GLASNIK MATEMATIČKI
Vol. 44(64)(2009), 241 – 254

STABILITY IS A WEAK SHAPE INVARIANT

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Abstract. We prove that the stability is a weak (and thus, a coarse as well) shape invariant in all (standard and abstract) cases.

1. Introduction

Recall that an inverse system $\boldsymbol{X}$ in a category $\mathcal{A}$ is said to be *stable* ([5, Chapter II, Section 9]) provided it is isomorphic in *pro*-$\mathcal{A}$ to a rudimentary system (P), $\boldsymbol{X} \cong (P)$. In the case of an abstract shape category $Sh_{(\mathcal{C},\mathcal{D})}$, an $X \in Ob\mathcal{C}$ is said to be *stable* provided it has the shape of a $P \in Ob\mathcal{D}$, $Sh(X) = Sh(P)$, i.e. if it admits a $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$ such that $\boldsymbol{X}$ is stable (with respect to *pro*-$\mathcal{D}$, $\boldsymbol{X} \cong (P)$). It is obvious that the stability is a shape invariant in any case of $Sh_{(\mathcal{C},\mathcal{D})}$. Recently N. Koceić Bilan and the author founded the coarse shape theory modeled on the coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ ([4]), and after that the author and B. Červar founded the weak shape theory modeled on the weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$ ([9]). The both are related to the shape theory such that there exists the following commutative diagram of functors:

$$(*) \qquad \begin{array}{ccccc} & & \mathcal{C} & & \\ & \swarrow S & \downarrow S^* & S_* \searrow & \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{J}{\rightarrow} & Sh^*_{(\mathcal{C},\mathcal{D})} & \underset{W}{\rightarrow} & Sh_{*(\mathcal{C},\mathcal{D})} \end{array}$$

where J and WJ are faithful keeping the objects fixed.

In a recent paper ([3]) Koceić Bilan proved that the stability is a *standard* coarse shape invariant, i.e. that it is an invariant in the case of $Sh^* \equiv Sh^*_{(HTop,HPol)}$. The aim of this paper is to prove that stability is a weak (and thus, a coarse) shape invariant in any (standard and abstract)

2000 *Mathematics Subject Classification.* 55P55, 57N25.
Key words and phrases. Shape, coarse shape, weak shape, stability.

case of $Sh_{*(\mathcal{C},\mathcal{D})}$. It further implies a few interesting consequences. Nevertheless, some basic notions related to the mentioned categories are needed.

2. Preliminaries

Given a category pair $(\mathcal{C},\mathcal{D})$, $\mathcal{D}\subseteq\mathcal{C}$ - a dense subcategory, the coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ is defined in [4] similarly to the shape category $Sh_{(\mathcal{C},\mathcal{D})}$, i.e. via a generalization of the pro-category *pro*-$\mathcal{D}$, called the pro*-category - *pro**-$\mathcal{D}$, by following the same original basic idea of [5]. Let $\boldsymbol{X}=(X_\lambda,p_{\lambda\lambda'},\Lambda)$ and $\boldsymbol{Y}=(Y_\mu,q_{\mu\mu'},M)$ be inverse systems in $\mathcal{D}$. A $*$-*morphism* (originally called - S^*-morphism), $(f,f^n_\mu):\boldsymbol{X}\to\boldsymbol{Y}$, consists of a function $f:M\to\Lambda$ and of a set of $\mathcal{D}$-morphisms $f^n_\mu:X_{f(\mu)}\to Y_\mu$, $n\in\mathbb{N}$, $\mu\in M$, such that, for every related pair $\mu\leq\mu'$ in M, there exist a $\lambda\in\Lambda$, $\lambda\geq f(\mu),f(\mu')$, and an $n\in\mathbb{N}$, so that, for every $n'\geq n$,

$$f^{n'}_\mu p_{f(\mu)\lambda}=q_{\mu\mu'}f^{n'}_{\mu'}p_{f(\mu')\lambda}.$$

If $(f,f^n_\mu):\boldsymbol{X}\to\boldsymbol{Y}$ and $(g,g^n_\nu):\boldsymbol{Y}\to\boldsymbol{Z}=(Z_\nu,r_{\nu\nu'},N)$ are $*$-morphisms, their *composition* is defined to be the $*$-morphism $(h,h^n_\nu):\boldsymbol{X}\to\boldsymbol{Z}$, where $h=fg$ and $h^n_\nu=g^n_\nu f^n_{g(\nu)}$. The *identity* $*$-morphism on an $\boldsymbol{X}$ is defined to be $(1_\Lambda,1^n_{X_\lambda}=1_{X_\lambda})$. The corresponding category is denoted by *inv**-$\mathcal{D}$.
An $*$-morphism $(f,f^n_\mu):\boldsymbol{X}\to\boldsymbol{Y}$ is said to be *equivalent to* a $*$-morphism $(f',f'^n_\mu):\boldsymbol{X}\to\boldsymbol{Y}$, $(f,f^n_\mu)\sim(f',f'^n_\mu)$, provided every $\mu\in M$ admits a $\lambda\in\Lambda$, $\lambda\geq f(\mu),f'(\mu)$, and an $n\in\mathbb{N}$, such that, for every $n'\geq n$,

$$f^{n'}_\mu p_{f(\mu)\lambda}=f'^{n'}_\mu p_{f'(\mu)\lambda}.$$

The relation $\sim$ is a natural equivalence relation on the class $Mor(inv^*\text{-}\mathcal{D})$. The corresponding quotient category $inv^*\text{-}\mathcal{D}/\sim$ is the pro*-category of $\mathcal{D}$, denoted by *pro**-$\mathcal{D}$. A morphism of $pro^*\text{-}\mathcal{D}(\boldsymbol{X},\boldsymbol{Y})$, i.e. the equivalence class $[(f,f^n_\mu)]$ of a $*$-morphism (f,f^n_μ), is denoted by $\boldsymbol{f}^*:\boldsymbol{X}\to\boldsymbol{Y}$. It is readily seen that there exists a faithful functor $\underline{J}:pro\text{-}\mathcal{D}\to pro^*\text{-}\mathcal{D}$ which keeps the objects fixed. Thus, we may consider *pro*-$\mathcal{D}$ to be a subcategory of *pro**-$\mathcal{D}$.

Recall that in the special case of inverse sequences of compact metric spaces there are characterizations of a $*$-morphism and of the appropriate equivalence relation (see [4], Theorems 3.1 and 3.2, and Definition 3.3; [6], Definitions 1 and 2). However, they are purely categorical, i.e. they do not depend on the terms (compacta, compact ANR's or compact polyhedra) but only on the inverse sequences as considered category objects. Therefore, they are valid for any category $\mathcal{D}$, i.e. for any $\mathcal{D}^{\mathbb{N}}$ and $tow^*\text{-}\mathcal{D}=\mathcal{D}^{\mathbb{N}}/\sim$. Here are the full analogues of Definitions 1 and 2 of [6].

