb{T},\omega)} < \varepsilon. \tag{2.5}$$

By the Weierstrass theorem, there exists a trigonometric polynomial T_0 such that

$$\left|f_0(x) - T_0(x)\right| < \varepsilon, \quad x \in \mathbb{T}.$$

Using this and formulas (1.2), (1.5) and Hölder inequality we get

$$\begin{aligned}
\|f_0 - T_0\|_{X(\mathbb{T},\omega)} &= \|(f_0 - T_0)\,\omega\|_{X(\mathbb{T})} \\
&= \sup\left\{\int_{\mathbb{T}} |f_0(x) - T_0(x)|\,\omega(x)\,|g(x)|\,dx : \|g\|_{X'(\mathbb{T})} \le 1\right\} \\
&\le \varepsilon \sup\left\{\int_{T} \omega(x)\,|g(x)|\,dx : \|g\|_{X'(\mathbb{T})} \le 1\right\} \\
&\le \varepsilon \sup\left\{\|\omega\|_{X(\mathbb{T})}\,\|g\|_{X'(\mathbb{T})} : \|g\|_{X'(\mathbb{T})} \le 1\right\} \le \varepsilon\,\|\omega\|_{X(\mathbb{T})},
\end{aligned}$$

which by (2.5) yields

$$\|f - T_0\|_{X(\mathbb{T},\omega)} \le \|f - f_0\|_{X(\mathbb{T},\omega)} + \|f_0 - T_0\|_{X(\mathbb{T},\omega)} < \left(1 + \|\omega\|_{X(\mathbb{T})}\right)\varepsilon,$$

and the assertion is proved. □

Corollary 2.7. *Under the assumptions of Lemma 2.6, the Fourier series of $f \in X(\mathbb{T},\omega)$ converges to f in the norm of $X(\mathbb{T},\omega)$.*

Proof. By Lemma 2.6 we have $E_n(f)_{X,\omega} \to 0$ $(n \to \infty)$ and then the proof follows from (2.4). □

Lemma 2.8. *Let $X(\mathbb{T})$ be a rearrangement invariant space with nontrivial Boyd indices α_X and β_X. If $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, and $f \in W_X^2(\mathbb{T},\omega)$, then the inequality*

$$\Omega_{X,\omega}^k(\delta, f) \le c\delta^2 \Omega_{X,\omega}^{k-1}(\delta, f''), \quad k = 1, 2, \ldots$$

holds with some constant c independent of δ.

Proof. Let's consider the function

$$g(x) := \prod_{i=2}^{k} (I - \sigma_{h_i})\, f(x).$$

Then $g \in W_X^2(\mathbb{T},\omega)$ and

$$(I - \sigma_{h_1})\, g(x) = (I - \sigma_{h_1})\left(\prod_{i=2}^{k} (I - \sigma_{h_i})\, f(x)\right) = \prod_{i=1}^{k} (I - \sigma_{h_i})\, f(x).$$

Hence

$$\begin{aligned}\prod_{i=1}^{k}\left(I-\sigma_{h_i}\right)f\left(x\right) &= g\left(x\right)-\sigma_{h_1}g\left(x\right)=g\left(x\right)-\frac{1}{2h_1}\int\limits_{-h_1}^{h_1}g\left(x+t\right)dt\\ &= \frac{1}{2h_1}\int\limits_{-h_1}^{h_1}\left[g\left(x\right)-g\left(x+t\right)\right]dt\\ &= -\frac{1}{4h_1}\int\limits_{-h_1}^{h_1}\left[g\left(x+t\right)-2g\left(x\right)+g\left(x-t\right)\right]dt\\ &= -\frac{1}{8h_1}\int\limits_{0}^{h_1}\int\limits_{0}^{t}\int\limits_{-u}^{u}g''\left(x+s\right)dsdudt.\end{aligned}$$

