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ON AUTOMORPHISMS OF ORDER p OF METACYCLIC p-GROUPS WITHOUT CYCLIC SUBGROUPS OF INDEX p

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Abstract. Let L be a metacyclic p-group, $p > 2$, without cyclic subgroups of index p and let $a \in \mathrm{Aut(L)}$ be of order p. We show that either a centralizes $\Omega_1(L)$ or $p = 3$ and the natural semidirect product $\langle a \rangle \cdot L$ is of maximal class so the subgroup L has very specific structure. This improves Lemma 4.9 from [MS].

According to [MS, Lemma 4.9], if $p > 3$ is a prime and a is an automorphism of order p of abelian group L of type (p^2, p^2), then a centralizes $\Omega_1(L)$ (the proof of this result is also reproduced in [AS, Lemma A.1.30]). The same conclusion is true provided L is abelian of type (p^m, p^n), $p > 3$ and $m \geq n > 2$ (it suffices to consider the restriction of a on $\Omega_2(L)$). Our aim is to improve this result as follows:

Theorem 1. *Suppose that L is a metacyclic p-group without cyclic subgroup of index p, $p > 2$. An element $a \in \mathrm{Aut}(L)$ of order p does not centralize $\Omega_1(L)$ if and only if $p = 3$ and the natural semidirect product $G = \langle a \rangle \cdot L$ is a 3-group of maximal class.*[1]

By Theorem 1, if $p > 3$ and L is a metacyclic p-group without cyclic subgroup of index p, then $\Omega_1(L)$ is centralized by A, where A is the subgroup generated by all elements of $\mathrm{Aut}(L)$ of order p. We claim that, if $W = A \cdot L$ is the natural semidirect product, then $|W : \mathrm{C_W}(L)|$ is a power of p. Indeed, if b is a p'-element of W, then b, as an element of A, centralizes $\Omega_1(L)$ and so

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[1]The proof of this theorem shows that if L has a cyclic subgroup of index p, then either $G = \langle a \rangle \cdot L$ is a group of maximal class and order p^4 or a group (b2) of Lemma 3(b). It is known that an outer automorphism of L of order p exists; see, for example, [Hup, Satz III.19.1]

the natural semidirect product $\langle b\rangle \cdot L$ has no minimal nonnilpotent subgroups (see [B2, Theorem 10.8]) so it is nilpotent [I, Theorem 9.18]; in that case b centralizes L.

Our proof of Theorem 1 uses fairly deep results of finite p-group theory and so it is essentially differed from the proof of [MS, Lemma 4.9] which is based on intricate computations with elements of L and the given automorphism a of L of order p.

COROLLARY 2. *Suppose that $p > 2$ and L is an abelian group of type (p^m, p^n), $m > 1, n > 1$. An element $a \in \mathrm{Aut}(L)$ of order p does not centralize $\Omega_1(L)$ if and only if $|m-n| \leq 1$, $p = 3$ and the natural semidirect product $G = \langle a\rangle \cdot L$ is a 3-group of maximal class.*

To deduce Corollary 2 from Theorem 1, it suffices to apply Remark 4, below.

We use standard notation of finite p-group theory (see [B1–B5]).

In Lemma 3 we gathered all known results which are used in what follows.

LEMMA 3. *Let $G > \{1\}$ be a p-group.*

(a) *If G is regular, then $\exp(\Omega_1(G)) = p$ and $|G/\mho_1(G)| = |\Omega_1(G)|$.*

(b) *Blackburn; see also [B1, Theorem 6.1]. If $p > 2$ and G has no normal elementary abelian subgroup of order p^3, then one of the following holds:*
   - (b1) *G is metacyclic.*
   - (b2) *$G = C\Omega_1(G)$, where $\Omega_1(G)$ is nonabelian of order p^3 and exponent p and C is cyclic (in particular, $G/\Omega_1(G)$ is cyclic and $\mho_1(C) \leq \mathrm{Z}(G)$).*
   - (b3) *G is a 3-group of maximal class not isomorphic to a Sylow 3-subgroup of the symmetric group of degree 3^2.*[2]

(c) *Blackburn; see also [B2, Theorems 9.5 and 9.6]. Let a p-group G of maximal class be of order greater than p^p. Then G is irregular, $\Omega_1(\Phi(G))$ is of order p^{p-1} and exponent p and $|G/\mho_1(G)| = p^p$. If, in addition, $|G| > p^{p+1}$, then there is in G a unique regular maximal subgroup, say G_1, and it is absolutely regular; all other maximal subgroups of G are of maximal class.*[3]

(d) *A p-group of maximal class and order $> p^3$ has no normal cyclic subgroup of order p^2, unless $p = 2$.*

(e) *Blackburn; see also [B3, Theorem 7.5]. Suppose that a non-absolutely regular p-group G has an absolutely regular maximal subgroup H. Then*

[2] A Sylow 3-subgroup of the symmetric group of degree 3^2 is the unique 3-group of maximal class that contains an elementary abelian subgroup of order 3^3.