A $*$-morphism of inverse sequences $(f,f^n_j):\boldsymbol{X}\to\boldsymbol{Y}$ of a category $\mathcal{D}$ consists of an increasing unbounded function $f:\mathbb{N}\to\mathbb{N}$ and of a collection of $\mathcal{D}$-morphisms $f^n_j:X_{f(j)}\to Y_j$, $n\in\mathbb{N}$, $j\in\mathbb{N}$, such that there exists an increasing and unbounded function $\gamma : \mathbb{N} \to \mathbb{N}$ (the commutativity radius) so that, for every $n \in \mathbb{N}$, the following diagram commutes:

$$\begin{array}{ccccccc} X_{f(1)} & \leftarrow & X_{f(2)} & \leftarrow & \cdots & \leftarrow & X_{f(\gamma(n))} \\ f_1^n \downarrow & & \downarrow f_2^n & & \cdots & & \downarrow f_{\gamma(n)}^n \\ Y_1 & \leftarrow & Y_2 & \leftarrow & \cdots & \leftarrow & Y_{\gamma(n)} \end{array} .$$

Two $*$-morphisms of inverse sequences $(f, f_j^n), (f', f_j'^n) : \boldsymbol{X} \to \boldsymbol{Y}$ are equivalent (homotopic), $(f, f_j^n) \sim (f', f_j'^n)$, if and only if there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$, $\sigma \geq f, f'$, (the shift function), and there exists an increasing and unbounded function $\chi : \mathbb{N} \to \mathbb{N} \cup \{0\}$ (the homotopy radius) such that, for every $n \in \mathbb{N}$ and every $1 \leq j \leq \gamma(n)$,

$$f_j^n p_{f(j)\sigma(j)} = f_j'^n p_{f'(j)\sigma(j)}.$$

The *coarse shape category* $Sh^*_{(\mathcal{C},\mathcal{D})}$ is now defined via its realizing category pro^*-$\mathcal{D}$ (by using all $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$ of all $\mathcal{C}$-objects X), i.e. quite analogously to $Sh_{(\mathcal{C},\mathcal{D})}$ via the ordinary *pro*-$\mathcal{D}$. There also exists a functor (the *coarse shape functor*) $S^* : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^*_{(\mathcal{C},\mathcal{D})}$ such that the diagram

$$\begin{array}{ccc} & \mathcal{C} & \\ S \swarrow & & \searrow S^* \\ Sh_{(\mathcal{C},\mathcal{D})} & \underset{T}{\rightarrow} & Sh^*_{(\mathcal{C},\mathcal{D})} \end{array}$$

commutes, where S is the ordinary (abstract) shape functor, and J is a faithful functor which keeps the objects fixed.
The most interesting case is the standard one (topological spaces and spaces having the homotopy types of polyhedra or ANR's, with the morphisms - homotopy classes of mappings), i.e. $\mathcal{C} = HTop \equiv Top/\simeq$ and $\mathcal{D} = HPol \equiv Pol/\simeq$ (or $HANR \equiv ANR/\simeq$). Then the notation is simplified to Sh^*. Further, in the special case of compact metrizable spaces and compact polyhedra, i.e. $\mathcal{C} = HcM \equiv cM/\simeq$ and $\mathcal{D} = HcPol \equiv cPol/\simeq$ (or $HcANR \equiv cANR/\simeq$), the coarse shape category of compacta $Sh^*(cM)$ can be realized via tow^*-$HcPol$ or tow^*-$HcANR$.

The weak shape category $Sh_{*(\mathcal{C},\mathcal{D})}$ is another generalization of the shape category $Sh_{(\mathcal{C},\mathcal{D})}$. Although similarly defined via the (reduced) pro$_*$-category $pro^\sim_*$-$\mathcal{D}$, its morphisms are much more sophisticated than those of $Sh^*_{(\mathcal{C},\mathcal{D})}$. We are going to describe the construction of its realizing category $pro^\sim_*$-$\mathcal{D}$ given in [9]. One should mention that the basic idea comes from [8], where the categories $\underline{S}(n)$, $n \in \mathbb{N} \cup \{\omega\}$, had been constructed having the inverse sequences of compact ANR's (or polyhedra) as the objects. The category $pro^\sim_*$-$\mathcal{D}$ is a generalization of the category $\underline{S}(1)$. First of all, let $inv^\sim$-$\mathcal{D}$ denote the reduced inv-category inv-$\mathcal{D}$, which is obtained by omitting morphism sets inv-$\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$ whenever $\boldsymbol{X}$ and $\boldsymbol{Y}$ have different index sets. In addition, we assume that all the index sets are directed, ordered and cofinite, having no maximal elements. The reduced pro-category $pro^\sim$-$\mathcal{D}$ is the corresponding quotient category of $inv^{\sim}$-$\mathcal{D}$. Clearly, $pro^{\sim}$-$\mathcal{D}$ is a subcategory of pro-$\mathcal{D}$. Further, for a fixed index set Λ, there is a full subcategory pro^{Λ}-$\mathcal{D} \subseteq pro^{\sim}$-$\mathcal{D}$, and in the case $\Lambda = \mathbb{N}$, $pro^{\mathbb{N}}$-$\mathcal{D} = tow$-$\mathcal{D}$.

A *ladder* of an $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ to a $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$, with $M = \Lambda$, over a segment

$$\boldsymbol{\mu} \equiv [\mu_1, \mu_2] = \{\mu \in \Lambda \mid \mu_1 \leq \mu \leq \mu_2\} \subseteq \Lambda,$$

denoted by $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, consists of an increasing (index) function

$$f : \boldsymbol{J} \to \boldsymbol{\lambda} = \boldsymbol{\mu},$$

where $\boldsymbol{J} \subseteq \boldsymbol{\mu}$ is an initial subset of $\boldsymbol{\mu}$, and of $\mathcal{D}$-morphisms

$$f_\mu : X_{f(\mu)} \to Y_\mu, \mu \in \boldsymbol{J},$$

such that, for every related pair $\mu \leq \mu'$,

$$f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}.$$

In the case $\boldsymbol{J} = \varnothing$ (i.e. $\mu_1 \notin \boldsymbol{J}$), $f_{\boldsymbol{\mu}}$ is said to be the *empty* ladder. The *identity ladder* on an $\boldsymbol{X}$ over a $\boldsymbol{\lambda}$, denoted by $1_{\boldsymbol{X}\boldsymbol{\lambda}}$, is given by $1_{\boldsymbol{\lambda}}$ and 1_{X_λ}, $\lambda \in \boldsymbol{\lambda}$. A ladder $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$ and a ladder $g_\nu : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, $N = M = \Lambda$, admit *composition* (in the usual way) provided $\boldsymbol{\mu} = \boldsymbol{\nu}$.

Two ladders $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, $f'_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, over the same segment $\mu = \boldsymbol{\lambda}$, are said to be *equivalent* (homotopic), denoted by $f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}}$, provided they both are empty or there exists a nonempty initial subset $\boldsymbol{J}^* \subseteq \boldsymbol{J} \cap \boldsymbol{J}'$ of $\boldsymbol{\mu}$ such that

$$(\forall \mu \in \boldsymbol{J}^*)(\exists \lambda(\mu) \in \boldsymbol{\mu}, \lambda \geq f(\mu), f'(\mu)) f_\mu p_{f(\mu)\lambda} = f'_\mu p_{f'(\mu)\lambda}.$$

This is an equivalence relation on the set of all ladders of $\boldsymbol{X}$ to $\boldsymbol{Y}$ over the same $\boldsymbol{\mu}$.