Now, according to (1.2), (1.5) and Fubini's theorem and getting the supremum under the integral sign we have

$$\begin{aligned}\left\|\prod_{i=1}^{k}\left(I-\sigma_{h_i}\right)f\right\|_{X(\mathbb{T},\omega)} &= \frac{1}{8h_1}\left\|\int\limits_{0}^{h_1}\int\limits_{0}^{t}\int\limits_{-u}^{u}g''\left(\cdot+s\right)dsdudt\right\|_{X(\mathbb{T},\omega)}\\ &= \frac{1}{8h_1}\sup\int\limits_{\mathbb{T}}\left|\int\limits_{0}^{h_1}\int\limits_{0}^{t}\int\limits_{-u}^{u}g''\left(x+s\right)dsdudt\right|\omega\left(x\right)\left|l\left(x\right)\right|dx\\ &\le \frac{1}{8h_1}\sup\int\limits_{\mathbb{T}}\left[\int\limits_{0}^{h_1}\int\limits_{0}^{t}\left|\int\limits_{-u}^{u}g''\left(x+s\right)ds\right|dudt\right]\omega\left(x\right)\left|l\left(x\right)\right|dx\\ &= \frac{1}{8h_1}\sup\int\limits_{0}^{h_1}\int\limits_{0}^{t}\left[\int\limits_{\mathbb{T}}\left|\int\limits_{-u}^{u}g''\left(x+s\right)ds\right|\omega\left(x\right)\left|l\left(x\right)\right|dx\right]dudt\\ &\le \frac{1}{8h_1}\int\limits_{0}^{h_1}\int\limits_{0}^{t}\left[\sup\int\limits_{\mathbb{T}}\left|\int\limits_{-u}^{u}g''\left(x+s\right)ds\right|\omega\left(x\right)\left|l\left(x\right)\right|dx\right]dudt\\ &= \frac{1}{8h_1}\int\limits_{0}^{h_1}\int\limits_{0}^{t}\left\|\int\limits_{-u}^{u}g''\left(\cdot+s\right)ds\right\|_{X(\mathbb{T},\omega)}dudt\\ &= \frac{1}{8h_1}\int\limits_{0}^{h_1}\int\limits_{0}^{t}2u\left\|\frac{1}{2u}\int\limits_{-u}^{u}g''\left(\cdot+s\right)ds\right\|_{X(\mathbb{T},\omega)}dudt,\end{aligned}$$

where the suprema above are taken over all functions $l \in X'(\mathbb{T})$ with $\|l\|_{X'(\mathbb{T})} \leq 1$. Taking into account the boundedness of σ_u we see that

$$\begin{aligned}\left\|\prod_{i=1}^{k}(I-\sigma_{h_i})f\right\|_{X(\mathbb{T},\omega)} &\leq \frac{1}{8h_1}\int_0^{h_1}\int_0^t 2u\,\|\sigma_u g''\|_{X(\mathbb{T},\omega)}\,dudt \\ &\leq c\frac{1}{8h_1}\int_0^{h_1}\int_0^t 2u\,\|g''\|_{X(\mathbb{T},\omega)}\,dudt = ch_1^2\,\|g''\|_{X(\mathbb{T},\omega)}\,.\end{aligned}$$

On the other hand, by the definitions of g and σ_{h_i} we have $g'' = \prod_{i=2}^{k}(I-\sigma_{h_i})f''$. Then from the last inequality we conclude that

$$\begin{aligned}\Omega_{X,\omega}^{k}(\delta,f) &= \sup_{\substack{0<h_i\leq\delta\\1\leq i\leq k}}\left\|\prod_{i=1}^{k}(I-\sigma_{h_i})f\right\|_{X(\mathbb{T},\omega)} \leq \sup_{\substack{0<h_i\leq\delta\\1\leq i\leq k}} ch_1^2\,\|g''\|_{X(\mathbb{T},\omega)} \\ &= c\delta^2\sup_{\substack{0<h_i\leq\delta\\2\leq i\leq k}}\left\|\prod_{i=2}^{k}(I-\sigma_{h_i})f''\right\|_{X(\mathbb{T},\omega)} = c\delta^2\Omega_{X,\omega}^{k-1}(\delta,f'')\end{aligned}$$

and this finished the proof. □

Corollary 2.9. *If $f \in W_X^{2k}(\mathbb{T},\omega)$ $(k = 1,2,\ldots)$, then*

$$\Omega_{X,\omega}^{k}(\delta,f) \leq c\delta^{2k}\left\|f^{(2k)}\right\|_{X(\mathbb{T},\omega)}$$

with some constant c independent of δ.

For an $f \in X(\mathbb{T},\omega)$ the $K-$functional is defined as

$$K(\delta,f;X(\mathbb{T},\omega),W_X^r(\mathbb{T},\omega)) := \inf_{\psi\in W_X^r(\mathbb{T},\omega)}\|f-\psi\|_{X(\mathbb{T},\omega)} + \delta\left\|\psi^{(r)}\right\|_{X(\mathbb{T},\omega)}$$

for $\delta > 0$.