[3] A p-group X is absolutely regular if $|X/\mho_1(X)| < p^p$; then X is regular, by Hall's regularity criterion [B2, Theorem 9.8(a)]. It follows that, if $p > 2$, then metacyclic p-groups are absolutely regular.

either G is irregular of maximal class or $G = H\Omega_1(G)$, where $\Omega_1(G)$ is of order p^p and exponent p.

(f) *Blackburn; see also [J, Theorem 7.1] and [BJ, Theorem 7.1]. If a 2-group G is minimal nonmetacyclic, then G is one of the following groups: (i)* E_8, (ii) $\mathrm{Q}_8 \times \mathrm{C}_2$, (iii) $\mathrm{D}_8 {*} \mathrm{C}_4$ (*central product) of order* 16, *(iv) a special group of order 2^5 with $|\mathrm{Z}(G)| = 2^2$.*

(g) *[B4, Proposition 19(a)]. If B is a nonabelian subgroup of order p^3 of a p-group G such that $\mathrm{C}_G(B) < B$, then G is of maximal class.*

(h) *If a metacyclic p-group G has a nonabelian subgroup B of order p^3, then either G is a 2-group of maximal class or $G = B$.*

(i) [BJ, Lemma 3.2(a)] *If G is a nonabelian two-generator p-group and $G' \le \Omega_1(\mathrm{Z}(G))$, then G is minimal nonabelian.*

(j) Blackburn; see also [B3, Theorem 7.6]. *If a p-group G has no normal subgroup of order p^p and exponent p, then it is either absolutely regular or of maximal class.*

(k) Huppert; see also [B5, Corollary 13]. *If $p > 2$ and G is such that $|G/\mho_1(G)| \le p^2$, then G is metacyclic.*

(l) Redei ([R]); see also [B2, Exercise 1.8a]. *If G is a metacyclic minimal nonabelian p-group of order p^m, then either $G \cong$* Q_8 or *$G = \langle a, b \mid a^{p^m} = b^{p^n} = 1,\, a^b = a^{1+p^{m-1}}\rangle$. If G is nonmetacyclic minimal nonabelian of order $> p^3$, then $\Omega_1(G) \cong \mathrm{E}_{p^3}$.*

Let us prove Lemma 3(d). Let $p > 2$ and X a p-group of maximal class and order $> p^3$. Then X has only one normal subgroup of order p^2; since this subgroup is abelian of type (p, p) [B2, Lemma 1.4], we are done.

Let us prove Lemma 3(h). Assume that $|G| > p^3$ and $\mathrm{C}_G(B) \not\le B$, where B is nonabelian of order p^3 and G is metacyclic. If $F \le \mathrm{C}_G(B)$ is of order p^2, then $\mathrm{d}(BF) > 2$ so BF is a nonmetacyclic subgroup of a metacyclic group G, a contradiction. Thus, $\mathrm{C}_G(B) < B$. Then, by Lemma 3(g), G is of maximal class so $|G : G'| = p^2$ which is impossible for metacyclic p-groups of order $> p^3$ with $p > 2$; in case $p = 2$, our G is of maximal class (Taussky).

Remark 4 (Blackburn). Suppose that G is a 3-group of maximal class and order $> 3^4$ and $G_1 < G$ is absolutely regular; then G_1 is noncyclic (Lemma 3(c)) and metacyclic (Lemmas (c,k)). Assume that G_1 has a cyclic subgroup of index 3. In that case, $\Omega_2(\mho_1(G_1))$ is cyclic of order 3^2, contrary to Lemma 3(d). Suppose that G_1 is abelian of type $(3^m, 3^n)$ with $m \ge n$. Then $\mho_n(G_1)$ is G-invariant and cyclic of order 3^{m-n} so $m - n \le 1$ (Lemma 3(d)). Now suppose that G_1 is nonabelian. Then G_1' is cyclic and G-invariant so $|G_1'| = 3$ (Lemma 3(d)). In that case, G_1 is minimal nonabelian and $G_1 = \langle a, b \mid a^{3^m} = b^{3^n} = 1,\, a^b = a^{1+3^{m-1}}\rangle$ (Lemma 3(i,l)). The center $\mathrm{Z}(G_1)$ *is abelian of type $(3^{m-1}, 3^{n-1})$ and G-invariant.* Let $k = \min\{m-1, n-1\}$. Then *$\mho_k(\mathrm{Z}(G_1))$ is G-invariant and cyclic of order* $3^{|m-n|}$ so $|m - n| \le 1$ (Lemma 3(d)).