A *hyperladder* of $\boldsymbol{X}$ to $\boldsymbol{Y}$, denoted by $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$, is a family of ladders $f_{\boldsymbol{\mu}} : \boldsymbol{X} \to \boldsymbol{Y}$, indexed by all the segments $\boldsymbol{\mu} = [\mu_1, \mu_2]$ in $M = \Lambda$, such that every related pair $\mu_1 \leq \mu'_1$ in Λ admits a $\lambda^1 \in \Lambda$, $\lambda^1 \geq \mu'_1$, so that, for every $\mu_2 \geq \lambda^1$, the ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$, assigned to $\boldsymbol{\mu} = [\mu_1, \mu_2]$, fulfills the requirements that $\mu'_1 \in \boldsymbol{J}$ (the domain of f) and $f(\mu'_1) \leq \lambda^1$. Briefly,

$$(\forall \mu_1 \in \Lambda)(\forall \mu'_1 \geq \mu_1)(\exists \lambda^1 \geq \mu'_1)(\forall \mu_2 \geq \lambda^1)$$

the index function $f : \boldsymbol{J} \to \boldsymbol{\lambda} = \boldsymbol{\mu} = [\mu_1, \mu_2]$ of the corresponding ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$ fulfills the following two conditions:

$$\mu'_1 \in \boldsymbol{J} \text{ and } f(\mu'_1) \leq \lambda^1.$$

The *identity hyperladder* on an $\boldsymbol{X}$, denoted by $(1_{\boldsymbol{X}\boldsymbol{\lambda}})$, is given by the family of all the identity ladders. A hyperladder $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ and a hyperladder $(g_{\boldsymbol{\nu}}) : \boldsymbol{Y} \to \boldsymbol{Z}$, $N = M = \Lambda$, are *composing* coordinatewise. All the (admissible) inverse systems in $\mathcal{D}$ and all the appropriate hyperladders form a category, denoted by $inv_*^{\sim}$-$\mathcal{D}$. Clearly, for each (admissible) fixed set Λ, there exists the corresponding full subcategory inv_*^{Λ}-$\mathcal{D} \subseteq inv_*^{\sim}$-$\mathcal{D}$.

Let $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$, $(f'_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ be a pair of hyperladders. Then $(f_{\boldsymbol{\mu}})$ is said to be *equivalent* (homotopic) *to* $(f'_{\boldsymbol{\mu}})$, denoted by $(f_{\boldsymbol{\mu}}) \simeq (f'_{\boldsymbol{\mu}})$, provided

$$(\forall \mu_1 \in \Lambda)(\forall \mu'_1 \geq \mu_1)(\exists \lambda^1_* \geq \mu'_1)(\forall \mu_2 \geq \lambda^1_*)$$

the corresponding ladders $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$ and $f'_{\boldsymbol{\mu}} \in (f'_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} = [\mu_1, \mu_2]$, are equivalent, $f_{\boldsymbol{\mu}} \simeq f'_{\boldsymbol{\mu}}$, such that, in addition,

$$\mu'_1 \in \boldsymbol{J}^* \subseteq \boldsymbol{J} \cap \boldsymbol{J}' \text{ and } \lambda(\mu'_1) \leq \lambda^1_*.$$

This is an equivalence relation on each set $inv^{\sim}_*\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})$, and it is compatible with the category composition. The equivalence class $[(f_{\boldsymbol{\mu}})]$ of an $(f_{\boldsymbol{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ is denoted by $\boldsymbol{f}_* : \boldsymbol{X} \to \boldsymbol{Y}$. These classes are composing by the rule

$$\boldsymbol{g}_* \boldsymbol{f}_* = [(g_{\boldsymbol{\nu}})][(f_{\boldsymbol{\mu}})] = [(g_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}})].$$

Consequently, there exists the appropriate quotient category ("$*$-reduced pro-category")

$$pro^{\sim}_*\text{-}\mathcal{D} \equiv (inv^{\sim}_*\text{-}\mathcal{D})/(\simeq).$$

Further, for each fixed Λ, there exists the corresponding quotient category

$$pro^{\Lambda}_*\text{-}\mathcal{D} \equiv (inv^{\Lambda}_*\text{-}\mathcal{D})/(\simeq),$$

which is a full subcategory of $pro^{\sim}_*\text{-}\mathcal{D}$.

Given a Λ, let $\mathcal{C}_\Lambda \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ determined by all the $\mathcal{C}$-objects (X) admitting $\mathcal{D}$-expansions $(\boldsymbol{p} : X \to \boldsymbol{X})$ over Λ. Now, one defines (in the usual way) the Λ-*weak shape category* $Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$ such that

$$Ob(Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}) = Ob\mathcal{C}_\Lambda \text{ and } Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}(X, Y) \approx pro^{\Lambda}_*\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y}).$$

There also exists a functor (the Λ-*weak shape functor*)

$$S^{\Lambda}_* : \mathcal{C}_\Lambda \to Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})}$$

such that the diagram

$$\begin{array}{ccc} & \mathcal{C}_\Lambda & \\ S^{\Lambda} \swarrow & & \searrow S^{\Lambda}_* \\ Sh^{\Lambda}_{(\mathcal{C},\mathcal{D})} & \underset{T^{\Lambda}}{\rightarrow} & Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})} \end{array}$$

commutes, where S^{Λ} is the restriction of the ordinary (abstract) shape functor, and T^{Λ} is a faithful functor which keeps the objects fixed. Further, the key fact is that, for every pair of (admissible) index sets Λ, Λ', there exists a functor

$$H^{\Lambda,\Lambda'} : Sh^{\Lambda}_{*(\mathcal{C},\mathcal{D})} \to Sh^{\Lambda'}_{*(\mathcal{C},\mathcal{D})},$$

which is a category isomorphism keeping the objects fixed ([9, Section 6]). Therefore, there exists a category $Sh_{*(\mathcal{C},\mathcal{D})}$, called the (*abstract*) *weak shape category*, such that

$$Ob(Sh_{*(\mathcal{C},\mathcal{D})}) = Ob\mathcal{C} \text{ and } Sh_{*(\mathcal{C},\mathcal{D})}(X, Y) \approx pro^{\sim}_*\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y})(\neq \varnothing).$$

There also exists the (*abstract*) *weak shape functor*

$$S_* : \mathcal{C} \to Sh_{*(\mathcal{C},\mathcal{D})}$$

such that the diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "S"'] \arrow[dr, "S_*"] & \\ Sh_{(\mathcal{C},\mathcal{D})} \arrow[rr, "T"'] & & Sh_{*(\mathcal{C},\mathcal{D})} \end{tikzcd}$$

commutes, where S is the ordinary (abstract) shape functor, and T is a faithful functor which keeps the objects fixed.

The most interesting case is the standard one, i.e. $\mathcal{C} = HTop$ and $\mathcal{D} = HPol$ (or $HANR$). Then the notation is simplified to Sh_*. Further, in the special case $\mathcal{C} = HcM$ and $\mathcal{D} = HcPol$ (or $HcANR$), the index set $\mathbb{N}$ suffices. Thus, the weak shape category of (metrizable) compacta $Sh_*(cM)$ can be realized via tow_*-$HcPol$ or tow_*-$HcANR$.

Finally, the shape, coarse shape and weak shape categories are related by diagram $(*)$ from above (Section 1) such that $WJ = T$.

3. The weak shape and stability

Our aim is to prove that the stability is a weak (and thus, a coarse) shape invariant in any (standard and abstract) case of $Sh_{*(\mathcal{C},\mathcal{D})}$. So we state the appropriate theorem:

THEOREM 3.1. *Let $(\mathcal{C},\mathcal{D})$ be a category pair such that $\mathcal{D} \subseteq \mathcal{C}$ is dense. Let $X, Y \in Ob\mathcal{C}$ have the same weak shape type, $Sh_{*(\mathcal{C},\mathcal{D})}(X) = Sh_{*(\mathcal{C},\mathcal{D})}(Y)$. If X is stable, then so is Y.*

An equivalent formulation in terms of the realizing category $pro^{\sim}_*$-$\mathcal{D}$ for $Sh_{*(\mathcal{C},\mathcal{D})}$ can be as follows (in the sequel, "$\boldsymbol{X}$ is a $\mathcal{D}$-expansion" is the abbreviation of "$\boldsymbol{p} : X \to \boldsymbol{X}$ is a $\mathcal{D}$-expansion with respect to $\mathcal{D}$ of a $\mathcal{C}$-object X").