Theorem 2.10. *Let $X(\mathbb{T})$ be a rearrangement invariant space with non-trivial Boyd indices α_X and β_X, and $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$. Then for $f \in X(\mathbb{T},\omega)$ and $k = 1,2,\ldots,$ the equivalence*

$$K\left(\delta^{2k},f;X(\mathbb{T},\omega),W_X^{2k}(\mathbb{T},\omega)\right) \sim \Omega_{X,\omega}^{k}(\delta,f) \tag{2.6}$$

holds, where the constants in this relation are independent of δ.

PROOF. Let ψ be an arbitrary function in $W_X^{2k}(\mathbb{T},\omega)$. By (1.6), Corollaries 3 and 5 we obtain

$$\begin{aligned}\Omega_{X,\omega}^{k}(\delta,f) &= \Omega_{X,\omega}^{k}(\delta,f-\psi+\psi)\\ &\leq \Omega_{X,\omega}^{k}(\delta,f-\psi)+\Omega_{X,\omega}^{k}(\delta,\psi)\\ &\leq c_1\|f-\psi\|_{X(\mathbb{T},\omega)}+c_2\delta^{2k}\left\|\psi^{(2k)}\right\|_{X(\mathbb{T},\omega)}.\end{aligned}$$

Taking the infimum over all $\psi\in W_X^{2k}(\mathbb{T},\omega)$, by definition of the $K-$functional we get

$$\Omega_{X,\omega}^{k}(\delta,f)\leq cK\left(\delta^{2k},f;X(\mathbb{T},\omega),W_X^{2k}(\mathbb{T},\omega)\right).$$

For the proof of the reverse estimation consider an operator L_δ on $X(\mathbb{T},\omega)$,

$$(L_\delta f)(x):=\frac{3}{\delta^3}\int_0^\delta\int_0^u\int_{-t}^{t}f(x+s)\,dsdtdu,\quad x\in\mathbb{T}.$$

Then

$$\frac{d^2}{dx^2}(L_\delta f)=\frac{c}{\delta^2}(I-\sigma_\delta)f$$

and hence

$$\frac{d^{2k}}{dx^{2k}}L_\delta^k=\frac{c}{\delta^{2k}}(I-\sigma_\delta)^k,\quad k=1,2,\dots\,. \tag{2.7}$$

The operator L_δ is bounded in $X(\mathbb{T},\omega)$. Indeed, using (1.5), (1.2) and the boundedness of σ_t in $X(\mathbb{T},\omega)$ we get

$$\begin{aligned}\|L_\delta f\|_{X(\mathbb{T},\omega)} &\leq \frac{3}{\delta^3}\int_0^\delta\int_0^u\left\|\int_{-t}^{t}f(\cdot+s)\,ds\right\|_{X(\mathbb{T},\omega)}dtdu\\ &= \frac{3}{\delta^3}\int_0^\delta\int_0^u 2t\|\sigma_t f\|_{X(\mathbb{T},\omega)}\,dtdu\\ &\leq c\frac{3}{\delta^3}\|f\|_{X(\mathbb{T},\omega)}\int_0^\delta\int_0^u 2tdtdu=c\|f\|_{X(\mathbb{T},\omega)}.\end{aligned}$$

Consider the operator

$$A_\delta^k:=I-\left(I-L_\delta^k\right)^k.$$

Then we have $A_\delta^k f\in W_X^{2k}(\mathbb{T},\omega)$ for $f\in X(\mathbb{T},\omega)$ and furthermore by (2.7) the inequality

$$\left\|\frac{d^{2k}}{dx^{2k}}A_\delta^k f\right\|_{X(\mathbb{T},\omega)}\leq c\left\|\frac{d^{2k}}{dx^{2k}}L_\delta^k f\right\|_{X(\mathbb{T},\omega)}\leq\frac{c}{\delta^{2k}}\left\|(I-\sigma_\delta)^k f\right\|_{X(\mathbb{T},\omega)}$$

holds. This inequality and the definition of $\Omega_{X,\omega}^{k}(\delta,f)$ yield

$$\delta^{2k}\left\|\frac{d^{2k}}{dx^{2k}}A_{\delta}^{k}f\right\|_{X(\mathbb{T},\omega)}\leq c\Omega_{X,\omega}^{k}(\delta,f). \tag{2.8}$$