Let G be a 3-group of maximal class and order $> 3^4$ and let $L < G$ be absolutely regular maximal subgroup of G (Lemma 3(c)). By Remark 4, L is either abelian or minimal nonabelian; in addition, L has no cyclic subgroup of index 3. In any case, the abelian subgroup $\Omega_1(L)$ of type $(3, 3)$ is contained in $\mathrm{Z}(L)$ (see Lemma 3(l)) so $\mathrm{C}_G(\Omega_1(L)) = L$ since $|\mathrm{Z}(G)| = 3$. Therefore, if $x \in G - L$ is of order 3 (note that, in general, such x need not exist), then x does not centralize $\Omega_1(L)$, and then such pair $\{x, L\}$ satisfies the hypothesis of Theorem 1.

Proof of Theorem 1. By Remark 4 and the paragraph following the remark, it suffices to prove that the natural semidirect product $G = \langle a \rangle \cdot L$ is a 3-group of maximal class (obviously, this semidirect product is not metacyclic). We have $|L| \geq p^4$ since the metacyclic subgroup L has no cyclic subgroup of index p.

Suppose that an element $a \in \mathrm{Aut}(L)$ of order p does not centralize $\Omega_1(L)$. Let G be defined as in the previous paragraph. By Lemma 3(a), $\Omega_1(L)$ and $L/\Omega_1(L)$ are abelian of type (p, p), and $\Omega_1(L) \triangleleft G$. Since $p > 2$, the subgroup $H = \langle a, \Omega_1(L) \rangle$ is nonabelian of order p^3 and exponent p, by assumption. We have $G = LH$ since $H \not\leq L$ and L is maximal in G. Clearly, G has no subgroup of order p^4 and exponent p (otherwise, the intersection of that subgroup with L will be of order $> p^2$ and exponent p, which is impossible).

Assume that G is regular. Then $\exp(\Omega_1(G)) = p$ (Lemma 3(a)) so, by the previous paragraph, $|\Omega_1(G)| = p^3 = |H|$ hence $\Omega_1(G) = H$. It follows that G has no elementary abelian subgroup of order p^3 so G is as in part (b2) of Lemma 3(b) (the group (b3) of Lemma 3(b) is irregular, by Lemma 3(c)). In that case, however, every metacyclic subgroup of that group has a cyclic subgroup of index p, contrary to the hypothesis.

Thus, G is irregular. In view of Remark 4 and the paragraph following it, one may assume that G is not a 3-group of maximal class. It follows from Lemma 3(c) that G is not of maximal class for all $p > 3$ (indeed, $\Phi(G) < L$ and $\Omega_1(\Phi(G))$ is of exponent p and order $p^{p-1} > p^2 = |\Omega_1(L)|$). As we have noticed, L is absolutely regular. Therefore, by Lemma 3(e), $G = L\Omega_1(G)$, where $\Omega_1(G)$ is of order p^p and exponent p. Since $L \cap \Omega_1(G) = \Omega_1(L)$ is abelian of order p^2, we get $p = 3$. It follows that $\Omega_1(G) = H = \langle x, \Omega_1(L) \rangle$ is nonabelian of order p^3 and exponent p so G has no elementary abelian subgroup of order p^3. In that case, G is an *irregular* 3-group of maximal class (since, as we have noticed, any group of part (b2) of Lemma 3(b) has no such a subgroup as L), contrary to the assumption. □

Remark 5. Here we consider a similar, but more complicated, situation for $p = 2$. Suppose that a metacyclic 2-group L without cyclic subgroups of index 2 is maximal in a 2-group G; then $\Omega_1(L)$ is a G-invariant four-subgroup (this follows immediately from Lemma 3(h)), and so G is not of maximal class. Let, in addition, $\Omega_1(L) \leq \mathrm{Z}(L)$. Suppose that there is an involution