THEOREM 3.2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\mathcal{D}$-expansions over the same index set Λ, and let $P \in Ob\mathcal{D}$. If $\boldsymbol{X} \cong \boldsymbol{Y}$ in pro^{Λ}_*-$\mathcal{D}$ and $\boldsymbol{X} \cong (P)$ in pro-$\mathcal{D}$, then $\boldsymbol{Y} \cong (P)$ in pro-$\mathcal{D}$.*

In order to prove Theorem 3.2, we need a rather long preparation including many various auxiliary facts. First of all, recall the main statement in the proof that stability is a standard coarse shape invariant ([3, Lemma 2]):

If a topological space X has the coarse shape of a polyhedron P, then X admits a sequential *$HPol$-expansion.*

The proof of this fact is purely categorical, i.e. it does not depend on $\mathcal{C} = HTop$ or $\mathcal{D} = HPol$. Therefore, its abstract analogue is true in general, i.e. it holds for every category pair $(\mathcal{C},\mathcal{D})$, where $\mathcal{D} \subseteq \mathcal{C}$ is dense. So we may state our first lemma.

LEMMA 3.3. *Let $(\mathcal{C},\mathcal{D})$ be a category pair such that $\mathcal{D}\subseteq\mathcal{C}$ is dense. Let $\boldsymbol{Y}$ be a $\mathcal{D}$-expansion that is isomorphic in pro*-$\mathcal{D}$ to a rudimentary system (P), $P\in Ob\mathcal{D}$. Then $\boldsymbol{Y}$ is isomorphic in pro-$\mathcal{D}$ to an inverse sequence $\boldsymbol{Y}'=(Y_j',q_{jj'}',\mathbb{N})$.*

Lemma 3.3 admits to generalize the main theorem (Theorem 5) of [3] to any abstract case. Thus, we establish our second lemma.

LEMMA 3.4. *Let $(\mathcal{C},\mathcal{D})$ be a category pair such that $\mathcal{D}\subseteq\mathcal{C}$ is dense, and let $\boldsymbol{Y}$ be a $\mathcal{D}$-expansion. Then $\boldsymbol{Y}$ is isomorphic in pro-$\mathcal{D}$ to a rudimentary system (P) if and only if $\boldsymbol{Y}$ is isomorphic to (P) in pro*-$\mathcal{D}$.*

Let us now recall Lemma II.9.2. of [5]:

Let $\boldsymbol{Y}=(Y_\mu,q_{\mu\mu'},M)$ be an inverse system in an arbitrary category $\mathcal{A}$. Denote by $\widetilde{\boldsymbol{Y}}=(\boldsymbol{Y}_{\underline{\mu}},\boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'},N)$ the inverse system in *tow*-$\mathcal{A}\subseteq$ *pro*-$\mathcal{A}$ (an object of *pro*-(*tow*-$\mathcal{A}$) $\subseteq$ *pro*-(*pro*-$\mathcal{A}$)) indexed by all increasing sequences $\underline{\mu}=(\mu_j)$ in M, where

- $\boldsymbol{Y}_{\underline{\mu}}=(Y_{\mu_j},q_{\mu_j\mu_{j'}},\mathbb{N})$, $\underline{\mu}\in N$, is the corresponding inverse sequence in $\boldsymbol{Y}$;
- $(N,\leq)$ is ordered coordinatewise;
- $\boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'}:\boldsymbol{Y}_{\underline{\mu}'}\to\boldsymbol{Y}_{\underline{\mu}}$, $\underline{\mu}\leq\underline{\mu}'$ in N, is the level morphism (of *tow*-$\mathcal{A}\subseteq$ *pro*-$\mathcal{A}$) induced by the bonding morphisms $q_{\mu_j\mu_{j'}}:Y_{\mu_{j'}}\to Y_{\mu_j}$ of $\boldsymbol{Y}$.

Let, for every $\underline{\mu}=(\mu_j)\in N$, $i_{\underline{\mu}}:\mathbb{N}\to M$ denote the function, $i_{\underline{\mu}}(j)=\mu_j$, $j\in\mathbb{N}$ and let $\boldsymbol{q}_{\underline{\mu}}=[(i_{\underline{\mu}},1_{Y_{\mu_j}})]:\boldsymbol{Y}\to\boldsymbol{Y}_{\underline{\mu}}$ denote the corresponding morphism of *pro*-$\mathcal{A}$. Then $\widetilde{\boldsymbol{q}}=(\boldsymbol{q}_{\underline{\mu}}):\boldsymbol{Y}\to\widetilde{\boldsymbol{Y}}$, $\underline{\mu}\in N$, is an inverse limit of $\widetilde{\boldsymbol{Y}}$ in *pro*-(*pro*-$\mathcal{A}$).

The next lemma brings a characterization of an isomorphism of any *pro*$_*$-$\mathcal{A}$, which is much more operative than that given in [9, Theorem 10].

LEMMA 3.5. *Let $\boldsymbol{f}_*=[(f_{\boldsymbol{\mu}})]\in pro_*^{\Lambda}$-$\mathcal{A}(\boldsymbol{X},\boldsymbol{Y})$. If $\boldsymbol{f}_*$ is an isomorphism, then every representing hyperladder $(f_{\boldsymbol{\mu}})$ of $\boldsymbol{f}_*$ fulfills the following condition:*

$(WI)'$ $\qquad(\forall\mu_1\in M=\Lambda)(\forall\mu_1'\geq\mu_1)(\exists\mu_1''\geq\mu_1')(\exists\lambda^1\geq\mu_1'')(\forall\mu_2\geq\lambda^1)$

the ladder $f_{\boldsymbol{\mu}}\in(f_{\boldsymbol{\mu}})$, where $\boldsymbol{\mu}=[\mu_1,\mu_2]$, has the following property:

$$(\forall\mu\in[\mu_1,\mu_1'])(\exists\mu'\in[\mu,\mu_1''])(\exists g^{\mu'}:Y_{\mu'}\to X_{f(\mu)})$$

which commutes with $f_{\boldsymbol{\mu}}$, i.e.

$$f_\mu g^{\mu'}=q_{\mu\mu'}\ \text{and}\ g^{\mu'}f_{\mu'}p_{f(\mu')\lambda}=p_{f(\mu)\lambda},$$

for some $\lambda=\lambda(\mu)\leq\lambda^1$.

Conversely, if there exists a representative $(f_{\boldsymbol{\mu}})$ of $\boldsymbol{f}_*$, having a unique cofinal index function, such that condition (WI)$'$ holds, then $\boldsymbol{f}_*$ is an isomorphism.