Since

$$I-L_{\delta}^{k}=(I-L_{\delta})\left(\sum_{j=0}^{k-1}L_{\delta}^{j}\right)$$

and L_{δ} is bounded in $X(\mathbb{T},\omega)$, we have

$$\begin{aligned}
\left\|\left(I-L_{\delta}^{k}\right)g\right\|_{X(\mathbb{T},\omega)} &= \left\|\left(\sum_{j=0}^{k-1}L_{\delta}^{j}\right)(I-L_{\delta})g\right\|_{X(\mathbb{T},\omega)}\leq c\left\|(I-L_{\delta})g\right\|_{X(\mathbb{T},\omega)}\\
&= c\left\|\frac{3}{\delta^{3}}\int_{0}^{\delta}\int_{0}^{u}\int_{-t}^{t}\left[g-g(\cdot+s)\right]dsdtdu\right\|_{X(\mathbb{T},\omega)}\\
&\leq \frac{3c}{\delta^{3}}\int_{0}^{\delta}\int_{0}^{u}2t\left\|\frac{1}{2t}\int_{-t}^{t}\left[g-g(\cdot+s)\right]ds\right\|_{X(\mathbb{T},\omega)}dtdu\\
&= \frac{3c}{\delta^{3}}\int_{0}^{\delta}\int_{0}^{u}2t\left\|(I-\sigma_{t})g\right\|_{X(\mathbb{T},\omega)}dtdu\\
&\leq \frac{3c}{\delta^{3}}\sup_{0<t\leq\delta}\left\|(I-\sigma_{t})g\right\|_{X(\mathbb{T},\omega)}\int_{0}^{\delta}\int_{0}^{u}2tdtdu\\
&= c\sup_{0<t\leq\delta}\left\|(I-\sigma_{t})g\right\|_{X(\mathbb{T},\omega)}
\end{aligned}$$

for every $g\in X(\mathbb{T},\omega)$. Applying this inequality $k-$times in

$$\left\|f-A_{\delta}^{k}f\right\|_{X(\mathbb{T},\omega)}=\left\|\left(I-L_{\delta}^{k}\right)^{k}f\right\|_{X(\mathbb{T},\omega)}=\left\|\left(I-L_{\delta}^{k}\right)\left(I-L_{\delta}^{k}\right)^{k-1}f\right\|_{X(\mathbb{T},\omega)},$$

we obtain

$$\begin{aligned}
\left\|f-A_{\delta}^{k}f\right\|_{X(\mathbb{T},\omega)} &\leq c_{1}\sup_{0<t_{1}\leq\delta}\left\|(I-\sigma_{t_{1}})\left(I-L_{\delta}^{k}\right)^{k-1}f\right\|_{X(\mathbb{T},\omega)}\\
&\leq c_{2}\sup_{0<t_{1}\leq\delta}\sup_{0<t_{2}\leq\delta}\left\|(I-\sigma_{t_{1}})(I-\sigma_{t_{2}})\left(I-L_{\delta}^{k}\right)^{k-2}f\right\|_{X(\mathbb{T},\omega)}\\
&\leq \ldots\leq c\sup_{\substack{0<t_{j}\leq\delta\\1\leq j\leq k}}\left\|\prod_{j=1}^{k}\left(I-\sigma_{t_{j}}\right)f\right\|_{X(\mathbb{T},\omega)}\\
&= c\Omega_{X,\omega}^{k}(\delta,f).
\end{aligned}$$

Since $A_\delta^k f \in W_X^{2k}(\mathbb{T},\omega)$, from the last inequality, the inequality (2.8) and the definition of the $K-$functional, we conclude that

$$\begin{aligned} K\left(\delta^{2k}, f; X(\mathbb{T},\omega), W_X^{2k}(\mathbb{T},\omega)\right) &\leq \left\|f - A_\delta^k f\right\|_{X(\mathbb{T},\omega)} + \delta^{2k}\left\|\frac{d^{2k}}{dx^{2k}} A_\delta^k f\right\|_{X(\mathbb{T},\omega)} \\ &\leq c\Omega_{X,\omega}^k(\delta, f), \end{aligned}$$

which gives the reverse estimation and hence the proof is completed. □

3. Proofs of the main results

Proof of Theorem 1.2. Let $\sum\limits_{k=0}^{\infty}(a_k \cos kx + b_k \sin kx)$ be the Fourier series of f and $S_n(x,f)$ be its nth partial sum i.e.,

$$S_n(x,f) = \sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx).$$

It is known that the conjugate function $\widetilde{f}$ has the Fourier expansion

$$\sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx).$$

If we denote

$$A_k(x,f) := a_k \cos kx + b_k \sin kx,$$

then by Corollary 2.7 we have

$$f(x) = \sum_{k=0}^{\infty} A_k(x,f)$$

in the norm of $X(\mathbb{T},\omega)$.