$a \in G - L$ that does not centralize $\Omega_1(L)$. Since $\langle x, \Omega_1(L)\rangle \cong \mathrm{D}_8$, it follows that G is not metacyclic (otherwise, G is of maximal class, by Lemma 3(h)). By hypothesis, $\mathrm{C}_G(\Omega_1(L)) = L$. If $E < G$ is elementary abelian of order 8, then $L \cap E = \Omega_1(L)$ so $\mathrm{C}_G(\Omega_1(L)) \geq LE = G$, a contradiction. Let H be a minimal nonmetacyclic subgroup of G; then $H \not\leq L$. Since H has no subgroup $\cong \mathrm{E}_8$, we get $|H| > 8$ and $\exp(H) = 4$ (Lemma 3(f)). If $\mathrm{Z}(H) \cong \mathrm{E}_4$, then $\mathrm{Z}(H)$ is contained in every abelian subgroup of H of order ≥ 8 (Lemma 3(f)) so, since $H \cap L$ contains an abelian subgroup of order 8 (Lemma 3(h)), we get $\mathrm{Z}(H) = \Omega_1(L)$ and $\mathrm{C}_G(\Omega_1(L)) \geq HL = G$, a contradiction. Thus, $\mathrm{Z}(H)$ is cyclic so, by Lemma 3(f), $H \cong \mathrm{D}_8 * \mathrm{C}_4$ is of order 16. A similar argument shows that if $A < G$ and $A \not\leq L$ is minimal nonabelian, then A has a cyclic subgroup of index 2. Indeed, A is metacyclic (Lemma 3(l)) so, if $|A| > 8$, then $|\Omega_1(A)| \leq 4$ and, if $\Omega_1(A) \cong \mathrm{E}_4$, then $\Omega_1(A) \not\leq \mathrm{Z}(A) = \varPhi(A)(\leq L)$ so $\Phi(A) = \mho_1(A)$ is cyclic. Now we construct a group $G = \langle a, L\rangle$, where $a \in G - L$ is an involution and L is metacyclic without cyclic subgroups of index 2 and such that $\Omega_1(L) \leq \mathrm{Z}(L)$ and $\Omega_1(L) \not\leq \mathrm{Z}(G)$. Let $G = Z \, wr \, C$ (wreath product), where Z is cyclic of order $2^n > 2$ and $C = \langle a\rangle$ is of order 2; then $|G| = 2^{2n+1}$ and $\mathrm{Z}(G)$ is cyclic of order $|Z| = 2^n$. Let $L = Z \times Z^a$ be the base of the wreath product G. We see that a does not centralize $\Omega_1(L)$ and L is abelian of type $(2^n, 2^n)$.

Suppose that an abelian 2-group L of type $(2^n, 2)$, $n > 2$, is maximal in a 2-group $G = \langle a, L\rangle$ and involution a does not centralize $\Omega_1(L)$. Then $H = \langle a, \Omega_1(L)\rangle \cong D_8$. We have $\mathrm{C}_G(\Omega_1(L)) = L$ so G has no subgroups $\cong \mathrm{E}_8$ (see Remark 5). Let $Z < L$ be cyclic of index 2. We claim that $H \cap Z = \mathrm{Z}(H)$. Indeed, $H \cap L = \Omega_1(L)$ is abelian of type $(2,2)$ so cyclic $H \cap Z < H \cap L$, and our claim follows, since $\Omega_1(Z) \triangleleft G$ (consider the kernel of representation of G by permutations of left cosets of Z and take into account that $|G : Z| = 4$ and $n > 2$). Thus, $G = HZ$, by the product formula. By the modular law, $H * \Omega_2(\mathrm{C}_G(H))$ is minimal nonmetacyclic of order 2^4. Assume that $M < G$ is minimal nonmetacyclic; then M is nonabelian, $2^3 < |M| \leq 2^5$ and $\exp(M) = 4$ (Lemma 3(f)) so $M \cap L$ (of order > 4) is abelian noncyclic (Lemma 3(g)). It follows that $\Omega_1(L) < M \cap L$ so, if $\mathrm{Z}(M)$ is noncyclic, we get $\mathrm{Z}(M)) \not\leq M \cap L$ (otherwise, $\Omega_1(L) = \mathrm{Z}(M) \leq \mathrm{Z}(G)$, a contradiction). It follows from Lemma 3(f) that $\mathrm{Z}(M)$ is cyclic so $M = \mathrm{D}_8 * \mathrm{C}_4$. As in Remark 5, if $A < G$ is minimal nonabelian, then A has a cyclic subgroup of index 2.

Suppose that a nonmetacyclic subgroup U is maximal in a p-group $G = \langle x, U\rangle$, where $o(x) = p > 2$ and $\Omega_1(U) \cong \mathrm{E}_{p^2}$; then $p = 3$ and U is of maximal class and order $> 3^3$ (Lemma 3(b)). Suppose, in addition, that there is an element of order 3 in $G - U$, and all such elements do not centralize $\Omega_1(U)$ (if there are no such elements, then G is of maximal class, by the same Lemma 3(b)). Then $\mathrm{C}_G(\Omega_1(U)) = L$ is maximal in G since $\Omega_1(U) \not\leq \mathrm{Z}(U)$, and G has no elementary abelian subgroups of order 3^3. Therefore, L is metacyclic and G is as in parts (b2) or (b3) of Lemma 3(b). However, a group of part (b2) has no maximal subgroup such as U. Thus, G is a 3-group of maximal class, and L is such as the subgroup G_1 in Remark 4.

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