Proof. First recall condition (WI) of [9, Theorem 10], for an $(f_{\boldsymbol{\mu}})$ of $\boldsymbol{f}_*$:

$$(WI) \qquad (\forall \mu_1 \in M = \Lambda)(\forall \mu \geq \mu_1)(\exists \mu' \geq \mu)(\exists \lambda \geq \mu')(\forall \mu_2 \geq \lambda)$$

there exists a $g^{\mu'} : Y_{\mu'} \to X_{f(\mu)}$ commuting with the ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} = [\mu_1, \mu_2]$, i.e.

$$f_\mu g^{\mu'} = q_{\mu\mu'} \text{ and } g^{\mu'} f_{\mu'} p_{f(\mu')\lambda} = p_{f(\mu)\lambda}.$$

The appropriate commutative diagram in $\mathcal{A}$ is given below.

$$\begin{array}{ccccccccccc}
X_{\mu_1} & \leftarrow & & X_{f(\mu)} & & \cdots & X_{f(\mu')} & \leftarrow & X_\lambda & \leftarrow & X_{\mu_2} \\
& & f_\mu \swarrow & & g^{\mu'} \nwarrow & f_{\mu'} \swarrow & & & & & \\
Y_{\mu_1} & \leftarrow & Y_\mu & & \leftarrow & Y_{\mu'} & & \leftarrow & Y_\lambda & \leftarrow & Y_{\mu_2}
\end{array}.$$

According to [9, Theorem 10], it suffices to prove that condition (WI) implies formally stronger condition (WI)$'$. Let $\mu_1 \in M$ and let $\mu_1' \geq \mu_1$. Consider a $\mu_0 \in [\mu_1, \mu_1']$ and the existing $\mu_0' \geq \mu_0$ and $\lambda_0 \geq \mu_0'$ by (WI). Observe that, by the defining property of a hyperladder, if $\mu_2 \geq \lambda_0$ is chosen sufficiently large, the term $f_\mu : P \to Y_\mu$ of the ladder $f_{\boldsymbol{\mu}}$ exists at each $\mu \in [\mu_1, \mu_0'] \subseteq [\mu_1, \mu_2] \subseteq M$. Moreover, for every such a μ_2, the corresponding index function takes its values $f(\mu) \leq \lambda_0$ for all $\mu \in [\mu_1, \mu_0']$. Then, by (WI), there also exist the appropriate morphisms $g^{\mu'}$ at all the corresponding indices μ', whenever $\lambda_0 \geq \lambda(\mu)$, for the same ladder $f_{\boldsymbol{\mu}}$. Therefore, in the case $\mu_0 = \mu_1'$, since every segment of M is a finite set, there exists a large enough $\mu_1'' \geq \mu_1'$ ("behind" all the "μ'"'s existing by (WI)) and there exists a $\lambda^1 \geq \mu_1''$ ("behind" all the "λ"'s existing by (WI)) such that, for every $\mu_2 \geq \lambda^1$, the ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$ has the desired property. ∎

Let us now state our main lemma.

Lemma 3.6. *Let $(\mathcal{C}, \mathcal{D})$ be a category pair such that $\mathcal{D} \subseteq \mathcal{C}$ is dense. Let $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be a $\mathcal{D}$-expansion, and let $\boldsymbol{P} = (P_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a trivial inverse system ($P_\lambda = P$ for all $\lambda \in \Lambda$, and $p_{\lambda\lambda'} = 1_P$ for all related pairs $\lambda \leq \lambda'$) in $\mathcal{D}$ such that $\Lambda = M$. Then, $\boldsymbol{Y}$ is isomorphic to $\boldsymbol{P}$ (equivalently, to (P)) in pro-$\mathcal{D}$ if and only if $\boldsymbol{Y}$ is isomorphic to $\boldsymbol{P}$ in pro$^\Lambda_*$-$\mathcal{D}$.*

Proof. The necessity part is trivially true. We have to prove the sufficiency. Let $\boldsymbol{f}_* : \boldsymbol{P} \to \boldsymbol{Y}$ be an isomorphism of *pro*$^\Lambda_*$-$\mathcal{D}$. Let $(f_{\boldsymbol{\mu}})$ be a representative of $\boldsymbol{f}_*$ having a unique increasing index function $f : M = \Lambda \to \Lambda$, $f \geq 1_\Lambda$, for all the nonempty ladders ([9, Lemma 6]). By Lemma 3.5,

$$(\forall \mu_1 \in M)(\forall \mu_1' \geq \mu_1)(\exists \mu_1'' \geq \mu_1')(\exists \lambda^1 \geq \mu_1'')(\forall \mu_2 \geq \lambda^1)$$

the ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} = [\mu_1, \mu_2]$, has the following property:

$$(\forall \mu \in [\mu_1, \mu_1'])(\exists \mu' \in [\mu, \mu_1''])(\exists g^{\mu'} : Y_{\mu'} \to P_{f(\mu)} = P)$$

such that

$$f_\mu g^{\mu'} = q_{\mu\mu'} \text{ and } g^{\mu'} f_{\mu'} = 1_P.$$

The proof now proceeds in four steps as follows:

- for every strictly increasing sequence $(\mu_n) \equiv \underline{\mu}$ in M such that the corresponding inverse sequence $\boldsymbol{Y}_{\underline{\mu}}$ $(\subseteq \boldsymbol{Y})$ is strongly movable, the hyperladder $(f_{\boldsymbol{\mu}})$ yields a level $*$-morphism $(1_{\mathbb{N}}, f_j^{\prime n})$ of the trivial sequence $\boldsymbol{P}' \equiv (P_n = P, 1_P, \mathbb{N})$ to the sequence $\boldsymbol{Y}' \equiv \boldsymbol{Y}_{\underline{\mu}}$ $(Y'_n \equiv Y_{\mu_n}$ and $q'_{nn'} \equiv q_{\mu_n \mu_{n'}})$;
- the equivalence class $\boldsymbol{f}'^* \equiv [(1_{\mathbb{N}}, f_j^{\prime n})] : \boldsymbol{P}' \to \boldsymbol{Y}'$ is an isomorphism of *tow**-$\mathcal{D}$;
- every strictly increasing strongly movable inverse sequence $\boldsymbol{Y}_{\underline{\mu}} \subseteq \boldsymbol{Y}$ is isomorphic in *pro*-$\mathcal{D}$ to the rudimentary system (P);
- the inverse systems $\boldsymbol{Y}$ and $\boldsymbol{P}$ are isomorphic in *pro*-$\mathcal{D}$.

STEP 1. Let $\underline{\mu} = (\mu_n)$ be a strictly increasing sequence in M such that the corresponding inverse sequence $\boldsymbol{Y}_{\underline{\mu}} \subseteq \boldsymbol{Y}$ is strongly movable. Consider any $n \in \mathbb{N}$. By the above condition, for μ_1 and "μ'_1" $= \mu_{n+1}$, i.e. for the segment $[\mu_1, \mu_{n+1}] \subseteq M$, there exist a "$\mu''_1$" $\equiv \mu_n^* \geq \mu_{n+1}$ and a "λ^1" $\equiv \lambda^n \geq \mu_n^*$ such that, for every "μ_2" $\equiv \mu_n^{**} \geq \lambda^n$, the ladder $f_{\boldsymbol{\mu}}$ (over the segment $\boldsymbol{\mu} = [\mu_1, \mu_n^{**}]$) has the above stated property. Especially,

$$(\forall j \in [1, n+1])(\exists \mu'_j \in [\mu_j, \mu_n^*])(\exists g^{\mu'_j} : Y_{\mu'_j} \to P_{f(\mu_j)} = P)$$

satisfying

$$f_{\mu_j} g^{\mu'_j} = q_{\mu_j \mu'_j} \text{ and } g^{\mu'_j} f_{\mu'_j} = 1_P.$$

Let us first assume that, for every $j \in \mathbb{N}$, μ_{j+1} is a strong movability index for μ_j. Since every $\mu \geq \mu_n^*$ can play the role of μ_n^* as well (see the proof of Lemma 3.5), we may assume that the chosen μ_n^* satisfies the strong movability condition for all pairs $\mu'_j, \mu_{j+1} \leq \mu_n^*$, $j = 1, \ldots, n$. Therefore, for every $j = 1, \ldots, n$, there exists a $\mathcal{D}$-morphism

$$s^{j+1} : Y_{\mu_{j+1}} \to Y_{\mu'_j}$$

such that

$$q_{\mu_j \mu'_j} s^{j+1} = q_{\mu_j \mu_{j+1}} \text{ and (related to } \mu_n^*) \ s^{j+1} q_{\mu_{j+1} \mu_n^*} = q_{\mu'_j \mu_n^*}.$$