Since for $k = 1, 2, \ldots,$

$$\begin{aligned} A_k(x,f) &= a_k \cos kx + b_k \sin kx \\ &= a_k \cos k\left(x + \frac{r\pi}{2k} - \frac{r\pi}{2k}\right) + b_k \sin k\left(x + \frac{r\pi}{2k} - \frac{r\pi}{2k}\right) \\ &= a_k \cos\left(kx + \frac{r\pi}{2} - \frac{r\pi}{2}\right) + b_k \sin\left(kx + \frac{r\pi}{2} - \frac{r\pi}{2}\right) \\ &= a_k\left[\cos\left(kx + \frac{r\pi}{2}\right)\cos\frac{r\pi}{2} + \sin\left(kx + \frac{r\pi}{2}\right)\sin\frac{r\pi}{2}\right] \\ &\quad + b_k\left[\sin\left(kx + \frac{r\pi}{2}\right)\cos\frac{r\pi}{2} - \cos\left(kx + \frac{r\pi}{2}\right)\sin\frac{r\pi}{2}\right] \\ &= \cos\frac{r\pi}{2}\left[a_k \cos k\left(x + \frac{r\pi}{2k}\right) + b_k \sin k\left(x + \frac{r\pi}{2k}\right)\right] \\ &\quad + \sin\frac{r\pi}{2}\left[a_k \sin k\left(x + \frac{r\pi}{2k}\right) - b_k \cos k\left(x + \frac{r\pi}{2k}\right)\right] \\ &= A_k\left(x + \frac{r\pi}{2k}, f\right)\cos\frac{r\pi}{2} + A_k\left(x + \frac{r\pi}{2k}, \widetilde{f}\right)\sin\frac{r\pi}{2} \end{aligned}$$

and

$$A_k\left(x, f^{(r)}\right) = k^r A_k\left(x + \frac{r\pi}{2k}, f\right),$$

we get

$$\begin{aligned}
\sum_{k=0}^{\infty} A_k(x,f) &= A_0(x,f) + \cos\frac{r\pi}{2}\sum_{k=1}^{\infty} A_k\left(x+\frac{r\pi}{2k}, f\right) \\
&\quad + \sin\frac{r\pi}{2}\sum_{k=1}^{\infty} A_k\left(x+\frac{r\pi}{2k}, \widetilde{f}\right) \\
&= A_0(x,f) + \cos\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^r}k^r A_k\left(x+\frac{r\pi}{2k}, f\right) \\
&\quad + \sin\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^r}k^r A_k\left(x+\frac{r\pi}{2k}, \widetilde{f}\right) \\
&= A_0(x,f) + \cos\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^r}A_k\left(x, f^{(r)}\right) \\
&\quad + \sin\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^r}A_k\left(x, \widetilde{f}^{(r)}\right).
\end{aligned}$$

Then

$$\begin{aligned}
f(x) - S_n(x,f) &= \sum_{k=n+1}^{\infty} A_k(x,f) \\
&= \cos\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k\left(x, f^{(r)}\right) + \sin\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k\left(x, \widetilde{f}^{(r)}\right).
\end{aligned}$$

Take into account that

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k\left(x, f^{(r)}\right) = \sum_{k=n+1}^{\infty}\frac{1}{k^r}\left[S_k\left(x, f^{(r)}\right) - S_{k-1}\left(x, f^{(r)}\right)\right] \\
&= \sum_{k=n+1}^{\infty}\frac{1}{k^r}\left\{\left[S_k\left(x, f^{(r)}\right) - f^{(r)}(x)\right] - \left[S_{k-1}\left(x, f^{(r)}\right) - f^{(r)}(x)\right]\right\} \\
&= \sum_{k=n+1}^{\infty}\left(\frac{1}{k^r} - \frac{1}{(k+1)^r}\right)\left[S_k\left(x, f^{(r)}\right) - f^{(r)}(x)\right] \\
&\quad - \frac{1}{(n+1)^r}\left[S_n\left(x, f^{(r)}\right) - f^{(r)}(x)\right],
\end{aligned}$$

and

$$\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k\left(x,\widetilde{f}^{(r)}\right) = \sum_{k=n+1}^{\infty}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)\left[S_k\left(x,\widetilde{f}^{(r)}\right)-\widetilde{f}^{(r)}(x)\right] -\frac{1}{(n+1)^r}\left[S_n\left(x,\widetilde{f}^{(r)}\right)-\widetilde{f}^{(r)}(x)\right],$$

by (2.4) we have

$$\begin{aligned}
&\left\|f-S_n(\cdot,f)\right\|_{X(\mathbb{T},\omega)}\\
&\leq \sum_{k=n+1}^{\infty}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)\left\|S_k\left(\cdot,f^{(r)}\right)-f^{(r)}\right\|_{X(\mathbb{T},\omega)}\\
&\quad+\frac{1}{(n+1)^r}\left\|S_n\left(\cdot,f^{(r)}\right)-f^{(r)}\right\|_{X(\mathbb{T},\omega)}\\
&\quad+\sum_{k=n+1}^{\infty}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)\left\|S_k\left(\cdot,\widetilde{f}^{(r)}\right)-\widetilde{f}^{(r)}\right\|_{X(\mathbb{T},\omega)}\\
&\quad+\frac{1}{(n+1)^r}\left\|S_n\left(\cdot,\widetilde{f}^{(r)}\right)-\widetilde{f}^{(r)}\right\|_{X(\mathbb{T},\omega)}\\
&\leq c_1\left\{\sum_{k=n+1}^{\infty}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)E_k\left(f^{(r)}\right)_{X,\omega}+\frac{1}{(n+1)^r}E_n\left(f^{(r)}\right)_{X,\omega}\right\}\\
&\quad+c_2\left\{\sum_{k=n+1}^{\infty}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)E_k\left(\widetilde{f}^{(r)}\right)_{X,\omega}+\frac{1}{(n+1)^r}E_n\left(\widetilde{f}^{(r)}\right)_{X,\omega}\right\}.
\end{aligned}$$

Since the sequence $\left\{E_n\left(f^{(r)}\right)_{X,\omega}\right\}$ is decreasing, we finally conclude that

$$\begin{aligned}
&\left\|f-S_n(.,f)\right\|_{X(\mathbb{T},\omega)}\\
&\leq c_1E_n\left(f^{(r)}\right)_{X,\omega}\left\{\sum_{k=n+1}^{\infty}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)+\frac{1}{(n+1)^r}\right\}\\
&\quad+c_2E_n\left(\widetilde{f}^{(r)}\right)_{X,\omega}\left\{\sum_{k=n+1}^{\infty}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)+\frac{1}{(n+1)^r}\right\}\\
&\leq c_3E_n\left(f^{(r)}\right)_{X,\omega}\left\{\sum_{k=n+1}^{\infty}\left(\frac{1}{k^r}-\frac{1}{(k+1)^r}\right)+\frac{1}{(n+1)^r}\right\}E_n\left(f^{(r)}\right)_{X,\omega}\\
&= \frac{2c_3}{(n+1)^r}E_n\left(f^{(r)}\right)_{X,\omega}.
\end{aligned}$$

This by the relation

$$E_n(f)_{X,\omega}\leq\left\|f-S_n(.,f)\right\|_{X(T,\omega)}$$

gives (1.8) and completes the proof of Theorem 1.2. □

COROLLARY 3.1. *For $f \in W_X^r(\mathbb{T},\omega)$ the inequality*

$$E_n(f)_{X,\omega} \leq \frac{c}{(n+1)^r}\left\|f^{(r)}\right\|_{X(\mathbb{T},\omega)}$$

holds with a constant c independent of n.

PROOF OF THEOREM 1.3. Let $\psi \in W_X^{2k}(\mathbb{T},\omega)$. Then by subadditivity of the best approximation and Corollary 3.1 we have

$$\begin{aligned} E_n(f)_{X,\omega} &= E_n(f-\psi+\psi)_{X,\omega} \leq E_n(f-\psi)_{X,\omega} + E_n(\psi)_{X,\omega} \\ &\leq c\left\{\|f-\psi\|_{X(\mathbb{T},\omega)} + \frac{1}{(n+1)^{2k}}\left\|\psi^{(2k)}\right\|_{X(\mathbb{T},\omega)}\right\}. \end{aligned}$$

Since this inequality holds for every $\psi \in W_X^{2k}(\mathbb{T},\omega)$, by the definition of the $K-$functional we get

$$E_n(f)_{X,\omega} \leq cK\left(\frac{1}{(n+1)^{2k}}, f; X(\mathbb{T},\omega), W_X^{2k}(\mathbb{T},\omega)\right).$$