Put

$$g_j = g^{\mu'_j} s^{j+1} : Y_{\mu_{j+1}} \to P_{f(\mu_j)}, j = 1, \ldots, n.$$

The first part ($j = 1$) of the appropriate diagram is filled up below (the existing term $f_{\mu'_1}$ and the morphism $g^{\mu'_1}$ are not drawn).

$$\begin{array}{ccccccccccc}
P_{\mu_1} \leftarrow & P_{f(\mu_1)} \leftarrow P_{f(\mu_2)} & & \cdots \leftarrow & & \cdots \leftarrow P_{\lambda^n} \leftarrow & P_{\mu_n^{**}} \\
f_{\mu_1} \swarrow & g_1 \nwarrow \quad \swarrow f_{\mu_2} & \cdots & f_{\mu_{n+1}} \swarrow & & & \\
Y_{\mu_1} \leftarrow & Y_{\mu_2} & & \leftarrow \ldots Y_{\mu_{n+1}} \leftarrow & & & \\
\nwarrow & \swarrow s^2 \quad \nwarrow & & \nwarrow & & \downarrow f_{\mu_n^*} & \\
Y_{\mu'_1} & & Y_{\mu'_2} & \cdots \quad Y_{\mu'_{n+1}} & \leftarrow Y_{\mu_n^*} & \leftarrow Y_{\lambda^n} \leftarrow & Y_{\mu_n^{**}}
\end{array}$$

Then, for every $j \in [1,n]$,

$$f_{\mu_j} g_j = f_{\mu_j} g^{\mu'_j} s^{j+1} = q_{\mu_j \mu'_j} s^{j+1} = q_{\mu_j \mu_{j+1}},$$

and

$$g_j f_{\mu_{j+1}} = g^{\mu'_j} s^{j+1} f_{\mu_{j+1}} = g^{\mu'_j} s^{j+1} q_{\mu_{j+1}\mu_n^*} f_{\mu_n^*} = g^{\mu'_j} q_{\mu'_j \mu_n^*} f_{\mu_n^*} = g^{\mu'_j} f_{\mu'_j} = 1_P.$$

Recall that each $P_\lambda = P$, $\lambda \in \Lambda$. Then, for each $n \in \mathbb{N}$ and every $j \in [1,n]$, by denoting $f^n_{\mu_j} \equiv f_{\mu_j}$ and $g^n_j \equiv g_j$, the following commutative diagram in $\mathcal{D}$ occurs (each top arrow is the identity 1_P)

$$\begin{array}{ccccccccc}
P_{\mu_1} & \leftarrow & P_{\mu_2} & \leftarrow \cdots \leftarrow & P_{\mu_n} & \leftarrow & P_{\mu_{n+1}} \\
f^n_{\mu_1} \downarrow & \nwarrow g^n_1 & \downarrow f^n_{\mu_2} & \nwarrow \quad \cdots & f^n_{\mu_n} \downarrow & \nwarrow g^n_n & \downarrow f^n_{\mu_n} \\
Y_{\mu_1} & \leftarrow & Y_{\mu_2} & \leftarrow \cdots \leftarrow & Y_{\mu_n} & \leftarrow & Y_{\mu_{n+1}}
\end{array}$$

Let us now denote $\boldsymbol{Y}' = \boldsymbol{Y}_{\underline{\mu}}$ by reindexing $\mu_n \mapsto n$, i.e. $\boldsymbol{Y}' = (Y'_n, q'_{nn'}, \mathbb{N})$, where $Y'_n = Y_{\mu_n}$ and $q'_{nn'} = q_{\mu_n \mu_{n'}}$. Let $\boldsymbol{P}' \equiv (P'_n = P, 1_P, \mathbb{N})$ be the trivial inverse sequence. Then, we have constructed a $*$-morphism (yielded by the hyperladder (f_μ))

$$(f', f'^n_j) : \boldsymbol{P}' \to \boldsymbol{Y}',$$

where the index function $f' = 1_{\mathbb{N}}$, and, for each $n \in \mathbb{N}$,

$$f'^n_j = f^n_{\mu_j} : P'_j \to Y'_j, \quad j = 1, \ldots, n+1,$$

($f'^n_j : P'_j \to Y'_j$ for all $j > n+1$ are chosen arbitrarily). Its commutativity radius is the function

$$\gamma : \mathbb{N} \to \mathbb{N}, \quad \gamma(n) = n+1.$$

Since $f' = 1_{\mathbb{N}}$, it is a level $*$-morphism $(1_{\mathbb{N}}, f'^n_j)$, as we claimed.

In the general case of a strictly increasing strongly movable $\boldsymbol{Y}_{\underline{\mu}}$, let μ_{k_n} in $\underline{\mu}$ be a strong movability index for μ_n, $n \in \mathbb{N}$, such that $\mu_{k_1} < \cdots < \mu_{k_n} < \cdots$. For every an $n \in \mathbb{N}$, apply the previous construction to μ_1 and "μ'_1" $= \mu_{k_{n+1}}$, i.e. to the segment $[\mu_1, \mu_{k_{n+1}}]$. Then the hyperladder $(f_{\boldsymbol{\mu}})$ will again yield a level $*$-morphism $(1_{\mathbb{N}}, f'^n_j) : \boldsymbol{P}' \to \boldsymbol{Y}'$, with $\gamma(n) = k_{n+1}$ (the difference comparing to the special case is that hereby, for a given $n \in \mathbb{N}$, $\mathcal{D}$-morphisms $g^n_j : Y_{\mu_{k_j}} \to P_{f(\mu_j)}$ exist for all $j = 1, \ldots, n$). This completes the proof of the first step.

Step 2. Observe that, in the special case of Step 1, we also have constructed a $*$-morphism

$$(g', g'^n_i) : \boldsymbol{Y}' \to \boldsymbol{P}',$$

where the index function $g' : \mathbb{N} \to \mathbb{N}$ is given by $g'(n) = n+1$, and, for each $n \in \mathbb{N}$,

$$g'^n_i = g^n_i : Y'_{i+1} \to P'_i, \quad i = 1, \ldots, n,$$

($g'^n_i : Y'_{i+1} \to P'_i$ for all $i > n$ are chosen arbitrarily). Its commutativity radius γ' is the identity function $1_{\mathbb{N}}$. Furthermore, it is obvious by definition (see the appropriate part of Section 2 or [4, Definition 3.8] and [6, Definition 2]) that the compositions

$$(g', g_i'^n)(f', f_j'^n) : \boldsymbol{P'} \to \boldsymbol{P'}$$

and

$$(f', f_j'^n)(g', g_i'^n) : \boldsymbol{Y'} \to \boldsymbol{Y'}$$

are equivalent to the corresponding $*$-identities. Thus, the equivalence class

$$\boldsymbol{f'^*} = [(f', f_j'^n)] : \boldsymbol{P'} \to \boldsymbol{Y'}$$

is an isomorphism of tow^*-$\mathcal{D}$. In the general case, $(g', g_i'^n)$ is given by $g'(n) = k_n$ and $g_i'^n = g_i^n : Y'_{k_i} \to P'_i$, $i = 1, \dots, n$, and its commutativity radius is the identity $1_{\mathbb{N}}$. Again, $\boldsymbol{g'^*} = [(g', g_i'^n)]$ is the inverse of $\boldsymbol{f'^*}$ in tow^*-$\mathcal{D}$, and the second step is finished.