According to Theorem 2.10 this implies

$$E_n(f)_{X,\omega} \leq c\Omega_{X,\omega}^k\left(\frac{1}{n+1}, f\right),$$

which completes the proof. □

PROOF OF THEOREM 1.4. Let $\sum\limits_{j=-\infty}^{\infty}\gamma_j(f)e^{ijx}$ be the exponential Fourier series of the boundary function of f and $S_n(x,f)$ be its nth partial sum, i.e.,

$$S_n(x,f) = \sum_{j=-n}^{n}\gamma_j(f)e^{ijx}.$$

Then for $f \in H_1(\mathbb{D})$, by Theorem 3.4 in [12] we have

$$\gamma_j(f) = \begin{cases} a_j(f), & j \geq 0 \\ 0, & j < 0 \end{cases}.$$

Let $T_n^*(x)$ be the polynomial of the best approximation to f from the class Π_n in the space $X(\mathbb{T},\omega)$. Then the relation (2.3) and Theorem 1.3 for every natural number n yield

$$\begin{aligned}\left\| f(z) - \sum_{j=0}^{n} a_j(f) z^j \right\|_{X(\mathbb{T},\omega)} &= \left\| f\left(e^{ix}\right) - \sum_{j=0}^{n} \gamma_j(f) e^{ijx} \right\|_{X(\mathbb{T},\omega)} \\ &= \|f - S_n(\cdot, f)\|_{X(\mathbb{T},\omega)} = \|f - T_n^* + T_n^* - S_n(\cdot, f)\|_{X(\mathbb{T},\omega)} \\ &\leq \|f - T_n^*\|_{X(\mathbb{T},\omega)} + \|S_n(\cdot, T_n^* - f)\|_{X(\mathbb{T},\omega)} \\ &\leq c\|f - T_n^*\|_{X(\mathbb{T},\omega)} = cE_n(f)_{X,\omega} \leq c\Omega_{X,\omega}^k\left(\frac{1}{n+1}, f\right)\end{aligned}$$

and the theorem is proved. ∎

Proof of Theorem 1.5. Let $f \in X(\mathbb{T}, \omega)$ and T_n $(n = 0, 1, 2, \ldots)$ be the polynomials of best approximation to f in the class Π_n.

Let $n = 1, 2, \ldots$ and $\delta := 1/n$. For any $m = 1, 2, \ldots$

$$\Omega_{X,\omega}^k(\delta, f) \leq \Omega_{X,\omega}^k(\delta, f - T_{2^{m+1}}) + \Omega_{X,\omega}^k(\delta, T_{2^{m+1}}). \tag{3.1}$$

We have

$$\Omega_{X,\omega}^k(\delta, f - T_{2^{m+1}}) \leq c_1\|f - T_{2^{m+1}}\|_{X(\mathbb{T},\omega)} = c_1E_{2^{m+1}}(f)_{X,\omega}. \tag{3.2}$$

On the other hand, using (2.2) and (2.3) we obtain

$$\begin{aligned}&\Omega_{X,\omega}^k(\delta, T_{2^{m+1}}) \leq c_2\delta^{2k}\left\|T_{2^{m+1}}^{(2k)}\right\|_{X(\mathbb{T},\omega)} \\ &\leq c_2\delta^{2k}\left\{\left\|T_1^{(2k)} - T_0^{(2k)}\right\|_{X(\mathbb{T},\omega)} + \sum_{i=0}^{m}\left\|T_{2^{i+1}}^{(2k)} - T_{2^i}^{(2k)}\right\|_{X(\mathbb{T},\omega)}\right\} \\ &\leq c_3\delta^{2k}\left\{\|T_1 - T_0\|_{X(\mathbb{T},\omega)} + \sum_{i=0}^{m}2^{(i+1)2k}\|T_{2^{i+1}} - T_{2^i}\|_{X(\mathbb{T},\omega)}\right\} \\ &\leq c_3\delta^{2k}\left\{E_1(f)_{X,\omega} + E_0(f)_{X,\omega} + \sum_{i=0}^{m}2^{(i+1)2k}\left(E_{2^{i+1}}(f)_{X,\omega} + E_{2^i}(f)_{X,\omega}\right)\right\} \\ &\leq c_4\delta^{2k}\left\{E_0(f)_{X,\omega} + \sum_{i=0}^{m}2^{(i+1)2k}E_{2^i}(f)_{X,\omega}\right\} \\ &= c_4\delta^{2k}\left\{E_0(f)_{X,\omega} + 2^{2k}E_1(f)_{X,\omega} + \sum_{i=1}^{m}2^{(i+1)2k}E_{2^i}(f)_{X,\omega}\right\},\end{aligned}$$