Step 3. By Steps 1 and 2, we have proved that, for every strictly increasing strongly movable inverse sequence $\boldsymbol{Y}_{\underline{\mu}} \subseteq \boldsymbol{Y}$, the chosen hyperladder $(f_{\boldsymbol{\mu}}) : \boldsymbol{P} \to \boldsymbol{Y}$ yields an isomorphism $\boldsymbol{f}^*_{\underline{\mu}} : \boldsymbol{P'} \to \boldsymbol{Y}_{\underline{\mu}}$ of tow^*-$\mathcal{D}$, where $\boldsymbol{P'} = (P'_n = P, 1_P, \mathbb{N})$. Since $\boldsymbol{P'}$ is trivially isomorphic in pro^*-$\mathcal{D}$ to the rudimentary system (P), it follows that $(P) \cong \boldsymbol{Y}_{\underline{\mu}}$ in pro^*-$\mathcal{D}$. By Lemma 3.4, (P) and every such $\boldsymbol{Y}_{\underline{\mu}}$ are mutually isomorphic in pro-$\mathcal{D}$ as well. This finishes Step 3.

Step 4. First, let us prove (in addition to Step 3) that, for every strictly increasing related pair $\underline{\mu} \leq \underline{\mu}'$ in N, $\boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'} : \boldsymbol{Y}_{\underline{\mu}'} \to \boldsymbol{Y}_{\underline{\mu}}$ is an isomorphism of tow-$\mathcal{D}$ provided both $\boldsymbol{Y}_{\underline{\mu}}$ and $\boldsymbol{Y}_{\underline{\mu}'}$ are strongly movable. According to the consideration in Step 1, for every $n \in \mathbb{N}$, let μ_{k_n} and $\mu'_{k'_n}$ be the strong movability indices for μ_n and μ'_n in $\underline{\mu}$ and $\underline{\mu}'$ respectively. Put $n' = \max\{k_n, k'_n\}$. Then, $\mu_{n'}$ and $\mu'_{n'}$ are the strong movability indices in $\underline{\mu}$ and $\underline{\mu}'$ (at the same level n') for μ_n and μ'_n respectively. To prove the statement, by the well known Morita lemma ([7]), it suffices, for every $n \in \mathbb{N}$, to construct a $\mathcal{D}$-morphism w making the following diagram (all other arrows are the appropriate bonding morphisms of $\boldsymbol{Y}$) commutative:

$$\begin{array}{ccc} Y_{\mu'_n} & \leftarrow & Y_{\mu'_{n'}} \\ \downarrow & w \nwarrow & \downarrow \\ Y_{\mu_n} & \leftarrow & Y_{\mu_{n'}} \end{array} .$$

Let $n \in \mathbb{N}$, and consider the segment $[\mu_1, \mu'_{n'}] \subseteq M$. By Lemma 3.5, there exist a $\mu^*_n \geq \mu'_{n'}$ and a $\lambda^n \geq \mu^*_n$ such that, for every $\mu^{**}_n \geq \lambda^n$, the ladder $f_{\boldsymbol{\mu}} \in (f_{\boldsymbol{\mu}})$ (over $\boldsymbol{\mu} = [\mu_1, \mu^{**}_n]$) has the following property:

$$(\forall \mu \in [\mu_1, \mu'_{n'}])(\exists \mu' \in [\mu, \mu^*_n])(\exists g^{\mu'} : Y_{\mu'} \to P_{f(\mu)} = P)$$

such that $f_\mu g^{\mu'} = q_{\mu\mu'}$ and $g^{\mu'} f_{\mu'} = 1_P$. Now, in the manner of Step 1 and with a similar notation, we obtain the following commutative diagram in $\mathcal{D}$:

$$\begin{array}{ccccccccc}
Y_{\mu'_n} & & & \leftarrow & & Y_{\mu'_{n'}} & & & \\
& f_{\mu'_n} \nwarrow & & & \swarrow g'_n & & & \nwarrow f_{\mu'_{n'}} & \\
\downarrow & & P & \leftarrow & & \downarrow & & & P \;. \\
& f_{\mu_n} \swarrow & & & \nwarrow g_n & & & \swarrow f_{\mu_{n'}} & \\
Y_{\mu_n} & & & \leftarrow & & Y_{\mu_{n'}} & & &
\end{array}$$

Put

$$w = f_{\mu'_n} g_n : Y_{\mu_{n'}} \to Y'_{\mu'_n}.$$

Then,

$$q_{\mu_n \mu'_n} w = q_{\mu_n \mu_{n'}} \text{ and } w q_{\mu_{n'} \mu'_{n'}} = q_{\mu'_n \mu'_{n'}}$$

go straightforwardly. Therefore, $\boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'}$ is an isomorphism of *tow*-$\mathcal{D} \subseteq$ *pro*-$\mathcal{D}$, as we claimed.

Let $\widetilde{\boldsymbol{Y}} = (\boldsymbol{Y}_{\underline{\mu}}, \boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'}, N)$ be the inverse system in *tow*-$\mathcal{D} \subseteq$ *pro*-$\mathcal{D}$, and let $\widetilde{\boldsymbol{q}} = (\boldsymbol{q}_{\underline{\mu}}) : \boldsymbol{Y} \to \widetilde{\boldsymbol{Y}}$ be the morphism of *pro*-(*pro*-$\mathcal{D}$), which is a limit of $\widetilde{\boldsymbol{Y}}$, obtained by means of $\boldsymbol{Y}$ according to [5, Lemma II.9.2] (see the consideration followed by Lemma 3.4). Notice that the inverse system $\boldsymbol{Y}$ is strongly movable ([9, Lemma 12 (iv)]). Then, for every increasing sequence $\underline{\mu}$ in M, a simple inductive construction yields a strictly increasing sequence $\underline{\mu}' \geq \underline{\mu}$ such that the corresponding inverse sequence $\boldsymbol{Y}_{\underline{\mu}'} \subseteq \boldsymbol{Y}$ is strongly movable. Let $\widetilde{\boldsymbol{Y}}' = (\boldsymbol{Y}_{\underline{\mu}}, \boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'}, N') \subseteq \widetilde{\boldsymbol{Y}}$ be the subsystem of $\widetilde{\boldsymbol{Y}}$ determined by all the strictly increasing strongly movable inverse sequences $\boldsymbol{Y}_{\underline{\mu}} \subseteq \boldsymbol{Y}$, and let $\widetilde{\boldsymbol{q}}' = (\boldsymbol{q}_{\underline{\mu}}) : \boldsymbol{Y} \to \widetilde{\boldsymbol{Y}}'$, be the corresponding morphism of *pro*-(*pro*-$\mathcal{D}$), where $\boldsymbol{q}_{\underline{\mu}} : \boldsymbol{Y} \to \boldsymbol{Y}_{\underline{\mu}}$, $\underline{\mu} \in N'$. Then, $\widetilde{\boldsymbol{Y}}'$ is a cofinal subsystem of $\widetilde{\boldsymbol{Y}}$, and thus, the restriction morphism $\widetilde{\boldsymbol{i}} : \widetilde{\boldsymbol{Y}} \to \widetilde{\boldsymbol{Y}}'$ is a natural isomorphism of *pro*-(*tow*-$\mathcal{D}$) $\subseteq$ *pro*-(*pro*-$\mathcal{D}$). Therefore, $\widetilde{\boldsymbol{q}}' : \boldsymbol{Y} \to \widetilde{\boldsymbol{Y}}'$ is an inverse limit as well. By that, and since we have already proven that every bonding morphism, $\boldsymbol{q}_{\underline{\mu}\,\underline{\mu}'} : \boldsymbol{Y}_{\underline{\mu}'} \to \boldsymbol{Y}_{\underline{\mu}}$ of $\widetilde{\boldsymbol{Y}}'$, $\underline{\mu} \leq \underline{\mu}'$ in N', is an isomorphism of *pro*-$\mathcal{D}$, we may infer that every $\boldsymbol{q}_{\underline{\mu}} : \boldsymbol{Y} \to \boldsymbol{Y}_{\underline{\mu}}$, $\underline{\mu} \in N'$, is an isomorphism of *pro*-$\mathcal{D}$ (see [5, Remark I.5.2]). Thus, by Step 3,

$$\boldsymbol{Y} \cong \boldsymbol{Y}_{\underline{\mu}} \cong (P) \cong \boldsymbol{P}$$

in *pro*-$\mathcal{D}$ holds, which finishes Step 4, and completes the proof of Lemma 3.6. ∎