because the sequence of best approximations $\left\{E_n(f)_{X,\omega}\right\}$ is monotone decreasing. By monotonicity of $\left\{E_n(f)_{X,\omega}\right\}$,

$$\begin{aligned}\sum_{l=2^{i-1}+1}^{2^i} l^{2k-1}E_l(f)_{X,\omega} &\geq \sum_{l=2^{i-1}+1}^{2^i} l^{2k-1}E_{2^i}(f)_{X,\omega}\\ &\geq E_{2^i}(f)_{X,\omega}\sum_{l=2^{i-1}+1}^{2^i}\left(2^{i-1}\right)^{2k-1}\\ &= E_{2^i}(f)_{X,\omega}\,2^{(i-1)2k},\end{aligned}$$

and hence

$$2^{(i+1)2k}E_{2^i}(f)_{X,\omega} \leq 2^{4k}\sum_{l=2^{i-1}+1}^{2^i} l^{2k-1}E_l(f)_{X,\omega} \tag{3.3}$$

holds for $i \geq 1$. So, we get the estimate

$$\Omega_{X,\omega}^k(\delta, T_{2^{m+1}}) \leq c_5\delta^{2k}\left\{E_0(f)_{X,\omega}+\sum_{l=1}^{2^m} l^{2k-1}E_l(f)_{X,\omega}\right\}. \tag{3.4}$$

If we select m such that $2^m \leq n < 2^{m+1}$, then by (3.3)

$$\begin{aligned}E_{2^{m+1}}(f)_{X,\omega} &= \frac{2^{(m+1)2k}E_{2^{m+1}}(f)_{X,\omega}}{2^{(m+1)2k}} \leq \frac{2^{(m+1)2k}E_{2^m}(f)_{X,\omega}}{n^{2k}}\\ &\leq \frac{2^{4k}}{n^{2k}}\sum_{l=2^{m-1}+1}^{2^m} l^{2k-1}E_l(f)_{X,\omega}.\end{aligned}$$

Combining (3.1), (3.2), (3.4) and using the last inequality completes the proof of Theorem 1.5. ∎

Proof of Corollary 1.6. Let

$$E_n(f)_{X,\omega} = \mathcal{O}\left(n^{-\alpha}\right), \quad \alpha > 0,\ n = 1, 2, \ldots,$$

for $f \in X(\mathbb{T}, \omega)$.

Let $\delta > 0$. If we choose the natural number n as the integral part of $1/\delta$, we get by Theorem 1.5

$$\begin{aligned}\Omega_{X,\omega}^k(\delta, f) &\leq \Omega_{X,\omega}^k\left(\frac{1}{n}, f\right) \leq \frac{c_1}{n^{2k}}\left\{E_0(f)_{X,\omega}+\sum_{m=1}^{n} m^{2k-1}E_m(f)_{X,\omega}\right\}\\ &\leq c_2\delta^{2k}\left\{E_0(f)_{X,\omega}+\sum_{m=1}^{n} m^{2k-1-\alpha}\right\},\end{aligned}$$

since

$$n \leq 1/\delta < n+1.$$

Hence, if $2k > \alpha$, then simple calculations yield $\Omega^k_{X,\omega}(\delta, f) = \mathcal{O}(\delta^\alpha)$. If $\alpha = 2k$, then

$$\sum_{m=1}^{n} m^{2k-1-\alpha} = \sum_{m=1}^{n} m^{-1} \leq 1 + \log(1/\delta),$$

and from this inequality we obtain

$$\Omega^k_{X,\omega}(\delta, f) = \mathcal{O}(\delta^\alpha \log(1/\delta)).$$

Finally if $\alpha > 2k$, then the series

$$\sum_{m=1}^{\infty} m^{2k-1-\alpha}$$

is convergent, hence the estimate

$$\Omega^k_{X,\omega}(\delta, f) \leq c_2 \delta^{2k} \left\{ E_0(f)_{X,\omega} + \sum_{m=1}^{\infty} m^{2k-1-\alpha} \right\} = \mathcal{O}\left(\delta^{2k}\right)$$

holds. □

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A. Guven
Department of Mathematics
Faculty of Art and Science
Balikesir University
10145, Balikesir
Turkey
E-mail: `ag_guven@yahoo.com`

D. M. Israfilov
Department of Mathematics
Faculty of Art and Science
Balikesir University
10145, Balikesir
Turkey
E-mail: `mdaniyal@balikesir.edu.tr`

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