Remark 3.7. There is another way to prove the assertion of Step 4. Namely, one readily verifies that, in general, $\widetilde{\boldsymbol{q}}\boldsymbol{q} : Y \to \widetilde{\boldsymbol{Y}}$ is a *tow*-$\mathcal{D}$ expansion if and only if $\boldsymbol{q} : Y \to \boldsymbol{Y}$ is a $\mathcal{D}$-expansion. In our special case, it further leads to the fact that $\boldsymbol{Y}$ and $\widetilde{\boldsymbol{Y}}$ (and $\widetilde{\boldsymbol{Y}}'$) are isomorphic in *pro*-(*pro*-$\mathcal{D}$). Then, one can construct a natural isomorphism of $\boldsymbol{Y}_{\underline{\mu}}$ to $\widetilde{\boldsymbol{Y}}'$, $\underline{\mu} \in N'$, in *pro*-(*pro*-$\mathcal{D}$). Finally, one can prove that $\boldsymbol{q}_{\underline{\mu}} : \boldsymbol{Y} \to \boldsymbol{Y}_{\underline{\mu}}$, $\underline{\mu} \in N'$, is an isomorphism of *pro*-$\mathcal{D}$. The difference comparing to the former way is in avoiding the notion of a limit.

Proof of Theorem 3.2. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems in $\mathcal{D}$ such that $M = \Lambda$, and let $\boldsymbol{X} \cong \boldsymbol{Y}$ in pro^{Λ}_{*}-$\mathcal{D}$. Let $\boldsymbol{X}$ be isomorphic in *pro*-$\mathcal{D}$ to a rudimentary system (P). Put $\boldsymbol{P} = (P_\lambda = P, 1_P, \Lambda)$. Then $\boldsymbol{X} \cong \boldsymbol{P}$ in *pro*-$\mathcal{D}$, and thus, $\boldsymbol{X} \cong \boldsymbol{P}$ in pro^{Λ}_{*}-$\mathcal{D}$ as well. By transitivity, $\boldsymbol{Y} \cong \boldsymbol{P}$ in pro^{Λ}_{*}-$\mathcal{D}$ holds. Then, by Lemma 3.6,

$$\boldsymbol{Y} \cong \boldsymbol{P} \cong (P)$$

in *pro*-$\mathcal{D}$ holds as well. □

At the end, we add a few consequences.

Corollary 3.8. *Let $(\mathcal{C}, \mathcal{D})$ be a category pair, where $\mathcal{D} \subseteq \mathcal{C}$ is a dense subcategory. Let $X, Y \in Ob\mathcal{C}$ be of the same weak shape type, $Sh_{*(\mathcal{C},\mathcal{D})}(X) = Sh_{*(\mathcal{C},\mathcal{D})}(Y)$. If X or Y is stable, then X and Y are of the same shape type, $Sh_{(\mathcal{C},\mathcal{D})}(X) = Sh_{(\mathcal{C},\mathcal{D})}(Y)$.*

Proof. Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be $\mathcal{D}$-expansions (over the same index set) of X and Y respectively. Then $Sh_{*(\mathcal{C},\mathcal{D})}(X) = Sh_{*(\mathcal{C},\mathcal{D})}(Y)$ means that $\boldsymbol{X} \cong \boldsymbol{Y}$ in $pro^{\sim}_{*}$-$\mathcal{D}$. Let X be stable. Then there exists a $P \in Ob\mathcal{D}$ such that $\boldsymbol{X} \cong (P)$ in *pro*-$\mathcal{D}$. By Theorem 3.2, $\boldsymbol{Y} \cong (P)$ in *pro*-$\mathcal{D}$ holds as well. Thus, $\boldsymbol{X} \cong \boldsymbol{Y}$ in *pro*-$\mathcal{D}$, i.e. $Sh_{(\mathcal{C},\mathcal{D})}(X) = Sh_{(\mathcal{C},\mathcal{D})}(Y)$. If Y is stable, the proof works in the same way. □

The next corollary follows immediately.

Corollary 3.9. *The classifications by the weak, by coarse and by ordinary shape type coincide on the class of all stable objects.*

Finally, in the special standard case of compact metrizable spaces, the strong movability suffices for the same conclusion.

Corollary 3.10. *Let X and Y be compact metrizable spaces such that X or Y is strongly movable. Then the following assertions are equivalent:*

(i) $Sh_*(X) = Sh_*(Y)$;
(ii) $Sh^*(X) = Sh^*(Y)$;
(iii) $Sh(X) = Sh(Y)$;

Proof. Clearly, we only need to prove that (i) implies (iii). First observe that the category pair $(HcM, HcPol)$ is a subpair of $(HTop, HPol)$. Suppose that X is strongly movable. Then, it is an $FANR$ (see [5, Theorem II. 9.16]). By [10, Lemma 2.13] (see also [10, Remark 2.14]; [1, Theorem 1.1]; [2, (6.3) Theorem]), there exists an ANR (equivalently, a polyhedron) P, generally *noncompact*, such that $Sh(X) = Sh(P)$. This means that X is stable with respect to *pro-HPol*. The conclusion now follows by Corollary 3.8. ∎

References

[1] D. A. Edwards and R. Geoghegan, *Shapes of complexes, ends of manifolds, homotopy limits and the Wall obstruction,* Ann. of Math. (2) **101** (1975), 521–535, Correction. Ibid. **104** (1976), 389.

[2] H. M. Hastings and A. Heller, *Splitting homotopy idempotents,* in: Shape Theory and Geometric Topology (Dubrovnik, 1981), Lecture Notes in Math. **870**, Springer, Berlin - New York, 1981, 23–36.

[3] N. Koceić Bilan, *On some coarse shape invariants*, submitted.

[4] N. Koceić Bilan and N. Uglešić, *The coarse shape*, Glas. Mat. Ser. III **42(62)** (2007), 145–187.

[5] S. Mardešić and J. Segal, Shape Theory, North Holland, Amsterdam, 1982.

[6] S. Mardešić and N. Uglešić, *A category whose isomorphisms induce an equivalence relation coarser than shape*, Topology Appl. **153** (2005) 448–463.

[7] K. Morita, *The Hurewicz and the Whitehead theorems in shape theory*, Sci. Rep. Tokyo Kyoiku Daigaku Sec. A, **12** (1974) 246–258.

[8] N. Uglešić and B. Červar, *A subshape spectrum for compacta*, Glas. Mat. Ser. III **40(60)** (2005), 347–384.

[9] N. Uglešić and B. Červar, *The concept of a weak shape type*, Int. J. Pure Appl. Math. **39** (2007), 363–428.

[10] N. Uglešić and B. Červar, *The S_n-equivalence of compacta,* Glas. Mat. Ser. III **42(62)** (2007), 195–211.

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Received: 22.9.2008.
Revised: 13.10.2008.